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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

I. *On the Magnetism of Basalt and the Magnetic Behaviour of Basaltic Bars when Heated in Air. By* G. E. ALLAN, *D.Sc. (Birm.), late Lecturer in Physics in the University of Birmingham* [*].

[Plates I. & II.]

THAT the magnetism of rocks has an important bearing not only from the geological but also from the physical standpoint, viz. in its relations to regional magnetic disturbances, has been shown by Prof. Sir A. Rücker [†], in a paper on this subject, in the first part of which are given determinations of the susceptibility of a large number of different kinds of rock, basaltic and others. The magnetic properties of the numerous ferriferous minerals have also been the object of many investigations [‡].

The subject matter of the present paper is confined chiefly to one kind of rock, viz. basalt; and the experiments described were made in order to find at what temperature basalt becomes non-magnetic. The effect of temperature on magnetite is, to a large extent, already known, and as the magnetism of basalt is taken to be due chiefly to the magnetite it contains, it might be assumed that the magnetic properties of this rock are always similar to, while feebler than, those of magnetite. But this, although in a certain measure true, is not always the case.

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[*] Read October 23, 1903.

[†] A. W. Rücker, Proc. Roy. Soc. vol. xlviii. p. 505 (1890).

[‡] *Congrès Int. de Physique,* Rapports 2, p. 470 (1900).

Messrs. Barton and Williams found in 1892 that magnetite has a maximum susceptibility at 325° C.[*] and a minimum at 557°[†], whilst P. Curie[‡], in 1895, using a rougher method, found its temperature of magnetic transformation to lie in the neighbourhood of 535° C.

The basalt specimens were obtained from two localities, Rowley Regis near Birmingham, and Linz on the Rhine. Two kinds were obtained at Rowley, being known locally as the blue and grisly grey. The former variety resembles the ordinary basalt, whilst the latter is formed of much larger crystals and is a dolerite. The German basalt, which I was enabled to obtain through the instrumentality of the late Herr Boden of Rotterdam, and which the Basalt Company of Linz gave me every facility to choose, was also a finely crystalline dark blue variety (being, indeed, similar in outward appearance to the Rowley blue basalt) and was obtained from the above Company's quarry on the Dattenberg.

The magnetic tests were made on bars cut from pieces of the above-named rocks to a pattern 10·5 cms. long by 1·5 cm. square, the expense of the cutting being defrayed by a grant made by the Birmingham Philosophical Society.

Method and Apparatus.

The magnetometric method was chosen as being probably the most convenient for testing different rock specimens under similar temperature conditions from time to time, and the temperatures were measured by means of a platinum thermometer. Accordingly a combined magnetizing solenoid and furnace was made (see fig. 1). The furnace was constructed by winding, on an unglazed porcelain tube PP, 37 cms. long and 2·7 cms. in diameter, 40 double coils of platinum wire, the coils covering 25 cms. length of the tube, and the winding being closer at the ends than in the middle. The porcelain tube, the internal diameter of which was

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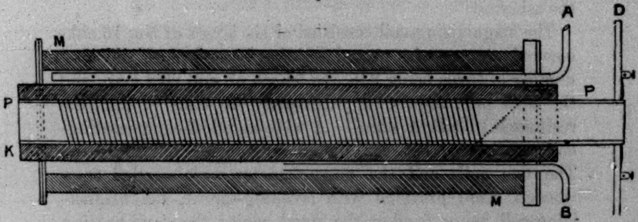
[*] E. H. Barton & W. Williams, Electrician, vol. xxix. p. 432 (1892).

[†] E. H. Barton & W. Williams, B. A. Report, Edinburgh, 1892.

[‡] P. Curie, *Ann. de Chim. et de Phys.* vol. v. ser. 7, p. 289 (1895).

2·3 cms., was supported within a wide glass cylinder, and the platinum wire was kept in position by means of a packing, K, of kaolin clay which filled up the space between the glass and the porcelain, and the ends of the glass cylinder were stopped up with a wet paste of kaolin clay, which was afterwards hardened by heat. The porcelain tube projected a few centimetres at one end, this projection being employed to carry a perforated disk, D, of biscuit ware to which the terminals of the heating-coil were attached. The glass cylinder fitted like a cartridge into the brass water-jacket on which the magnetizing-coil, MM, was wound. Water, entering the water-jacket at a low pressure by the tube A, was distributed by means of the small holes along the tube and drawn off through the tube B. A bar of basalt could thus be placed within the furnace and heated to any required temperature in a magnetic field while the magnetizing-coil was kept cool by a slow current of water. The resistance of the furnace-wire when cold was 12 ohms, and an E.M.F. of 110 volts could raise the temperature inside the furnace to 800° C., the current then being 3·75 to 3·5 amperes.

Fig. 1.—Combined magnetizing-coil M M and electric furnace P P with water-jacket A B.



The furnace described above differs from that of Ledeboer* in having within the heating-coil a tube to hold the specimen, but, on the other hand, it appears to be similar to one employed by Prof. Holborn †.

It might be remarked here that in the furnaces of Holborn, Barton & Williams, S. W. Richardson, and others asbestos is employed as the heat-insulating material. This, in the ordinary commercial form, whether of sheet, twine, or wool, was found to be quite unsuitable for use in working with basalt as it gave rise to considerable magnetic disturbances on account of the iron it contained. After testing various substances as heat-insulators, kaolin clay was substituted for the asbestos.

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* Ledeboer, *Journ. de Physique*, p. 199 (1888).

† Nature, p. 350, Feb. 17, 1902.

The magnetizing-coil consisted of six layers of No. 16 silk-covered copper wire wound on the water-jacket, the total number of turns being 956, the length of coil 29·2 cms., and the field per ampere approximately 40 C.G.S. *, and the resistance 1·93 B. A. ohms. A wide-mouthed compensating-coil was placed between the magnetometer and the scale. The two coils and the magnetometer were carried on a wooden stand provided with levelling-screws, the magnetizing-coil being fixed on the stand, whilst the stand holding the compensating-coil was movable by means of a screw, and was held in position by brass springs.

The magnetometer consisted of a small piece of steel spring, 1 cm. long, stuck on the back of a cross cut out of thin sheet copper. The cross held in front a mirror, 1 cm. in diameter, which had been cut from a thin convex lens silvered on one side, and was suspended by a fine quartz fibre 30 cms. long, the needle hanging inside a block of bronze that served to damp its vibrations. The mirror had a focal length of 1 metre, and cast the image of a cross-wire on a transparent scale at a distance of 175·5 cms. The arrangement of the apparatus, so as to allow the necessary readings and manipulations to be made by one observer, is shown in fig. 2, where F is the furnace and magnetizing solenoid and C the compensating-coil. M is the magnetometer with its lamp L and scale S_2. The coil C_2 neutralizes the disturbing effect of the current supplied to the magnetizing-coil, and C_3 that due to the furnace current. The current in the magnetizing-coil was measured by the amperemeter A, and regulated by the rheostat R. The resistance measurements were made by means of the box K M N, along with the divided ohm O O,

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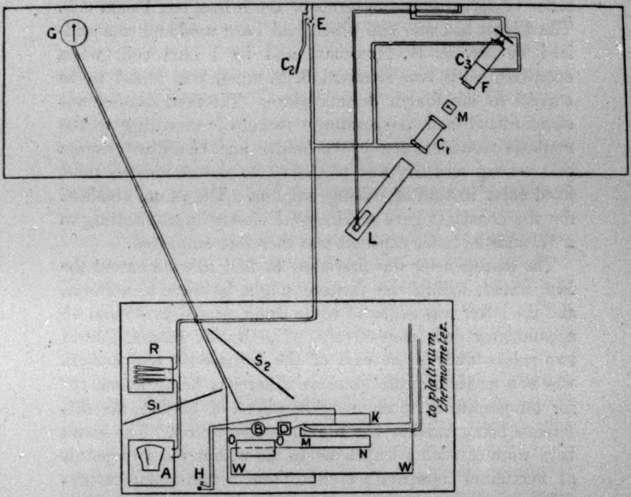
* Corrected for the end effect. See Maxwell, Elec. and Mag. vol. ii. section 676; and J. S. Townsend, Phil. Trans. p. 533 (1896).

the bridge-wire W W, and the galvanometer G, with its scale S_1; B is a Leclanché cell and D a resistance of 60 ohms. An observer standing near the key H could thus take all the readings, the commutator E being manipulated by two threads passing between E and H.

Fig. 2.—Diagram to show arrangement of apparatus.



For the platinum thermometer a length of about 110 cms. of pure platinum wire, 0·15 mm. in diameter, obtained from Messrs. Johnson & Matthey, was wound on a flat strip of mica and connected with thicker leads, compensating leads being also employed. The strip of mica on which the thermometer was wound was nearly the same length as the bars of basalt, so that the temperature measured would be the mean temperature of the part of the furnace occupied by the bar. No trouble was experienced from laying the bare thermometer on the basalt, and measurements of the surface conductivity of the bars before and after heating showed that, when cold at least, the insulation was quite sufficient. Further, the constants of the thermometer did not change by any greater amount than could be attributed to the handling the latter underwent in being placed in the furnace at the beginning of the experiments. Resistances were measured by means of an old but very accurate Elliott B. A. box combined with a divided ohm and a metre bridge, the values of the two latter being expressed in terms of the Elliott box 1 ohm coil. The Elliott box was one which had been used and standardized by Mr. R. H. Housman, and its 1 ohm coil, when compared with two standard B. A. units, was found to be correct to the fourth decimal place. The thermometer was standardized and temperatures reckoned according to the methods recommended by Callendar and Griffiths *, except that boiling mercury was employed to determine the third fixed point instead of boiling sulphur. The values obtained for the constants gave a value of δ closely approximating to 1·57, and the latter constant was therefore employed.

The thermometer was first used to find to what extent the temperature within the furnace might be taken as uniform, and the latter was explored while being heated by 25 and 48 accumulator cells respectively. The results showed that at two points towards the ends of the furnace the temperature was at a maximum, the extreme difference being about 10° for temperatures in the neighbourhood of 400° C., the difference being smaller for higher temperatures. The basalt bars were of such a length as to lie within the two points at maximum temperature, and the actual temperature-difference between the ends of the bar and the middle probably did not exceed 6° or 7°.

In employing the magnetometric method it was, of course, necessary to reduce all magnetic disturbances to a minimum. These were found to be caused chiefly by the two circuits which supplied currents to the magnetizing-coil and the furnace from a distant battery of accumulators, and the small compensating-coils C_2 and C_3 were included in the two circuits and adjusted to neutralize the disturbances mentioned.

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* H. L. Callendar, Phil. Trans. p. 161 (1887); Callendar & Griffiths, Proc. Roy. Soc. vol. xlix. p. 56 (1890); E. H. Griffiths, 'Nature,' vol. liii. p. 39 Nov. 14th, 1895.

It was also found impossible to carry out extended magnetic observations except on Sundays or during the night. A 4-minute electric tram service passing at a distance of over 300 yards did not cause any serious inconvenience.

To test the apparatus and thermometer together, a small strip of mild transformer steel, obtained from Dr. D. K. Morris, the critical temperature of which was known to lie between 770° and 780°, was placed in the furnace with the thermometer and heated up. The temperature of the critical point was found to be 776° C. The apparatus was therefore considered to give satisfactory results, the difficulty of course remaining, however, with regard to the difference of temperature that must in all probability exist between the outside and inside of a bar with such low heat-conducting power as that of basalt, when it is being heated or cooled.

Before testing the effect of temperature on basalt, some of the bars were passed through cycles of magnetization. A 4-coil manganin rheostat with mercury cups was made, the resistances of the coils being such that currents of about 0·25, 0·5, 0·75, and 1 ampere could be obtained by putting the coils in series, by steps. The current was measured by a Weston ammeter of the type called the Century Standard Testing Set, which gave 100 divisions per ampere.

The three curves given, viz. for Rowley Blue No. 2, Rowley Dolerite No. 2, and Dattenberg No. 1, show that hysteresis exists in these bars, but only in a very limited degree; and curves Nos. I. and II. (Pl. I.) form an illustration of the point emphasised by Prof. Sir A. Rücker in the paper mentioned, "that powerful permanent magnetization affords no proof of high permeability." The dolerite when first obtained and tested had the greatest effect on a long compass-needle, and appeared to be much more magnetic than the Rowley blue basalt from the same locality.

The Magnetic Behaviour of Bars of Basalt Heated in Air.

Profs. Roberts-Austen and Rücker, when measuring the specific heat of basalt*, found that, "in the processes of heating and cooling, the basalt undergoes more or less important changes in constitution, and that frequent heatings and coolings, and the nature of the flame—whether oxidizing or reducing—appeared to affect the results very seriously."

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* Roberts-Austen and Rücker, Phil. Mag. Oct. 1891, p. 353.

The bars, being heated in air, were therefore subject to change of constitution, and it appeared that the greatest change occurred during the first heating, and diminished rapidly after each subsequent heating, the magnetic properties being tolerably constant at about the sixth heating. A reddish, rust-like coating was formed on the surface of the bar, the colour increasing in intensity with the number of heatings, and the coating forming soonest on the Rowley specimens. The above changes are accompanied by loss of water, more than 0·5 gram of water, or about 0·88 per cent., being given off in some cases.

The procedure by means of which the following results were obtained may be shortly described. Each experiment took about seven hours to complete, and was carried out mostly on Sundays, as during the week the compensation of the magnetic coil was liable to change from hour to hour owing to the wall tremors caused by the motion of machinery situated on the ground-floor beneath. Great care had to be taken to prevent any motion of the compensating-coil, since there was no means of detecting any disturbance due to this cause until the end of an experiment. Further, the effect to be measured was small, and, also, a motion of the compensating-coil was accompanied by a change in the magnetic field round the needle. Compensation having been effected, the middle of the bridge was found, this chiefly depending on the plug resistance of the circuit. The bar of basalt to be tested was then, with the platinum thermometer on the top, pulled carefully into its position in the furnace by means of thread, and the ends of the furnace stopped with cotton-wool to prevent air currents. Water was then passed slowly through the water-jacket, and corresponding thermometer readings and magnetometer deflexions, due to the reversal of a current of about 1·15 amperes in the magnetizing-coil, were taken as the temperature was gradually raised or lowered. A small damping-coil, not shown in fig. 2, was employed to prevent violent oscillation of the needle on reversal of the current.

The following numbers are given as an example of the series of readings taken during an experiment.

Feb. 15th, 1903.

Time.	Current in magnetizing-coil.	Scale-reading.	Platinum thermometer.	Observed deflexion.
h m	amp.	mm.	ohms.	mm.
11 19 A.M.	±1·17	214·0 215·8	7·4	Coil effect −1·8
32	Rowley Blue No. 2 in furnace, with near end 8·5 cms. from end of furnace.			
	∓1·17	207·0 215·5	...	8·5
	10 cells on furnace.			
37	±1·169	215·8 206·3	8·0	9·5
55	1·167	206·2 215·6	9·0	9·4
	1·167	215·6 206·3	9·5	9·3
	20 cells.			
12 11 P.M.	1·165	205·4 214·8	11·0	9·4
28	1·164	214·5 204·7	12·5	9·8
34	1·163	204·2 214·0	13·0	9·8
	25 cells.			
41	1·163	214·3 204·0	14·0	10·3
50	1·162	203·9 214·3	15·0	10·4
1 0	1·162	213·8 203·5	15·5	10·3
	30 cells.			
9	1·161	203·3 214·2	16·5	10·9
19	1·161	214·3 202·5	17·5	11·8
	32 cells.			
27	1·163	202·0 214·5	18·0	12·5
32	1·164	215·2 201·6	18·2	13·6
38	1·163	200·8 215·0	18·4	14·2
45	1·162	215·6 201·6	18·5	14·0

Time.		Current in magnetizing-coil.	Scale-reading.	Platinum thermometer.	Observed deflexion.
h	m	amp.	mm.	ohms.	mm.
			35 cells.		
1	49 P.M.	1·160	216·5 202·3	18·7	14·2
	51	1·160	202·0 216·6	19·0	14·6
	56	1·160	217·8 202·0	19·3	15·8
2	1	1·159	201·7 216·7	19·5	15·0
			37 cells.		
	6	1·183	216·5 204·0	19·8	12·5
	10	1·183	204·5 215·0	20·0	10·5
	14	1·183	214·2 206·3	20·2	7·9
	21	1·183	208·2 213·0	20·4	4·8
			40 cells.		
	26	1·159	212·6 209·5	20·6	3·1
	32	1·159	209·5 212·5	21·1	3·0
	43	1·159	213·3 210·5	21·5	2·8
			50 cells.		
	49	1·150	211·2 214·3	23·0	3·1
			40 cells.		
	53	1·157	214·3 211·3	23·0	3·0
3	10	1·158	211·5 214·6	22·2	3·1
			35 cells.		
	17	1·159	214·5 211·7	21·5	2·8
			32 cells.		
	25	1·180	212·0 215·1	20·7	3·1
	27	1·180	215·5 212·3	20·4	3·2
	29	1·180	211·0 215·8	20·1	4·8
	32	1·182	218·0 209·5	19·9	8·5
	36	1·181	206·5 220·4	19·7	13·9
	41	1·182	223·0 206·7	19·5	16·3
	44	1·182	205·5 221·8	19·4	16·3
	49	1·183	223·1 206·3	19·3	16·8
	56	1·183	205·8 222·7	19·2	16·9

Time.	Current in magnetizing-coil.	Scale-reading.	Platinum thermometer.	Observed deflexion.
	amp.	mm.	ohms.	mm.
h m		30 cells.		
4 3 P.M.	1·160	224·0 207·3	19·0	16·7
8	1·160	207·2 222·5	18·7	15·3
		25 cells.		
16	1·160	225·0 212·0	18·0	13·0
23	1·160	213·4 226·4	17·4	13·0
30	1·160	228·5 216·0	17·0	12·5
		Furnace current off.		
33	1·160	216·2 229·0	16·0	12·8
38	1·160	231·0 218·5	14·0	12·5
45	1·162	218·9 231·4	12·0	12·5
56	1·163	230·2 218·3	10·0	11·9
5 11	1·163	215·6 227·3	8·5	11·7
34	±1·165	225·7 214·3	7·7	11·4
		Thermometer and basalt removed from furnace.		
40	±1·168	216·0 217·5	...	Coil effect −1·5

From the above figures, which are those for the sixth heating of Rowley Blue No. 2, it may be seen that whilst the temperature was made to vary rapidly at the beginning and at the end of the experiment, previous work having shown that there was very little change in the susceptibility of the bar under test up to about 300° C., the rate of heating or cooling was slowed down as the temperature of maximum or minimum susceptibility was approached, the temperature change in the furnace being reduced to 2° or less per minute. The effect due to the coils alone having been almost constant throughout the experiment, 1·7 mm. were added to each deflexion. The deflexions were then calculated for a current of 1·15 amperes in the magnetizing coil, and these plotted with the temperatures.

The experiments on Rowley blue basalt No. 2 show that during the first heating the susceptibility increases to a maximum at about 340° C., and then diminishes, at first slowly, then rapidly, until a minimum is reached at about 500° C. As the bar cools it does not regain its magnetic qualities, but, after passing through a maximum point at about the same temperature as during the heating, reaches the temperature of the air with about 1/7 of its initial strength. The further heating of the bar develops a maximum point, or it may be two maxima (see Curve VIII.) about 480°, the minimum being near 540°.

Curve IV. (fig. 4) was obtained as the result of a test made on the bar Rowley Blue No. 1, which had previously been used in furnace trials. The curve had retained the same character for at least six previous heatings. There is no sudden change, and the susceptibility increases gradually to a maximum at about 185°, and then diminishes to a minimum in the neighbourhood of 550°.

The next three Curves (IX., X., and XI. in fig. 6) represent the results obtained for two bars cut from a piece of Dattenberg basalt. It will be seen at once that their behaviour is very different from that of the Rowley basalt, being distinguished by a low temperature of maximum susceptibility and a gradual loss of magnetic strength with rise of temperature. For four out of five bars from this kind of rock the temperature of maximum strength was considerably below 100° C. The Dattenberg rock also does not deteriorate with heat so rapidly as that from Rowley, presumably because the former rock is of more recent formation. The bar Dattenberg No. 1 showed maximum permeability at 32° C. when being heated, and 35° on cooling. The curve for the first heating is not given, as it was marred by motion of the compensating-coil. During the third heating a maximum appeared at about 45° C., and on cooling the maximum had not been reached at 24°. The second bar gave a maximum at 143°, whilst the cooling curve was very similar to that obtained for Rowley Blue No. 1 (fig. 5). The effect of heating was to raise the susceptibility of the cold bar to nearly 2·4 times its initial value; an increase in strength due to heating may also be noted in Curves VI., VII., and VIII. The two bars were cut from a slice taken transversely from a basaltic column, and their temperature-magnetization curves are distinguished from the others by the fact that whilst an ill-defined minimum is reached at a comparatively low temperature, the bar retains considerable magnetic susceptibility at that temperature.

With a more numerous series of experiments it is probable that an explanation of the difference between the two Dattenberg bars Nos. 1 and 2 might have been forthcoming. At present it can only be surmised that No. 2, which was the only one of its kind to exhibit exceptional behaviour, was from a piece of rock which had deteriorated by weathering. Some of the basaltic columns on the Dattenberg, after lying exposed for a long time to the action of the weather, deteriorate to such an extent that a large column can be shattered into fragments with one blow of a hammer.

The curves for Dattenberg bars Nos. 3, 4, and 5, shown in fig. 7, are similar to those for bars 1 and 2. Three bars were cut from a strip of rock taken lengthwise from the edge of a basaltic column. When in position bars 3 and 5 lay near two adjacent and almost parallel faces of the column, which both showed signs of weathering; bar 4 was taken from between the other two. Whilst the initial susceptibility is different for each bar, being highest for the inside one, the temperature of maximum strength also varies, being 68°. 48°, and 57° respectively for bars 3, 4, and 5. The temperature of minimum permeability is not distinct, but all the bars show little sign of magnetization at about 250° C. These three curves were taken for the purpose of comparing the magnetic properties of bars taken from adjacent positions in the rock; and it holds for these bars and for Nos. 1 and 2, that the bar with the lowest temperature of maximum susceptibility has also the highest initial susceptibility.

Curves Nos. XIII. and XIV. are for bars Nos. 1 and 2 of Rowley dolerite. They each show feeble signs of a maximum at 109°, and another at 263°; but whilst the strength of bar No. 1 then decreases to a minimum at 445° followed by a maximum at 467°, magnetism disappearing almost entirely at 597°, bar No. 2 goes on increasing in strength, and has another maximum at 362°, reaching a minimum at a little over 600°. Bar No. 1 is unique in the series in this respect that no other exhibited so great a variation of magnetic strength as that shown in Curve XIII. between the temperatures 380° and 520°, the susceptibility of the bar at 425° being almost half the value it had at 380° and 470°. During the cooling, bar No. 1 showed a maximum susceptibility at 488°, whilst bar No. 2 was strongest at 446°. In their general character the two cooling curves are similar to those for Rowley Blue No. 2 (see Curves V., VI., and VII.). It may be noted that in the heating curves for Rowley Blue No. 2 (Curve IV.), and for Rowley dolerite No. 2 (Curve XIV.) which are very similar, and also in that for dolerite No. 1 (Curve XIII.), there is a brow in the dip of the curve corresponding to the period at which each bar has almost completely lost its magnetism.

The last curve given is for a bar of magnetite from Magnet Cove, Arkansas, cut to the same pattern as the bars of basalt. This one was placed 3·3 cms. farther from the magnetometer needle than the basalt bars, and the field employed was 3·6 C.G.S., or 1/13 of the field used to magnetize the basalt. The curve for ascending temperature is somewhat irregular, and there appears to be a maximum at or near 280°, and also at 475°, just before the magnetite begins to lose magnetism. During both the heating and the cooling the thermometer indicated a sudden change of temperature at 552°. In magnetite, therefore, the magnetic transformation is accompanied, as in iron, by a sudden change of temperature. The value 552° and the general form of the heating curve are in good agreement with the results obtained by Messrs. Barton and Williams, who give, in the volume mentioned*, the value 557° for the temperature of the minimum and also a curve, drawn to an arbitrary scale of permeability, which, however, shows no maximum at the higher temperature. Heating has the effect of considerably reducing the susceptibility of the magnetite, and during the cooling a maximum was reached at 344° C. The magnetite, when withdrawn from the furnace, was coated in places with the same red oxide (?) as the bars of basalt, the coating being present chiefly at one spot where a part of the original exposed surface of the magnetite still remained.

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* B. A. Report, Edinburgh, 1892.

Microscopic examination of the rocks was next proceeded with with a view to elucidating the effect of heat on the bars. Sections of the three different kinds of rock having been procured, the chips from which these sections had been taken were placed in the furnace and raised to 700°–800° for an hour. Sections of the heated chips were also obtained, and the photomicrographs given in figs. 10–15 (Pl. II.) show parts of the sections, figs. 10, 12, and 14 being unheated rock, and figs. 11, 13, and 15 heated rock. No change could be detected in the chief magnetic constituents of the rocks, viz. magnetite in the Dattenberg basalt, and ilmenite in the Rowley rocks, the magnetite being easily distinguished in fig. 14 as black opaque granules, often in well-formed crystals, scattered uniformly throughout the section, whilst the ilmenite appears in fig. 10 as larger black masses, and in fig. 12 in the form of long black needles and plates. The observed change had taken place in the alteration products of the rocks, such as chlorite and serpentine. The general appearance after heating indicated a cracking of the crystals, the cracks being filled with the same reddish powder as had appeared on the outside of the bars. This powder being opaque the micrographs of the heated rocks appear darker in consequence. Heat appeared to have little effect on the Dattenberg specimens, it being difficult to find any difference between the heated and the unheated sections when examined either by eye or under the microscope. The redness which was so prominent in the other heated sections was wanting in that of the Dattenberg basalt. This agrees with what was found in regard to its magnetic behaviour, viz. that heating produces less permanent change in the magnetic properties of Dattenberg basalt than in those of Rowley rock. Fig. 15 is from a thin negative and the effect of heat on the section is exaggerated. The photomicrographs, for which I am indebted to Mr. J. D. Whittles, are taken with a magnification of 40 diameters.

Susceptibility of Basalt.

The magnetometric method, when employed to test bars of the size used in the foregoing experiments, does not lead to accurate values of the susceptibility. The results below are therefore to be taken as giving a comparison of magnetic intensities of the different specimens rather than as a series of accurate absolute values of the susceptibility.

The bars employed, and their dimensions, are given in the following table.

Bar.	Weight.	Density.	Length.	Volume.
Rowley Blue.	gms.		cms.	c.c.
No. 1 …………	63·054	2·895	10·4	21·77
No. 2 …………	63·437	2·864	10·4	22·14
Rowley Dolerite.				
No. 1 …………	59·261	2·727	10·2	21·73
No. 2 …………	58·642	2·715	10·4	21·46
Dattenberg.				
No. 1 …………	73·809	3·006	10·4	24·56
No. 2 …………	66·3	2·99	10·4	22·2
No. 3 …………	67·010	2·956	10·4	22·68
No. 4 …………	70·017	3·024	10·4	23·16
No. 5 …………	69·706	3·006	10·4	23·19
Magnetite …………	117·49	4·628	10·4	25·39

To determine the susceptibilities of the rocks, a constant was found, from experiments on two bars, which, when multiplied by the deflexion produced by any bar, gave the magnetic moment of the latter. For example, the bar Rowley dolerite No. 2 was placed in the magnetizing coil, with its centre 27·8 cms. from the magnetometer-needle. By reversing a field of 46 C.G.S. the bar produced a deflexion of 32·2 mms. on a scale at a distance of 175·5 cms. The value of H at the magnetometer-needle was taken as 0·12, and was obtained by vibrating a short magnet in the position occupied by the magnetometer, and at a place where H had been measured with a magnetometer somewhat after the Kew pattern.

Employing the formula for the magnetic moment of the bar

$$M = H \frac{(r^2 - \lambda^2)^2}{2r} \tan \theta,$$

and putting $H = \cdot 12$, $r = 27\cdot 8$ cms., λ (= half the pole-distance) $= 4\cdot 2$ if we assume that the poles are 1 cm. from the ends of the bar, and

$$\tan\theta = \frac{1{\cdot}61 \text{ cm.}}{2 \times 175{\cdot}5 \text{ cms.}},$$

$$M = \frac{{\cdot}12(27{\cdot}8^2 - 4{\cdot}2^2)^2 \times 1{\cdot}61}{2 \times 27{\cdot}8 \times 2 \times 175{\cdot}5} = 5{\cdot}65.$$

Hence, the volume of the bar being 21·5 c.c., the intensity of magnetization is $\frac{5{\cdot}65}{21{\cdot}5} = 0{\cdot}263$, and the susceptibility $= 0{\cdot}0057$.

Since the moment of the bar is proportional to the deflexion produced, and as all the bars were placed at the same distance from the needle, subjected to the same magnetic field, and had the same length, with one exception, we shall not be far wrong in taking 3·6, the mean of experiments on two bars, as the magnetic moment per 20 mms. double deflexion. Employing this constant, the following values of K, the susceptibility, are found.

Bar.	Initial Deflexion, H=46 C.G.S.	K at beginning.	K at end.	Number of Heatings.
Rowley Blue.	mm.			
No. 1	31·7	·0053	·0057	One
No. 2	36·2	·0064	·0023	Six
Rowley Dolerite.				
No. 1	32·2	·0058	·0013	One
No. 2	32·2	·0059	·0009	One
Dattenberg.				
No. 1	35·2	·0056	·0039	Two
No. 2	7·5	·0013	·0031	One
No. 3	14·2	·0025		
No. 4	28·3	·0048		
No. 5	22·5	·0038		
Magnetite	61·3 (H=3·6)	·0166	·0126	One

The next table gives the initial, maximum, minimum, and final values of the susceptibility for each heating, along with the temperatures of the maxima and minima. The last column gives the percentage increase in susceptibility between the temperature of the air and the temperature of maximum susceptibility during heating.

Bar.	K at beginning.	K maximum, during heating.	K minimum.	K at end.	Temp. of maximum K.	Temp. of minimum K.	Per cent. increase in K.
Rowley Blue.							
No. 1 ...... (10th heating)	·0053	·0056	·0005	·0057	185	550	6
No. 2 ...... (1st heating)	·0064	·0087	·0009	·0009	340	500	36
No. 2 ...... (6th heating)	·0017	·0031	·0008	·0022	480	540	82
Rowley Dolerite.							
No. 1	·0058	·0087	·0004	·0013	260	597	50
No. 2	·0059	·0089	·0009	·0009	360	619	51
Dattenberg.							
No. 1 ...... (2nd heating)	·0056	·0059	·0011	·0056	32	500	5
No. 1 ...... (3rd heating)	·0058	·0060	·0010	·0039	45	510	3
No. 2	·0013	·0042	·0010	·0031	140	260	223
No. 3	·0025	·0030	·0002	...	68	250	20
No. 4	·0048	·0050	·0002	...	48	250	4
No. 5	·0038	·0042	·0004	...	57	250	11
Magnetite......	·0166	·0214 ·0206	·0003	·0126	280 475	552	29

The values found for the susceptibility of Rowley basalt are much greater than those given by Prof. Rücker in the paper mentioned at the beginning; but differences quite as great present themselves in the values found for the five Dattenberg bars.

In the same paper Prof. Rücker gives a theory of regional magnetic disturbances, the latter being explained by the magnetism of the rocks. By assuming that the temperature increases 1° for every 90 feet, a depth of 12 miles is found, below which magnetic matter ceases to exist. Now A. Harker has objected *, on geological grounds, to the assumption of a thickness of 12 miles, and puts the depth at $\frac{1}{50}$ of this amount. Prof. Rücker shows, however, that the average range of vertical disturbance being about ·00140 C.G.S., differences of this order may be obtained by supposing that the slabs of magnetic matter are 16 kiloms. (10 miles) thick, and that the upper surfaces are 4 kiloms. from the surface of the earth. We may, however, arrive at a similar result by assuming that iron exists in the crust of the earth chiefly in the form of magnetite and iron ores, and that the temperature at which these cease to be magnetic is 555° C. With a rise of temperature of 1° per 90 feet of depth this would give a depth of 15 kiloms, or $9\frac{1}{2}$ miles, as the thickness of the magnetic floor, a value which corresponds with that calculated from a knowledge of the range of vertical disturbance.

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* A. Harker, Proc. Camb. Phil. Soc. vol. x. pt. 5 (1900).

In concluding this description of the experiments, which were carried out in the Physical Laboratory of Birmingham University, I take the opportunity of expressing my thanks to Prof. Poynting for suggesting the investigation, providing the necessary apparatus and space, and for much encouragement and assistance during the progress of the experiments.

I am also indebted to all the members of the Geological Department of the same University for their cordial assistance with the geological side of the investigation.

Physical Laboratory,
Birmingham University,
May, 1903.

Discussion.

Dr. R. T. Glazebrook expressed his interest in the paper and said that Dr. Allan had given interesting data which would be of value in the subject of terrestrial magnetism.

Prof. W. F. Barrett, in a letter sent to the Secretaries, said that Dr. Allan's paper was one of great interest and considerable importance, and he wished to congratulate the Author on the results of his laborious investigation. Several years ago he (Prof. Barrett) examined the magnetic properties of various specimens of columnar basalt which had been taken from the Giant's Causeway in Co. Antrim, and a note on the results of this investigation was published in the Proceedings of the Royal Dublin Society for Dec. 1889. Each block was found permanently magnetized with a strongly-marked north and south pole, the magnetic axis running diagonally through the block and inclined to the horizon approximately at the angle of dip. As the blocks formed part of vertical columns in the Causeway, their magnetization was undoubtedly due to the Earth's magnetic field, the concave end of each block—for the ends are not plane but slightly concave or convex—was (in all the blocks examined) found to be a north-seeking pole, and must therefore have been downwards when *in situ* at the Causeway. The weight of the blocks varied from 24 to 37 kilogrammes, and the specific gravity of a fragment of one of them was found to be 2·86. With such large irregular and feebly-magnetized masses only a very approximate estimate of the magnetic moment was possible by the ordinary method. This was, however, attempted, the blocks being placed on a turn-table with their centres 60 cms. distant from the reflecting magnetometer. As might be expected, the magnet moment per gramme was found to be very small, and almost alike in each case. The volume of the blocks being known, their permanent intensity of magnetization was found to be less than that estimated by Everett for the Earth, regarded as a uniformly magnetized body, viz., 0·079. Nothing could be inferred from this, as the blocks had been removed a long time from the Causeway and had been lying about in different magnetic positions to that which they originally occupied. The long retentivity of the direction of their original magnetization in spite of rough usage was, however, somewhat remarkable.

II. *The Bending of Magnetometer Deflexion-Bars.* *By* C. Chree, *Sc.D., LL.D., F.R.S.* (*From the National Physical Laboratory.*)[*]

In May 1901 I communicated to the Society a paper[†] making various applications of Elastic Solid Equations to Metrology. Amongst the examples treated was the bending of magnetometer deflexion-bars. As explained (*l. c.* pp. 52–54), the deflecting magnet is carried by the deflexion-bar at an appreciable height above the C. G. of the cross-section, and the bending of the bar when in use, under its own weight and that of the magnet with its carriage, results in an increase of the distance between the deflecting and deflected magnets. To keep the instrument properly level, there ought to be a counterpoise on the other arm of the deflexion-bar, at the same distance as the deflecting magnet from the centre. In the absence of such a counterpoise, supposing the instrument originally level, the weight of the magnet and carriage causes a slight tilting. In consequence of this, the point of suspension of the deflected magnet moves towards the deflecting magnet, thus reducing the horizontal distance between them. This compensating effect increases with the length of the suspension of the deflected magnet; it also depends on the pattern and massiveness of the magnetometer. The tilting is objectionable for several reasons, and suitable counterpoises exist in some instruments. In others, however, there is no regular counterpoise, the only equivalent being a thermometer, usually considerably lighter than the magnet and carriage, whose position varies according to the ideas of the observer, who may even put it on the same arm as the magnet. I have investigated the tilting effect in one or two cases, but refer to it at present only to put observers on their guard. If it exists, but is overlooked, the corrections deduced from a pure bending experiment are not strictly applicable.

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[*] Read October 23, 1903.

[†] Proc. Physical Soc. vol. xviii pp. 1–55.

Since the publication of my first paper on the subject measurements of the bending effect have been made on over twenty magnetometers at the National Physical Laboratory; and it is now the regular practice to take the effect into account in framing the certificates, as these record lengths to 0·001 cm.

The earlier measurements were made by Mr. F. E. Smith, the bar being symmetrically loaded, and the consequent increase in distance between two carried points being read by microscopes.

After regular work was commenced at Bushy House, the measuring-apparatus was transferred there, and for some time magnetometers were sent over from Kew to Bushy, so that the measurements might be made by Mr. Smith on the bars supported as in use. The transport of heavy magnetometers being troublesome, and the degree of accuracy absolutely necessary being comparatively small, I devised the following method which enables the measurements to be taken at Kew. As actually carried out, the method makes no claim to the highest precision; but it is, I believe, novel and capable of development, and it could easily be applied by the owner of any of the older magnetometers which have been verified at Kew, the great majority of which are probably situated where there is no ready access to physical laboratories.

The magnetometer is set up exactly as in an ordinary deflexion experiment, with the deflecting magnet on its carriage at any convenient position on the deflexion-bar, and equal weights are hung up symmetrically, one on each arm. The consequent increase of distance between the two magnets diminishes slightly the deflecting force on the deflected magnet, and a slight change of reading is observed in the magnetometer-telescope. Putting the weights off and on, one gets in a few minutes sufficient measurements to supply a good mean. As a check, it is well to observe at two distances. Also if one takes a distance as large as 30 cms.—usually the shorter of the two distances employed in determining the horizontal force—it is advantageous to employ two deflecting magnets simultaneously, situated at equal distances on the two arms, on carriages of equal height. When only one magnet or carriage is available, a distance of 25 cms. is a convenient one in this country, with a deflecting magnet of normal moment (say 800 to 1000 C.G.S.). The smaller the horizontal force at the place, the larger is the distance at which deflexions may conveniently be taken, and the more favourable, *ceteris paribus*, are the conditions for the experiment. One can of course increase the bending effect most easily by increasing the weights applied. But if one uses weights much larger than those used at Kew (660 grms. each approximately) there is undue risk of tilting the magnetometer over—which would probably entail disastrous results. It is also desirable that the experimental weights should together not much exceed the weight of the bar (usually under a kilogram).

To a first approximation the deflexion-angle u due to a magnet of moment m, at distance r, when the horizontal force is H, is given by

$$r^3 \sin u = 2m/\mathrm{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The deflecting and deflected magnets are supposed to be mutually perpendicular, as in the ordinary horizontal force experiment. Answering to an increment δr in r, we have an increment δu in u; and by (1)—assuming m and H unchanged—the two increments are connected by the relation

$$3(\delta r/r) + \cot u \, \delta u = 0,$$

or

$$\delta r = -\tfrac{1}{3} r \cot u \, \delta u. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the present case r is a distance read off from the deflexion-bar, u is the inclination of the deflected magnet to the magnetic meridian, as given by the azimuth circle of the magnetometer, and $-\delta u$ is the change in u answering to loading by given weights*. In practice $-\delta u$ represents a number of scale-divisions, as read off on the magnetometer telescope.

Supposing α to be the value in minutes of arc of a scale-division, and $(-\delta u)$ to represent the number of scale-divisions, the working equation answering to (2) is

$$\delta r = \tfrac{1}{3} r(-\delta u) \,.\, \alpha \times \cdot 000291 \cot u. \quad . \quad . \quad . \quad (3)$$

The angle u varies of course with changes in natural declination or horizontal force, with change of temperature in the deflecting magnet, and with any artificial magnetic disturbances peculiar to the locality. Except at times of pronounced magnetic storm, natural magnetic changes or changes of temperature present no real difficulty, as loadings and unloadings occupy so little time that a few repetitions secure a mean from which slow regular disturbing effects are satisfactorily eliminated. Large irregular magnetic disturbances, such as may be experienced near an electric railway, would unquestionably be troublesome; but sites where such disturbances exist are in any case unsuitable for magnetometers.

The degree of accuracy obtainable by the method may be judged by reference to the results obtained by Mr. T. W. Baker on the first occasion when it was tried. Independent measurements were made in this instance by Mr. F. E. Smith, who employed the microscope apparatus previously in use.

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* Necessarily δr is positive and δu negative, *i. e.* the deflexion-angle is always diminished by loading.

	Magnetic Method at Kew. Weights used each 658 grms. Using				Direct Measurement at Bushy House. Weights used each 493 grms.	
	one magnet	two deflecting magnets.				
Distance where bending effect observed ... cms.	25	27·5	30	40	30	40
$-\delta u$ (minutes of arc)	2·72	4·67	2·57	1·05		
Calculated δr, ... cm.	·0088	·0098	·0098	·0156		
Calculated $EI\times 10^{-8}$	363	351	368	265	359	370
Means	361				364	

Here E is Young's modulus for the bar, and I the moment of inertia of the cross-section about a perpendicular to the plane of bending through the C. G. What either method really gives directly is EI/h, where h is the height of the magnet's centre above that of the cross-section. In the magnetic method h represented of course the exact height at which the magnet is found in actual use. In Mr. Smith's measurements h had a slightly different value, so the value of EI formed the most convenient basis for comparison.

Excluding the observations at 40 cms. in the magnetic method—to which we shall return presently—we have an exceedingly close agreement between the results from the two methods. This is partly fortuitous, as the third significant figure is probably ornamental in either case. For one thing, there is an uncertainty of probably at least 1 or 2 per cent. in the values of h. Again δu was actually observed to only 0·1 scale-division, or $0'\!\cdot\!1$ approximately, and only four or five observations with weights off and on were made at each distance. Finally, electric tram disturbances at Kew, though not absolutely large, are sufficient to interfere appreciably with the accuracy of the method when the change of deflexion-angle is small. This applies more particularly to the observations at 40 cms., where the change of deflexion-angle was only about 1′. In fact it was decided on the spot that these observations were too uncertain, and the observations at 27·5 cms. were taken to replace them.

The formula employed in calculating EI/h from δr is

$$EI/h = W'(-a^2 + 2cr - r^2)/(2\delta r), \quad . \quad . \quad . \quad . \quad (4)$$

where W' is the mean weight of the two applied weights (which should be equidistant from the centre of the bar and at least nearly equal),

$2a$ the distance between the two supports of the bar,
$2c$,, ,, ,, weights W',
r the distance from the centre of the bar where δr is observed.

If all the lengths are measured in cms., and W' is in grammes, then E is in grammes weight per sq. cm.

In applying the result it is not really necessary to calculate EI/h, supposing the observation made on the actual magnet of the magnetometer in its own carriage. Also, for practical purposes, an experiment at a single suitable distance would suffice, though two distances are better. The general formula giving the change in the distance x between the deflecting and deflected magnets due to bending under the ordinary conditions of use is

$$\delta x = \frac{h}{EI}\left[\frac{W}{2}(x^2 - a^2) + \frac{w}{6}\{l(l^2 - 3a^2) - (l - x)^3\}\right], \quad . \quad (5)$$

where

$2l$ = whole length of deflexion-bar,
$2wl$ = whole weight of deflexion-bar,
W = weight of magnet and carriage;

while a, h, and EI have the same meanings as before.

Thus, combining (4) and (5), we have

$$\delta x = \delta r\left[W(x^2 - a^2) + \tfrac{1}{3}w\{l(l^2 - 3a^2) - (l - x)^3\}\right] \div \left[W'(-a^2 + 2cr - r^2)\right], \quad . \quad . \quad (6)$$

where δr is given by (2) or (3).

Bending experiments have now been made on magnetometers by four makers. So far as elastic quality is concerned, all the bars from any one of the makers have been sufficiently alike to be grouped together. There have been differences, however, in the weights carried, which render it necessary to arrange the bars in seven groups when considering the size of the bending effect. In every case, except that of a single special Dover bar, there have been at least two specimens in each group. What is here called a "Cooke-Elliott" bar is really an old Elliott bar adapted for use with Cooke magnets &c. in India. So far as elastic property is concerned, these may be classed with ordinary Elliott bars. The means of the results obtained appear in the following table.

Maker or type of bar.	$\mathrm{EI}\times10^{-6}$.	$\mathrm{E}\times10^{-7}$.	Increase of distance between magnets due to bending at distances stated. (Unity $= 1\mu \equiv$ ·001 mm., or ·0001 cm.)							
			22·5	25	26·25	30	35	40	45	50 cms.
Cambridge Instrument Co.	375	110	...	27	...	31	36	41	45	51
Cooke India Office pattern, older	561	82	30	...	35	42	49	58	...	...
Cooke India Office pattern, newer			24	...	28	32	38	45	...	...
Dover .... usual ...	379	83	...	35	...	41	46	51	56	...
Dover .... special ...			...	46	...	55	64	73	82	92
Cooke-Elliott	336	102	43	...	50	58	70	84	...	...
Elliott usual			...	30	...	35	40	45	50	56

The great majority of the measurements on which the calculations depend have been made by Mr. F. E. Smith, to whose skill and care as an observer the consistency in the results obtained in bars from the same maker owes much. Owing to the shape of the Cooke bars, the calculation of I in their case, and hence that of E, is exposed to greater uncertainties than with the others.

In actual magnetic work distances are measured only to ·001 cm., or to 10μ, and to this degree of accuracy the results in the table are probably representative of the great majority of old magnetometers by the respective makers.

In my previous paper I remarked on the fact that in many cases the increase in distance due to the bending varied roughly as the distance itself. This accidental phenomenon—accidental in so far as it is due to an undesigned relationship between the weight of the bar and that of the magnet and carriage—is shown by the bars of all the different makes, but will be most readily seen in the table in the case of the bars by the Cambridge Instrument Co. and by Messrs. Elliott Bros. In the former case the bending increased the distance by almost exactly 1 part in 10,000 at all distances.

It is not at all improbable that the method, or modifications of it, might be found useful in a variety of elasticity measurements, especially when made on non-magnetic materials. The deflecting magnet might be replaced by a coil traversed by an electric current. It would be easy to carry a small coil or light magnet at the end of an arm, so as to bring it very close to the magnet or suspended coil meant to be deflected. In this way one could arrange that an elastic displacement, largely multiplied if desired, should produce a large effect on the field of a galvanometer-needle. Readings of the deflexion-angle of a calibrated galvanometer, or of the movement of a compensating coil situated at some distance, and movable by a fine screw so as to secure a balance, might afford an exceedingly sensitive method.

Discussion.

Prof. S. P. **Thompson** said that Dr. Chree had pointed out an important matter in magnetometry, and had told us of the existence of various sources of error. He asked why instrument-makers could not design a better instrument, referring especially to the deflexion-bar itself, which he said was not of a good form from an engineering point of view. Could not the bending be considerably reduced by making the bar in the form of a girder? Would it not be an advantage to lessen the weight of the carriage by making it of aluminium or of some light alloy? Prof. Thompson said that although the error referred to by the author of the paper seemed large, yet it could not be greater than errors arising from inaccuracy in the setting of the carriage or from other causes inherent in a deflexion-experiment.

Dr. W. **Watson** expressed his interest in the paper, and said he had never before appreciated the magnitude of the error due to the bending of the deflexion-bar. In the many determinations of horizontal magnetic force which he had carried out, he had increased the effect by placing the thermometer near to the magnet on the same side of the magnetometer. He did not think it would be an advantage to lighten the carriage too much, because of the wear and tear which an ordinary magnetometer must be capable of withstanding. With regard to the accuracy of the setting of the carriage, he said that it was not easy to set it to 0·1 mm., but that inaccuracies of setting to a certain extent compensated each other. Dr. Watson also pointed out that the position of the magnet in the carriage was not always absolutely the same.

Dr. GLAZEBROOK pointed out that the method described in the paper had been used by Dr. Chree to determine Young's Modulus by suspending known weights on the bar and observing the magnetic effect. A large number of small errors entered into the determination of horizontal force, and they were being investigated one by one at the National Physical Laboratory.

Dr. CHREE, replying to Prof. Thompson, said that various attempts had been made to reduce the bending effect. The primary difficulty was that the suspended magnet's box occupied part of the space through which a straight bar would naturally pass. By using a bar with two rectangular bends one could make the distance of the magnet from the neutral axis small, but the introduction of corners was a complication both practically and theoretically, and a bent bar was less conveniently measured. In some of the magnetometers used by the United States Coast and Geodetic Survey he understood the bar consisted of two parts, with a rectangular bend, pinned together at the centre of the instrument. This arrangement had advantages, but a great deal would depend on the continued tightness of the pinning. As to the form of the cross-section of the bar, a type which would appeal more to engineers was the girder pattern in the magnetometers used in the Indian Survey. In this case, however, the bars themselves and the magnet carriages were both heavier than in other patterns, and the bending-effect was consequently not reduced. So long as the bending-effect could be accurately determined, its magnitude seemed to him of minor importance.

There were many causes of error in determinations of horizontal force, some of which he had considered in detail in a paper published some years ago in the Royal Society's 'Proceedings.' One of the most troublesome questions, and one to which he had recently been giving special consideration, was connected with the formula for the action of the deflecting on the suspended magnet.

III. *The Conductometer.* *By* Rollo Appleyard[*].

This is a direct-reading instrument, intended for the comparison of electrical conductivity[†] of copper and other wires, for a range within, say, 5 per cent. above and 5 per cent. below 100.

In comparing two wires, either may be regarded as the standard. Suppose that balance is obtained with two samples of equal length upon a straight bridge-wire, divided into 100 parts, the position of balance being L divisions: (A) assuming the two wires to be of the same mean diameter, but of different conductivities; (B) assuming the wires to be of equal conductivity, but of unequal diameters. In case (A) it is found from the conditions of balance that a change of 1 per cent. conductivity between +5 per cent. and −5 per cent. corresponds on the average with $\frac{1}{400}$ of the total length of the bridge-wire. If therefore the middle of the bridge-wire is marked "100" and divisions are marked off from that point to right and left, each equal to $\frac{1}{400}$ of the length of the bridge-wire, these approximately correspond to successive increments of 1 per cent. conductivity. Or again, if the standard wire is not 100 per cent., move the whole scale thus graduated so that the mark on it corresponding to the conductivity of the standard is in coincidence with the electrical middle of the bridge-wire. I have proved that this arrangement is still direct-reading, and that its indications may be trusted to within a considerable degree of accuracy.

In case (B), suppose the two wires have diameters d and $(d+y)$ respectively. Then an expression for L in terms of y and d can be found. If in this expression y is given some definite value, say 1 mil, and if d is then given successive values corresponding to the whole range of diameters of wires in common use, a table or curve can be constructed showing at once the amount by which the slider must be moved, *i. e.* the *deviation* to be applied, to compensate for 1 mil, or for any fraction of 1 mil *difference* of diameters of the two wires, for any value of d within the range. Provided that y and d are in the same unit, the expression for L is perfectly general.

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[*] Read December 11, 1903.

[†] See "The Electrical Conductivity of Copper," Electrical Review, June 19, July 3 & 10, and August 14 1903

These partial operations (A) and (B), when combined, represent the complete corrections to be applied within the required limits of accuracy. I have proved that the errors involved by carrying out this method for a range from 95 to 105 per cent. conductivity never exceed 0·1 per cent. In the conductometer, the two partial operations are carried out by setting two scales with reference to the electrical middle of the bridge-wire. The first scale, marked "100" at the middle, is divided into say 10 divisions, each $\frac{1}{400}$ of the length of the bridge-wire, every such division representing 1 per cent. conductivity. This scale is set to correspond with the conductivity of the standard wire, as above explained. Or the standard wire may be replaced by a resistance-coil of the same material, corresponding to the resistance of a wire of length equal to that of the test-wire of diameter d, and of say 100 per cent. conductivity, in which case the "100" mark of this scale is placed in coincidence with the middle of the bridge-wire, and is there clamped so long as that kind of wire is being tested. A second scale is divided into graduations each equal to $\frac{1}{100}$ of the total length of bridge-wire. The middle point of this scale is marked "0." A sliding contact for the bridge-wire can be set and clamped to this scale, at a point along it corresponding to the *deviation*, as above explained. This setting is to right or left of the "0," according to whether the test-wire is of greater or of less diameter than the standard wire. The *deviation* can be calculated from the mass, as well as from the diameter of the wire; the appropriate scale-setting is then given simply by "fifty times the difference of the masses, divided by the sum of the masses," for any equal lengths whatever of tesi-wire and standard wire, the masses being expressed in any single unit of mass whatever. For routine testing, where large quantities of copper have to be dealt with, the average time of a test for a long series of tests has by this instrument been reduced to 18 seconds, with the *diameter* method. The theory and the mechanical details of the instrument are fully described in the 'Proceedings of the Institution of Civil Engineers,' vol. cliv. Session 1902–1903, part iv. It is clear that the principle above described for compensating for differences of diameter can be applied to potentiometer work generally. If conductivity is assumed constant, the instrument can be used as a very sensitive micrometer.

IV. *On a Method of Mechanically Reinforcing Sounds.*

By T. C. Porter, *M.A.*[*]

[Plate III.]

It is now about ten years since a friend, Mr. A. J. Jex-Blake, first drew my attention to the fact that if a small tuning-fork be struck and then held *in* the flame of a bunsen-burner the loudness of its note is very materially increased; at that time the explanation, though simple, did not occur to me, and although I mentioned the fact to two or three physicists, they did not suggest the cause.

That the phenomenon is not a case of ordinary resonance is proved by the fact that no increase in the loudness of the note is observed if the fork is held over the burner, either with or without the gas turned on; nor when the length of the tube of the burner is altered so that it would naturally respond, when filled with the mixture of gas and air, to the pitch of the fork employed. There is, in this case, some resonance, but it is very much fainter than the reinforcement we are considering.

Further, if the fork be held in the luminous flame from the same burner, caused by stopping up its holes, the sound is slightly louder, so that it is the action of the rarefactions and condensations of the sound-waves in the *burning* mixture of gas and air which gives rise to the increased loudness. The following experiment shows this admirably. A piece of wire gauze is supported about three-quarters of an inch above the bunsen, and the issuing mixture of gases ignited above the gauze, and the supply of gas and air so adjusted that the flame is all blue, and nearly quiet; if the fork be then held in the flame there is a very marked increase of loudness, whilst if it be held, as it easily can with a little care, between the gauze and the top of the burner, there is scarcely any augmentation of the sound.

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[*] Read December 11, 1903.

If the various parts of a bunsen flame be explored with a sounding fork it will be found that there is the greatest effect in the hottest part of the flame, *i. e.* in that part where the most rapid chemical action is proceeding, and if experiments are made to compare the effects of the luminous and nonluminous flames of the same bunsen-burner, it appears that the latter is the more energetic, though the reinforcement caused by the former is very considerable. The effect of the sound-pulses is probably therefore to change the continuous flame of the burner into one which is more or less discontinuous, each condensation and rarefaction being attended by a corresponding increase and diminution of the rate at which the gas and air are burning, and in many cases it is easy to convince oneself that this is really the case by viewing the flame in the rotating mirror, when the appearance of the flame is similar to that of the flame in experiments such as the chemical harmonicon or Koenig's vowel flames; indeed the explanation of the effect of the sounds upon the flames in those experiments, and in those to be described, is very similar. In the well-known experiments just mentioned, however, and in Tyndall's and Rayleigh's work on sensitive flames, the flames *themselves* do not sensibly reinforce the sounds to which they are sensitive. In the chemical harmonicon, for example, in which the flame burning in an open tube may be made to give a loud note by a weak note of the same pitch sung outside the tube, it is the vibration of the *air* in the tube to which the *loudness* is due, and the note itself is chiefly determined by the dimensions of the tube. In my experiments, however, each sound-wave determines what may be called an explosion, which

Fig. 3.

Curves I., II., and III.—Hysteresis in Rowley Blue No. 2, Rowley Dolerite No. 2, and Dattenberg No. 1.

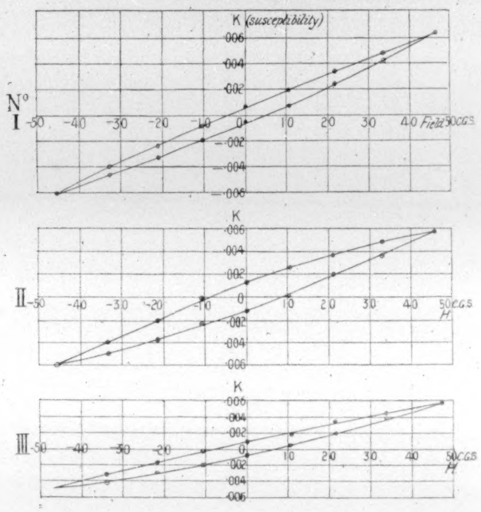


Fig. 4.

Curve IV.—Rowley Blue No. 1. Tenth heating.

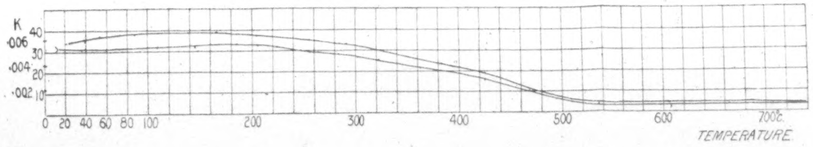


Fig. 5.

Curves V., VI., VII., VIII.—Rowley Blue No. 2, 1st, 3rd, 5th, & 6th heatings.

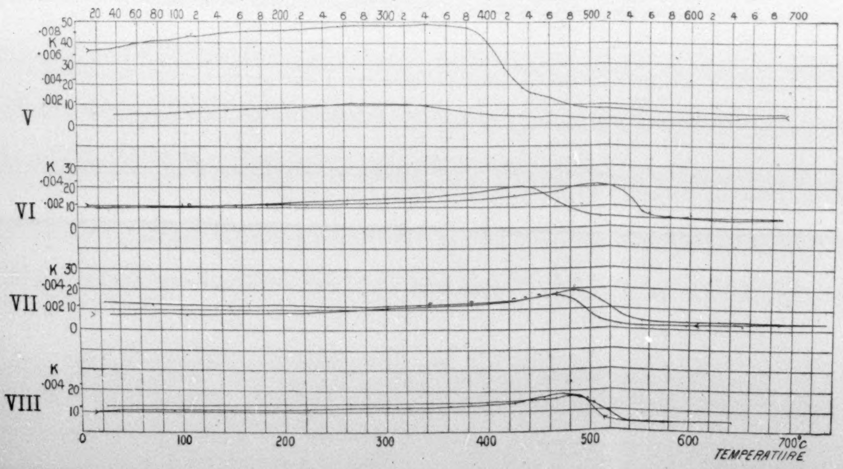


Fig. 6.

Curves IX., X., and XI.—Dattenberg bars No. 1 (2nd & 3rd heatings); No. 2 (1st heating.)

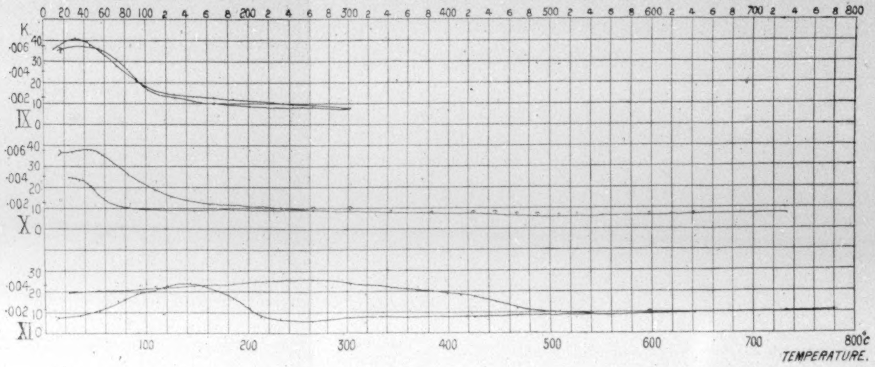


Fig. 7.

Curves XII.—Dattenberg, Nos. 3, 4, & 5. Curves with rising temperature only.

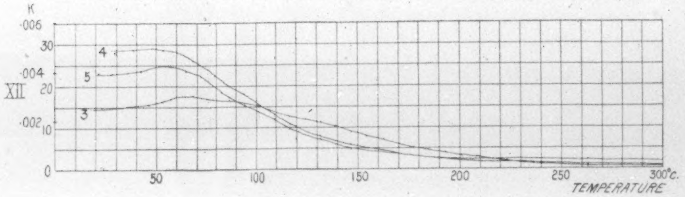


Fig. 8.

Curves XIII. and XIV.—Rowley Dolerite No. 1 (1st heating) and No. 2 (2nd heating).

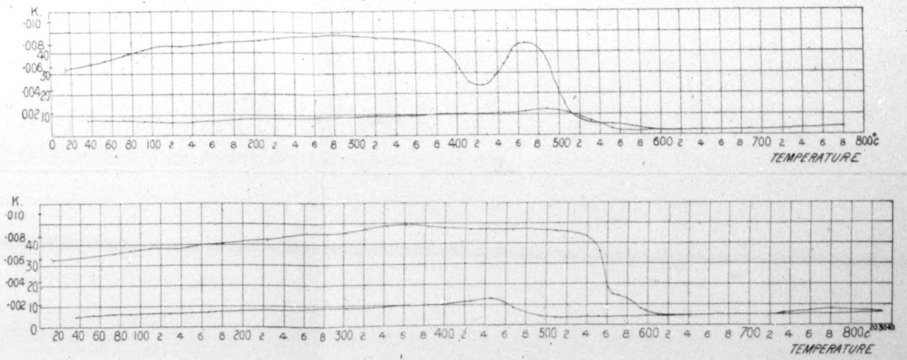


Fig. 9.

Curve XV.—Magnetite No. 1 (1st heating).

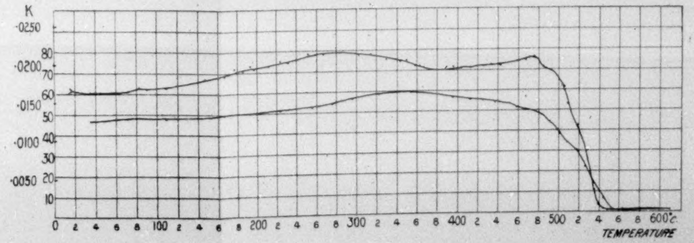
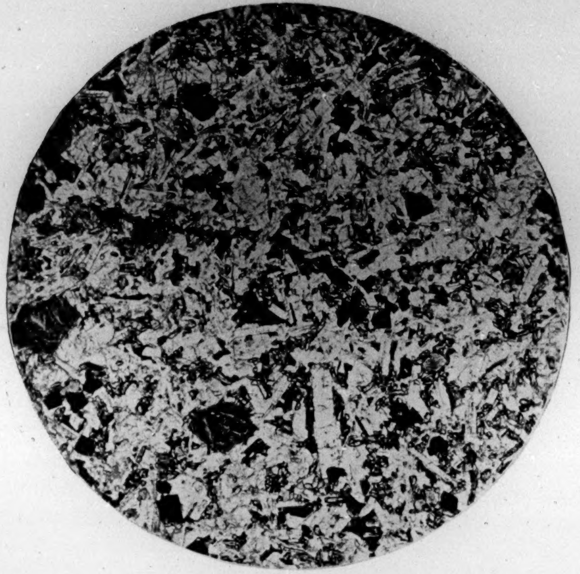
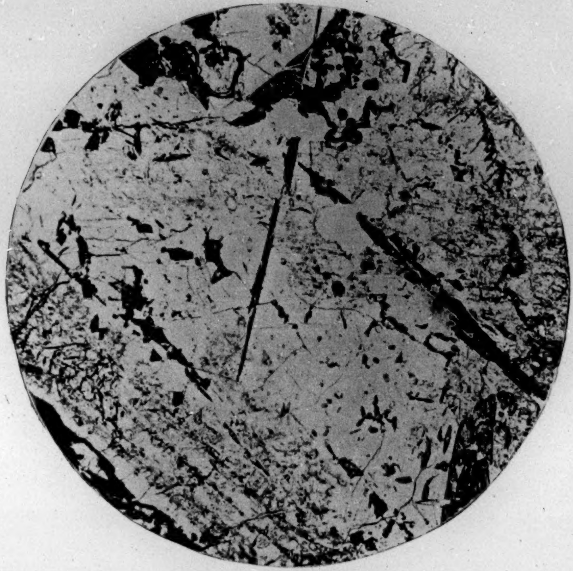


Fig. 10.



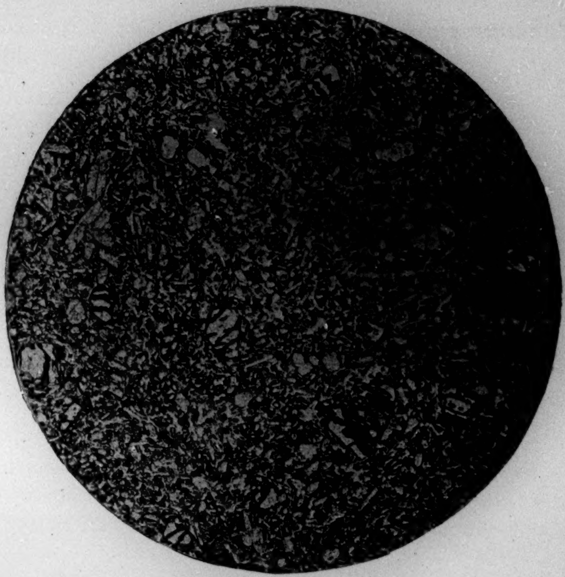
Rowley Blue, unheated.

Fig. 12.



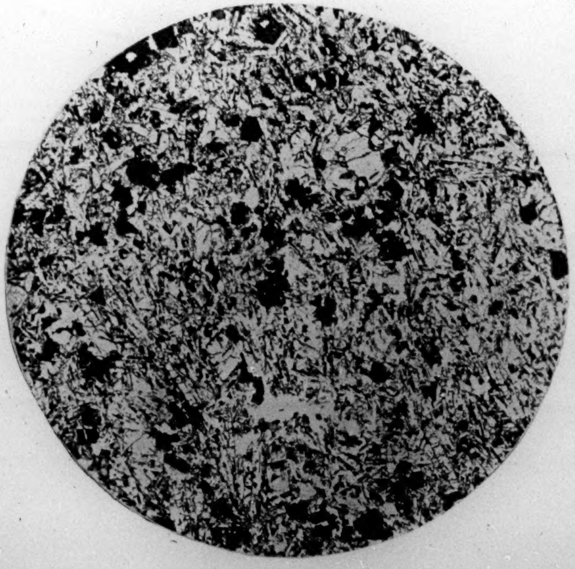
Rowley Dolerite, unheated.

Fig. 14.



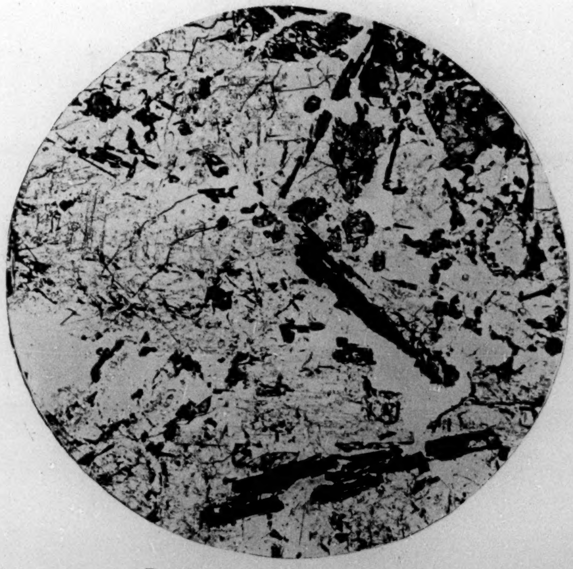
Dattenberg Basalt, unheated.

Fig. 11.



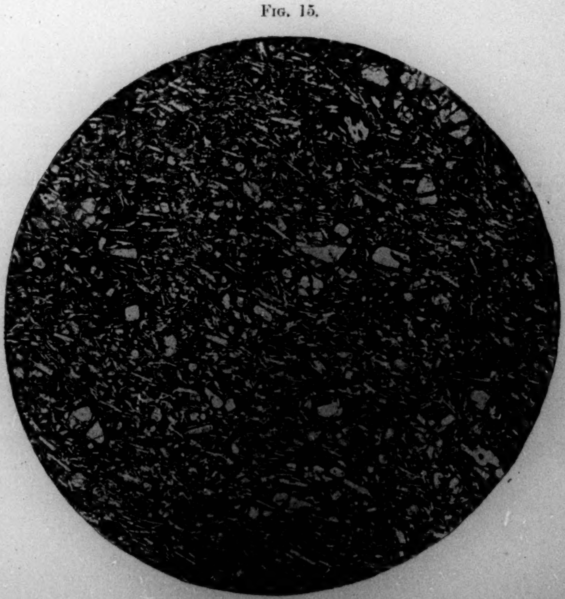
Rowley Blue, after heating.

Fig. 13.



Rowley Dolerite, after heating.

Fig. 15.



Dattenberg Basalt, after heating.

FIG. 3.

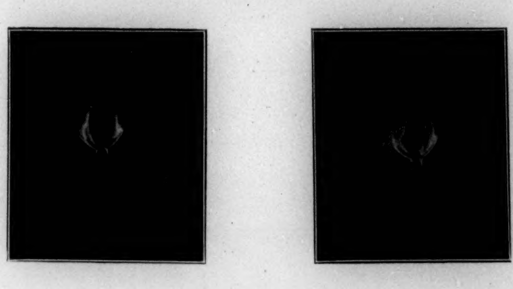


FIG. 4.

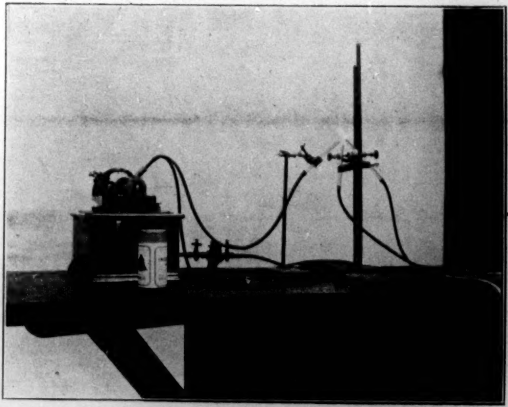
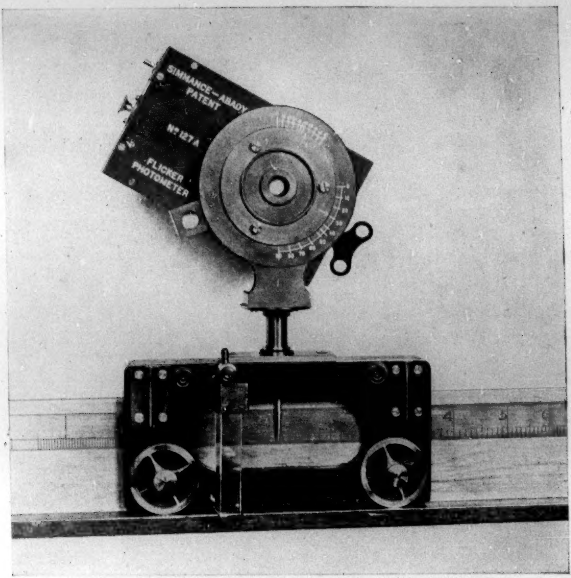


Fig. 1.


SIMMANCE-ABADY
PATENT
No 127 A
FLICKER
PHOTOMETER

SIMMANCE-ABADY PHOTOMETER (Angle-form)

Fig. 2.

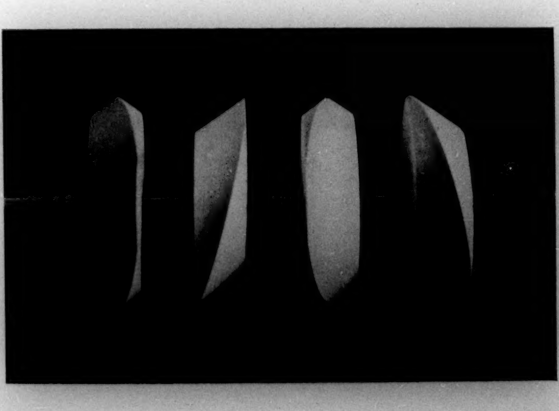
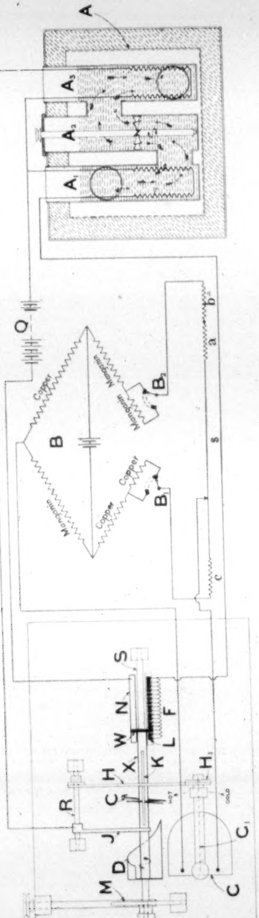


FIG. 1.


A
A₁
A₂
A₃
Q
B
Copper
Manganin
Copper
Manganin
Copper
Copper
Manganin
Manganin
B₁
B₂
a
b
s
c
S
N
W
L
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X
K
H
H₁
R
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HOT
COLD
J
D
C₁
C
M

FIG. 2.

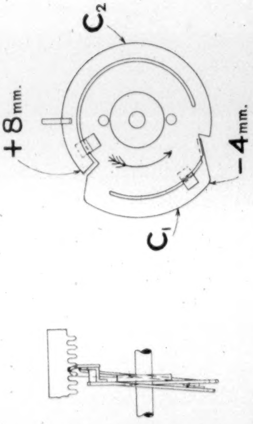
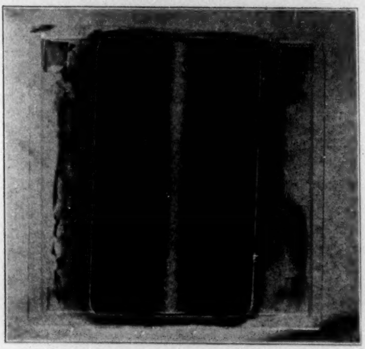
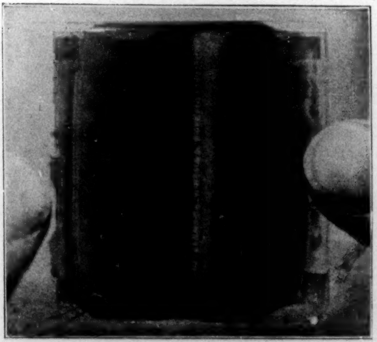

C₂
+8mm.
−4mm.
C₁

FIG. 3.



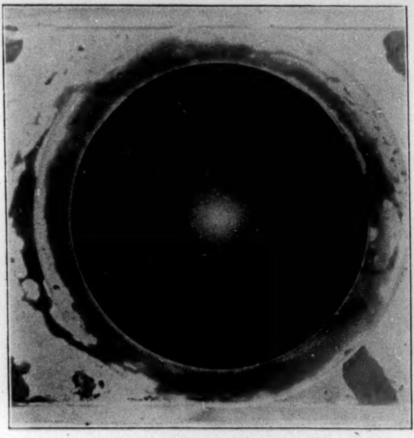
Cylindrical lens lying on a plane with fuchsin-glycerine solution between them. (Transmitted light.)

FIG. 4.



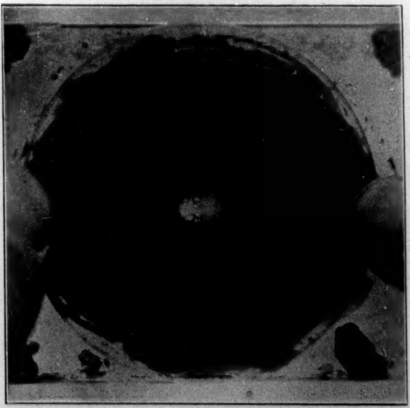
The same lens being rolled from left to right. (Transmitted and reflected light.)

Fig. 6.



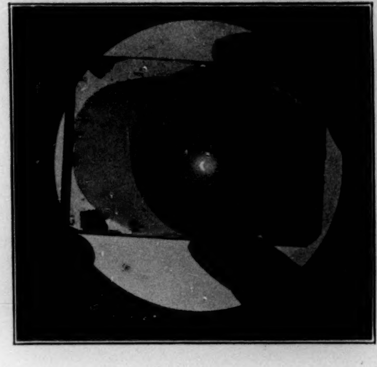
Convex lens lying on a plane with fuchsin-glycerine solution between them. (Transmitted light.)

Fig. 7.



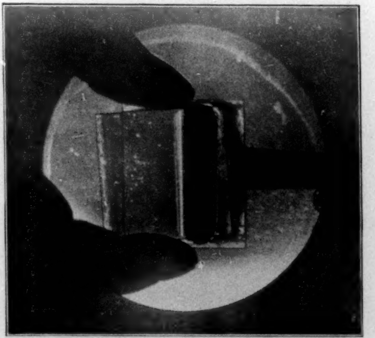
The same lens being rolled. (Transmitted and reflected light.)

Fig. 8.



A lens being drawn along a plane from left to right.
(Transmitted light.)

Fig. 9.

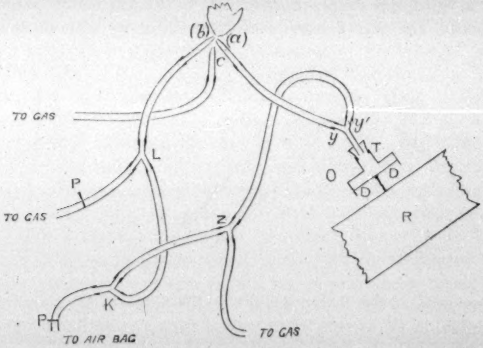


Biprism lying on a plane with fuchsin-glycerine solution between them
The biprism is being pushed from left to right.
(Transmitted light.)

may give to the air a pulse of very much greater amplitude than that of the corresponding sound-pulse, and the succession of these explosions may thus reinforce the sound enormously. Three of the conditions for the maximum effect are obviously (*a*) that the mixture of gas and air shall be that for maximum force of explosion, (*b*) that each explosion shall consume as great a mass of the explosive mixture as possible, and (*c*) that the form of the flame during the explosion shall be such as to spread the disturbance through the air as advantageously as possible.

I have used for the source of the sounds an ordinary "Home" Edison-Bell phonograph, with the "reproducer" sold to be used with the instrument. In this, as every one knows, the roughnesses of the "record" make a rod vibrate up and down, and these vibrations are communicated purely mechanically to a thin disk of glass or mica, which in turn transmits them to the air on the side of the disk remote from the rod; the aerial disturbances are then conducted by means of a tube, usually to a trumpet, but in the experiments here described the reinforcement of the sounds is obtained by the combustion of coal-gas and air, and the method in which I have obtained the best results will be next described. The arrangement can most easily be understood from fig. 1.

Fig. 1.



R is the "record," DD is the vibrating disk, T is the short metal tube on the upper side of the "reproducer," which is generally connected by a short piece of tubing with a trumpet. In this case, however, it is closed with a pierced cork, into which fits a "Y" connector, one branch of which, *y*, is connected by tubing to the jet "*a*," whilst the other branch, *y*′, is joined to a second Y-fork, called in the diagram Z; of the two remaining branches of this one leads to an ordinary gas-supply nozzle, whilst the other is joined to a third Y-piece, the stem of which is connected with a large gas-bag, containing air under very considerable pressure (generally a cwt. and a half being placed on the pressure-boards). The third arm of this Y-piece, namely K, leads to a fourth Y-piece "L," the stem of which is joined to the jet "*b*," whilst its other arm is connected to a second gas-supply nozzle. A small third jet *c* rises vertically beneath the other two, and almost touches them; this is joined by tubing to a third gas-supply nozzle. P, P, P, are screw pinch-cocks. The jets I have used have hitherto been made of glass, and in fig. 2 they are drawn life-size—in (*a*) a vertical section, in (*b*) as seen when viewed from a point immediately above them. In both these figures the relative positions and dimensions of the jets have been carefully depicted. The third jet *c*, fig. 1, is an ordinary glass jet, and, so far as I have seen, need not be of any special dimensions or shape, but *b* and *a* should be made as nearly as possible as drawn. Ordinary black indiarubber-tubing serves well for the connexions, and for the three-way pieces, those sold by the Edison-Bell Phonograph Co., 39 Charing Cross-road, serve

Fig. 2 (*a*).

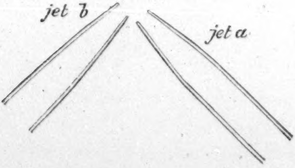
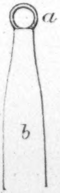


Fig. 2 (*b*).



very well. In using the apparatus the gas should be turned on first, and lit at the jets (*a*) and (*b*), and these, if of glass, should be allowed a little time to grow warm before the pinch-cock admitting air from the gas-bag to the jets is unscrewed: this should be done very gradually, and the gas and air which issues at jet (*a*) adjusted until the issuing flame is quite blue. The jet *c* should then be lighted and turned down very low. Jet *b* is then brought into the position in which it gives the loudest sound, and the final adjustment of the gas and air passing through (*b*) made by very gradually opening the pinch-cock which regulates the supply of air from the bag: if the reinforcement of the sound is still unsatisfactory, the supply of gas to (*b*) should be gradually shut off. In some instances the best results are obtained when almost pure air is issuing from *b*. If the jets hiss and roar, as they are at times wont to do, *c* may be turned up a little higher, or the position of jet *b* may be very carefully shifted further into, or away from the flame from jet *a*, or moved slightly from or towards jet *a*: in practice the adjustments are not difficult, and when all is right the reinforcement of the tones of the phonograph is very striking, being quite equivalent to the use of a small trumpet, and easily heard all over a fairly large room. Fig. 3 (Pl. III.) is the stereoscopic photograph of the flame, ·37 times life-size, when it is in a condition to "play" well, and fig. 4 is a photograph of the general arrangement of the apparatus taken whilst the flame was "playing."

It will naturally be supposed that if pure oxygen is used instead of air the results will be better: I have made the experiment, with the expected result, but the difference is not so great as one would expect, and in practice the manipulation of the jets is made so much more difficult, to say nothing of the increased cost, that I do not recommend the experiment. Ethylene and acetylene in place of coal-gas I have not tried, but there is no reason for supposing that they would yield much better results than those already obtained.

It would be unnecessary and tedious to describe all the experiments which led me to adopt the above arrangement, but there are one or two of interest that it may be as well to mention. If jet (*a*) in fig. 1 is used alone, the branches of the three-way piece Z being put into connexion with the air-bag and the gas supply respectively, a common blowpipe-flame can easily be obtained from (*a*), and this will be found to sing, but not nearly so loudly as the flame already spoken of: the blue tip of the inner cone can be seen shooting in and out as it emits the different sounds. If now a thin sheet of platinum-foil be brought into this part of the flame, the sound is very markedly reinforced, the tones given out by the hot platinum being very "round," smooth, and pleasant to the ear, though, as I have already said, they are not loud. I attribute this reinforcement to the rapid expansion and contraction of the platinum in response to the variation in the heat of the flame caused by the sound-waves from the phonograph, and not to simple reflexion of these waves from the platinum surface. The well-known surface action of the metal may also play some part in the phenomenon, although there is no reinforcement, if, whilst the platinum is in position and sounding, the blowpipe-flame is blown out: the platinum, under these circumstances, remains at a yellow heat, from the combustion of the unignited gas and air on its surface, but there is no sound until the flame is rekindled.

Finally, the action of the flame in reinforcing sounds originally of equal loudness but of different pitch deserves notice. This can be generally stated as follows:—The smaller the flame the more rapid the current of air or of gas, or of both; or the more jet (*b*) is made to encroach on jet (*a*) in fig. 2 (*b*), the higher is the pitch to which the flame most readily responds and reinforces, and *vice versâ*: but it is very remarkable how wide a range of pitch the tones may have which are reinforced at one and the same time, so that it is possible and easy in the case of a phonograph record in which the high-pitched notes have been recorded too strongly, to lessen their loudness, or, indeed, to eliminate them altogether without seriously interfering with the other notes: on the other hand, if it is the base which is too prominent, this may be reduced, whilst additional strength is added to the treble.

I do not think that it is too much to say that these facts* place at our disposal a method of reinforcing sounds possessing some special advantages, and may prove capable of increasing faint sounds so that many can hear them at once, to a degree beyond that attained by any other method at present known.

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* *Note.*—Since the reading of this paper, the writer has been much indebted to Mr. Chichester Bell for calling his attention to a communication of Mr. Bell's, read before the Royal Society, and published in the 'Transactions' of that Society, 1886, Part 2, and entitled "The Sym-

Eton, Oct. 1903.

DISCUSSION.

Mr. CHICHESTER A. BELL, in a letter to the Secretaries, expressed his interest in the subject of the paper and pointed out that the literature dealing with the vibrations of jets and flames was very extensive. He believed that all the phenomena described by various writers, including the author, were referable to the simple fact that any impulse communicated in any way whatsoever to a jet at the orifice from which it escapes, produces a disturbance which tends to increase as the jet travels onwards until finally it causes complete disintegration. He had investigated the matter at length in a paper on "The Sympathetic Vibrations of Jets," published in the Phil. Trans. for 1886, and shown that jets or flames or combinations of them may be made to reproduce sounds of all kinds and of a wide range of period. He felt

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pathetic Vibrations of Jets." It appears that as early as 1866 Kundt obtained musical tones by the impact of two flames, and of an air-jet against a flame. Barrett and Tyndall were apparently the first to notice that a sensitive flame sometimes reproduces the tones by which it is affected. In 1875 Decharme found that carbon dioxide blown through a jet against a flame gave a feeble effect, and attributed this to the decomposition of the CO_2—a chemical explanation. He also found that pure oxygen gave only a feeble effect in comparison with air, and that nitrogen gave no effect at all. Mr. Bell himself, by boring a small hole in a telephone-plate, and forcing through it, at a gentle pressure, a stream of air which impinged on a small flame, *reinforced* the otherwise inaudible sounds of the telephone till they could be distinctly heard over a small room. The explanation of the reinforcement,—founded by Mr. Bell on a long series of delicate and beautiful experiments, including the measurement of the relative pressures at different points in and near the jets,—seems to be a purely physical one. The actual experiments as described in the present paper are, however, new, so far as the author can ascertain—though Mr. Bell has experimented on the use of flames in connexion with the graphophone, and took out a patent thereon in 1886.

assured, however, that Mr. Porter had worked the matter out for himself *ab initio.*

Mr. W. DUDDELL suggested that the flame was in a condition of instability, so that any disturbance created in it tended to increase in amplitude. If the energy for the increased sound came from the tuning-fork, and the flame simply acted as a trumpet, then the fork would be rapidly damped; but if the flame was unstable the tuning-fork would not be so rapidly damped. He asked the author if he had tried experiments on this point.

Prof. A. S. HERSCHEL referred to the ways in which energy could be transmitted by a gas (1) by a flow, and (2) by the passage of undulations. The second method was concerned in the experiments shown by the author. In reply to a question by Prof. Ayrton, he said he did not think that the increased sound was due to an increase in the amplitude of vibration of the tuning-fork.

Mr. D. J. BLAIKLEY asked if the flame could be so adjusted or used as to restore to its original form a compound wave which in passing from the phonograph had suffered distortion by strengthening of the upper partials through unequal resonance; and further, if the flame had any such power, did its action go so far as to reduce the wave-form to that of a pure tone? (The Chairman pointed out that in the experiments shown with the phonograph the distinctive character of the notes emitted by the various instruments could be easily recognized.)

Prof. H. L. CALLENDAR referred to Prof. Dixon's experiments on the propagation of explosive waves through gases. These waves had been photographed, and proved that the passage of a detonation-wave through an explosive mixture caused a local heating, shown by a corresponding brilliance in the photograph. Complicated effects were produced by the meeting of a detonation- and a retonation-wave, but in every case there was an increase of chemical action in the flame. He suggested that the behaviour of the platinum foil in the blowpipe-jet might be ascribed to the meeting of such waves.

Mr. PORTER, replying to Mr. Duddell, said a tuning-fork was unquestionably rapidly damped when placed in a flame.

With regard to Prof. Callender's observations, he said he had looked in vain for anything corresponding to the existence of loops or nodes in the flame. Referring to Mr. Blaikley's remarks, he said that a small flame reinforced the high notes better than the lower ones and *vice versa*.

V. *The Simmance-Abady "Flicker" Photometer.*

By Messrs. SIMMANCE & ABADY *.

[Plate IV.]

THIS photometer is of the alternating-light type, recently called in France "scintillation-photometer," but more generally known in America and England as "flicker."

A very long experience with all the known variations of "flicker" photometers, and the construction of many others of different patterns, resulted in the design of the present instrument, and enabled actual rules to be laid down, which when adhered to produce a photometer which is most sensitive to degrees of lights of the same colour, and also enable the intensities of the most violently contrasted tints to be compared and balanced.

These rules are as follows:—The light effects must be in juxtaposition without any apparent division line, and must move, oscillate, or rotate so that the point of juncture of the rays of the two lights passes and returns entirely across the vision-field. Any hiatus, or longer exhibition of one light than the other, biases the result. The observation surface or surfaces upon which the light rays fall, must be at exactly the same distance from the eye, at exactly the same angle in relation to the line of sight, and must be of pure white such as is afforded, for example, by a clean chalk, plaster of Paris, magnesium carbonate, or barium sulphate ; any tint affects the accuracy of the result. The observation surfaces must also themselves in turn occupy the field of vision; an apparent movement or optical illusion does not afford accurate results.

* Read December 11, 1903.

Sir W. Abney, whose work in colour photometry is so well known, has placed on record his opinion that in using the "flicker" photometer, the operator was in some way testing acuteness of sight, and not relative intensities. Acuteness of sight naturally does come into play in this, as in all other matters in which sight is used at all, but when we remember that with lights of the same tint we apparently see in the photometer a white surface which is alternately illuminated by the two lights, we must conclude that when this alternation is invisible (that is to say, when we can see no change) the reason is because both lights are of the same power. Naturally, abnormally acute vision would perceive more minute changes of intensity, but this is a factor always present in any investigation of the sort.

It appears quite certain that the relative intensities of two lights, whether of the same colour or different, *can* be accurately and easily gauged by the method. An interesting experiment has recently been conducted by Mr. Roxburgh and Mr. Young, of the Ophthalmic Hospital. Readings upon this photometer were made independently by, firstly, Mr. Jacques Abady, who is an experienced photometrist, but suffering from astigmatism; secondly, a gentleman with a pupil dilated by treatment with atropin; thirdly, a patient from the hospital, absolutely colour-blind; and, fourthly, by the writer, whose sight is considerably impaired, but who has some experience with photometer reading. All the results coincided, and this not only when the lights were of the same white tint, but with red against white, green against white, and green against red. A test of very dark blue against red showed slight discrepancies, owing, probably, to the great obscuration of the light by the blue, almost black, glass, but even these were no more in importance than would probably have occurred when using a Lummer-Brodhun or Bunsen disk with lights of the same colour.

The method of reading with this photometer resolves itself into the simple question as to whether the disk of light seen through the sighting tube is moving or is motionless. An observer who makes a reading for the first time, when he fully understands that he is required to do no more than this, can read with as great facility and accuracy as an experienced operator, and it is this facility which gives value to an apparatus intended for testing the high intensity lights in vogue at the present time, and also colour effects.

As to the causes of the phenomenon, they, no doubt, are simple on the surface, but may be more complex when closely considered. The pupil of the eye cannot adjust itself to the rapidly changing intensities, and when the balance is struck, the light, being unchanging and motionless, permits of pupil adjustment. The colours then blend by persistence of the impression, but should the intensities again change, this retentive sense is dulled by the anxiety of the nerves controlling the pupil orifice to perform their work, an endeavour, however, frustrated by the rapid changing. It appears certain that an abnormal excitation of the nerves controlling the diaphragm has the effect of lessening the susceptibility of the retina to retention of impressions, whether this excitation is caused by glancing at a bright light or by gazing too long at the changing disk of the photometer. When the nerves are over-strained in this way the "flicker" does not disappear at all in many cases, although the point of equality of intensity is always unmistakable. Thus in using the photometer it is suggested that the instrument shall be moved along the bar rather rapidly, the while observing the disk through the sighting-glass. The "motionless" point will then be apparent at once, and by a little practice (which in this, like everything else, generates confidence) an operator will swing the disk-box into its balance of intensity without an effort and without giving the sensitiveness of his retina time to suffer from the numbing effect of the flickerings or throbbing. These conditions point to the possibility of the eyes becoming under other circumstances of stress unable to appreciate the disappearance of the flicker. Obviously, if fatigue can produce such effects as described, certain conditions of health will act similarly. It may be that a bad liver will produce a constant flicker effect ; it is quite certain, however, that no circumstances short of partial or absolute blindness can prevent the point of equality of intensity being appreciated, although it may evidence itself in different appearances of the disk according to the varying conditions of the retina as mentioned above.

The writer would submit, with all deference to the expressed opinions of many of the world's greatest scientists, that the Purkinje phenomenon does not affect the accurate working of the form of Flicker photometer shown, if indeed it affects any form of alternating photometer. This Purkinje effect may be briefly described in this way :—A right-angled wedge receives upon one face the reddish light from a pentane standard, and on the other the bluish light from an arc-lamp—we then adjust the relative positions of the arc-lamp, the pentane standard, and the wedge until we obtain what we consider is an equal illumination on the two adjacent sides of the wedge. If we then move in both these lights to half their distance from the wedge, we find that the retinal stimulation or apparent brightness of the two surfaces is no longer the same. It is quite clear from this (and also we may gather the cause from Fechner's law and Von Helmholtz) that the retinal sensation of brightness does not increase according to the same law for all colours. It is obvious, therefore, that no unaided eye can judge accurately in photometrical comparisons so long as the disturbing effect of colour is present, unless the distances between the receiving surfaces and the lights are arranged according to some rule which must take the place of that of inverse squares. To put this in another way, it follows that even though an expert photometrist can with a Bunsen or Lummer-Brodhun apparatus arrive at what he is certain is (and what is no doubt) a fair balance of the two tints he is comparing, yet grave doubts must exist as to whether the relative candle-powers of the two sources can be taken as being indicated by this balance. The Purkinje effect is purely a colour effect and colour does not enter into consideration in this "Flicker" photometer.

No one would dream of considering a colour-blind person to be affected by the Purkinje law, and yet, as has been shown, such a person can read accurately with this photometer. The experiments made to prove the Purkinje phenomenon have all been carried out with coloured surfaces, never, so far as is generally known, with a scintillation photometer, but conclusive trials have shown that with this instrument as now shown, no alterations in distance disturb the true readings. A red light compared with a green at 3 metres gives the same comparative intensities to each as at 2 metres, and an arc-light tested against a 10-candle pentane lamp at 40 feet indicates the same candle-power as when the distances are reduced to 15 feet. Thus the question of colour does not interfere with the results.

The accuracy of the photometer for coloured lights is confirmed by the following experiments.

Two standard lamps of exactly the same power were used, one at each end of the photometer-bar. A coloured glass screen was interposed, to intercept the rays from one light, which was then measured against the unscreened light. This gave value for one coloured screen. A second colour was then substituted for the first and its value measured against the same unscreened light. Thus the following values were obtained:—

(*a*) Unscreened lights.
(*b*) Candle-power of one light screened with one colour.
(*c*) Candle-power of one light screened with the other colour.

It is obvious, therefore, that one obtained the theoretical ratios of the one light screened with the first colour in terms of the other light screened with the second colour. Then the one colour was compared against the other, and the results show how nearly this ratio agrees with the theoretical ratios. Various initial powers of lights and various lengths of bars were used for these experiments. All the readings on the bar were computed according to the law of inverse squares; and it was therefore assumed that the absence of "flicker" or point of equality indicated by the Simmance-Abady photometer is *candle-power* of intensity of light.

The following tests are the mean of three persons' readings, and the three individual readings all made independently showed practically no variations.

	Power in Candles.	Theoretical Ratio. $\left(\frac{\delta}{\theta}\right)$.	Actual Reading on Photometer.
Signal Green ...... δ	21·8	1·4 nearly	1·39
Signal Red......... θ	15·6		
Yellow .... δ	9·75	4·67	4·75
Blue .... θ	2·088		
Yellow .... δ	10·01	6·9	6·86
Red.................. θ	1·45		
Green............... δ	2·16	1·08	1·08
Blue... .... θ	2·0		
Purple .... δ	2·0	1·29	1·275
Red.................. θ	1·55		
Purple .... δ	3·81	1·95	1·9
Signal Green...... θ	2·0		

To detail briefly the construction of this Simmance-Abady photometer:—It consists of a wheel of a white material (Pl. IV. fig. 1) with a specially shaped periphery, which wheel is caused to revolve before an eyepiece by means of a suitable motor. At right angles to the line of sight and parallel with the axis of the revolving wheel are the two lights undergoing examination, the rays of which fall upon the shaped periphery of the wheel, enabling the effect of each light to be seen in turn through the eyepiece.

Pl. IV. fig. 2 shows the four cardinal positions of the wheel in sequence, illuminated by two unequal lights. It will be seen how the light effects travel across the line of vision and alternate as the wheel revolves.

The revolving wheel has its periphery formed by two equal conical surfaces or other surfaces of revolution. The axes of the two conical surfaces are each parallel to the axis of the revolving wheel, at exactly equal distances therefrom, and the three axes lie in the same plane. The vertices of the conical surfaces are at equal distances on opposite sides of the line of sight. In manufacturing, the wheel is first made as a disk of uniform thickness. It is then chucked in a lathe eccentrically, and a straight line motion being given to the cutting or grinding tool, the conical surfaces are generated. The wheel is then rechucked eccentrically on the opposite side, and the other conical surface is generated by the cutting or grinding tool, having the same straight line motion as before. The result is that the periphery of the wheel has its two sides uniformly sloping at equal opposite angles, while the ridge of intersection of the two surfaces crosses and recrosses the axis of vision during each revolution. This wheel is driven with a motion communicated from a suitable motor, either spring, electric, or any form which will produce absolutely regular motion—and at the same time afford means for easily adjusting the speed. In the instrument shown an expansion governor is utilized driven at a carefully arranged ratio speed, but the connexion between the spindle carrying the governor and that carrying the reflecting wheel is not rigid, but effected by means of a coiled spring. The most perfect accuracy of centreing is essential. Outside the box are the remontoir, stopping, and starting lever, and speed adjustment, a spring motor being used.

The peculiar shape of the reflecting-wheel affords means for using it in manners quite impossible in any other form of colour photometer, that is to say, it may, at will, test lights at various angles from the horizontal. Having found the careful cutting of the angles of the wheel so vital, it necessarily follows that when lights out of the horizontal are being tested, the box containing the wheel must be just as carefully turned on its axis for preserving the arranged conditions. A double quadrant scale (one scale being numbered at double the actual angle) and a small sighting or view-finding attachment, enable the angle formed by the horizontal of the one light and the altitude of the other to be accurately ascertained, and the box to be placed at the correct angle of bisection.

The photometer is made to suit any bar or scale. The standards of dimensions are those of the Lummer-Brodhun apparatus and the ordinary Bunsen disk-box.

Discussion.

Prof. W. E. Ayrton said it was an important question to settle exactly what the photometer measured. The fact that different observers, using the same photometer, obtained the same results for the ratio between two lights did not prove that the instrument actually compared intensities. He had come to the conclusion that when the distances between the sources of light and the photometer were great the results were not affected by the Purkinje effect.

The Rev. T. C. PORTER expressed his interest in the instrument, and said there was no doubt that the flicker photometer measured intensity independent of colour. The results obtained directly from a flicker photometer agreed with those deduced by indirect methods from observations with ordinary instruments.

Mr. L. GASTER expressed his admiration for the photometer, and explained that this type of instrument was becoming more important because of the great number of improvements made during the last few years in the manufacture of incandescent lamps and arc-light carbons by the addition of other substances to the pure carbon. A great step towards the simplification of the process of photometry had been attained by Prof. C. P. Mathews, by the aid of whose photometer one is enabled to get in one reading, an illumination upon the screen, proportional to the mean spherical intensity of the source.

Dr. W. WATSON said the authors had compared a red light with a white light and also a blue light with a white light. He asked if they had compared a red with a blue directly, and if so, how the results agreed with those deduced from the comparison of both with a white light.

Mr. W. DUDDELL asked what was meant by the equality of two lights of different colours. Was it judged by the eye, the bolometer, or the flicker effect? On account of the Purkinje effect the inverse square law could not be used, and he asked for the real fundamental relation underlying the use of the instrument.

Mr. J. ABADY, replying on behalf of the Authors, said the discussion had turned mainly on two points: (1) Does the instrument measure candle-power? and (2) is it independent of the Purkinje effect? If a Bunsen disc were taken and used to compare the candle-powers of two lights of the same colour, the grease-spot vanished at a certain point. If a flicker photometer was substituted for the Bunsen disc, the flicker disappeared at the *same* point. In both cases candle-power had been measured. In reply to Prof. Ayrton he said that results obtained from the photometer by different observers agreed among themselves and also agreed with results obtained from the Bunsen disc photometer. With regard to the Purkinje effect, he did not think it affected the readings of a flicker photometer, because the same values for the ratio between the intensities of two different coloured lights were obtained by varying the distances of the lights from the instrument within wide limits. Answering Dr. Watson, they had carried out the experiments suggested, and the results from the direct and the indirect comparison were in absolute agreement.

VI. *On the Viscosity of Pitch-like Substances.* *By* Prof. F. T. TROUTON, *F.R.S.*, *and* Mr. E. S. ANDREWS, *B.Sc.*[*]

THE various methods which have been proposed for measuring viscosity meet with difficulties when it is attempted to apply them to the measurement of the viscosity of bodies such as pitch. The girder method has been applied to examine the viscosity of ice as well as methods depending on direct extension and compression; but these apparently did not lead readily to a numerical determination of the coefficient of viscosity. The application of Stokes' method, depending on the rate at which a spherical body—say a lead bullet—sinks through the material, seems apparently to have been prevented by the difficulty of knowing exactly its velocity in the middle of the substance, the terminal effects leaving considerable uncertainty. As described later, this particular difficulty was surmounted by the use of Röntgen rays in some experiments made to compare the coefficient obtained by this method and that by the method described in this paper.

To obviate some of the difficulties, a method was proposed involving the torsion of a cylindrical bar. In this method a constant torque was applied to a cylinder of the substance, and the relative motion of the ends observed. From these and the dimensions of the body the viscosity was calculated.

From symmetry we may assume that any two planes in

[*] Read June 12, 1903.

the body, lying at right angles to the axis of the cylinder, move over each other, about the common axis, remaining plane all the while.

Let δx be the distance apart of the two planes. Then, if μ is the coefficient of viscosity of the material of the cylinder (supposed independent of the velocity), and $\delta\omega$ the relative angular velocity of the planes, we have

$$T=2\pi\mu\frac{\delta\omega}{\delta x}\int_0^R r^3 dr,$$

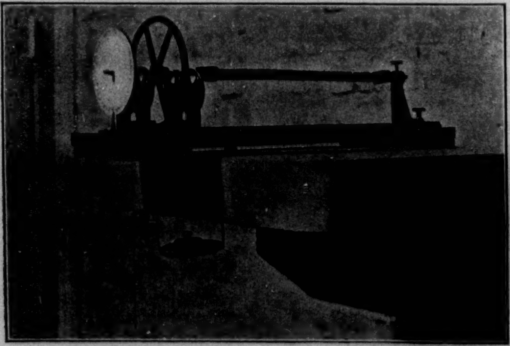
where T is the torque applied.

Thus we have

$$\mu=\frac{2T}{\pi U R^4},$$

where U is the relative angular velocity per centimetre of length of the cylinder, and R is its radius.

Fig. 1.



The form finally adopted for applying the method consisted in a shaft turning freely on anti-friction wheels with a pulley attached, from which hung a weight for the purpose of applying the constant torque (see fig. 1). The shaft carried a square socket for the purpose of gripping the squared end of the cylinder of the substance, which was made to fit exactly. A similar but fixed socket prevented the other end from turning. The rate of rotation was observed by means of a divided circular disk carried on the shaft. With this apparatus the experiments described below were undertaken to test whether (1) the rate of rotation was proportional to the torque; (2) the rate of rotation of cylinders of the same material was inversely as the fourth power of the radius.

Incidentally two unsuspected effects were at once disclosed by the use of this apparatus. One is that the coefficient of viscosity of bodies such as pitch is a function of the time, observations showing that the velocity of flow for a given stress diminishes with time from its initial value down to a constant quantity. The second is that on removing the stress there is a flow back in the opposite direction, which gradually diminishes to zero with time.

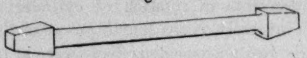
The method lends itself also to the determination of the coefficient of viscosity at different temperatures, as the cylinder can be conveniently surrounded by a jacket and kept at any required temperature. This mainly arises from its not being necessary to have access to the bar while under observation.

In this way the coefficient for soda-glass was determined at temperatures ranging between 500° C. and 700° C., and that of pitch from 0° C. to 15° C.

Experiments with Pitch.

The greatest difficulty experienced here was the finding of a convenient method of preparing suitable cylinders. After trying various methods, the most successful was the simple one of rolling the pitch between boards while still soft and warm after recent heating. This was done on a square wooden board by means of a narrower board with end guides —hanging down over the side of the square board—which enabled it to be drawn parallel to itself backwards and forwards over the square board. Two thin strips of desired thickness acted as distance-pieces to regulate the final diameter of the cylinder. With a little practice very perfect cylinders can be rolled in this way. The next stage was to put squared ends to the rods. This was done by softening the pitch and thickening and squaring the ends as shown in fig. 2. The ends were slightly tapered to fill the sockets on the apparatus which were also tapered. It is desirable to have the thickness of the squared ends larger than that of the cylinder, in order that as little as possible of the observed flow may take place at the squared ends: otherwise the rate

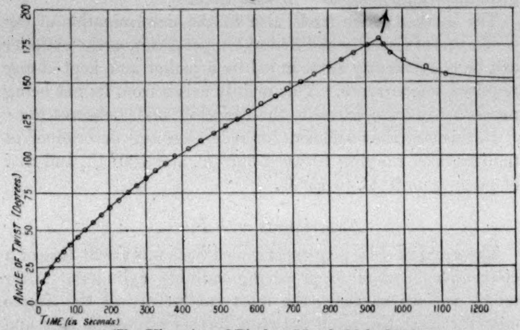
Fig. 2.



of rotation would have to be observed between two marked points on the bar.

The results obtained with a specimen of pitch sold under the name of "British Pitch" are plotted in fig. 3, and show the two effects mentioned above.

Fig. 3.



Torsional Force removed.

The Viscosity of Pitch. March 16th, 1903.

On first applying the torque there is a rapid flow or movement, which gradually diminishes and finally reaches a steady state. This steady state of rotation would apparently go on indefinitely but for the bending produced in the cylinder by its own weight, which finally introduces complications. Indeed, by judiciously keeping the cylinder horizontal by slight upward pressure occasionally applied, it is possible to twist the cylinder for hours together.

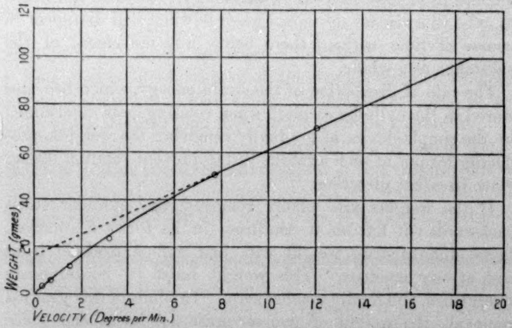
If a white line be painted along the specimen before twisting, a beautiful spiral line of many twists is ultimately obtained. Probably with arrangements to make the experiments under a fluid of the same density as pitch so as to remove the vertical component of force on the pitch, the rotation could be kept going as long as one pleased.

Elastic Viscous Recovery.—This second effect is shown at the end of the curve, where on removing the turning couple the cylinder of itself turns back some little distance, at first rapidly, then slowing down gradually to rest.

Evidently, to do this, energy must have been stored in the substance in the form of elastic strain. The effect may perhaps be looked upon as complementary to the initial state when the rate of displacement is abnormally great in comparison to the final or steady rate of rotation. In this initial stage a store of elastic energy is gradually accumulated, which is preserved intact during the state of steady rotation, and is given out on removal of the stress to produce the return flow.

Different Torques applied to the same-sized Cylinders.—Experiments were made to ascertain if the connexion between the torque and the rate of rotation was that deduced from the simple theory given above. These experiments show that though for practical purposes the theory is sufficient, yet in reality the phenomenon is more complicated than therein assumed. The results are shown in fig. 4, where the final or

Fig. 4.



Curve showing relation between Weight and Viscosity.

steady angular velocity is plotted against the weight producing it. It will be seen that for small torques the rate of rotation is proportionally less than that for larger ones, but that finally above a certain value the curve develops into a straight line. For all values of the torque above this value, on subtracting a small constant quantity we get proportionality. This is probably partially due to the purely elastic strain, and will be better understood from later consideration in this paper.

Determinations with different-sized Cylinders.—Determinations made with different-sized cylinders were in satisfactory agreement with the inverse-fourth-power law. As an example, the values obtained with two cylinders of the same kind of pitch and determined at the same temperature are given here:—

R. (cm.).	T. (dyne cm.).	l. (cm.).	U. (steady value).	μ.
·36	$9{\cdot}65 \times 10^4$	21·5	$7{\cdot}87 \times 10^{-4}$	$9{\cdot}9 \times 10^{10}$
·67	$1{\cdot}04 \times 10^6$	24·5	$7{\cdot}96 \times 10^{-4}$	$10{\cdot}1 \times 10^{10}$

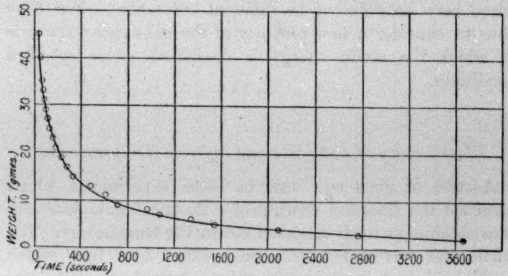
The Rate of Dissipation of Strain Energy.—The apparatus enables us to show that the strain energy, which we have seen to exist in a viscous substance when flowing, will disappear in course of time without there being any movement of the substance as a whole.

The rate of dissipation of the strain energy, which becomes stored in the cylinder of pitch when rotating, was determined by the simple device of gradually removing the weights, producing torque, at such a rate as just to prevent rotation taking place in either direction.

If the weights are wholly removed, the cylinder turns backwards : if left on it continues in its forward rotation. Some intermediate weight will just suffice to hold it at rest at any moment. This weight must be continuously diminished, and finally reaches zero. To admit of this gradual removal, the weight of course must not be in one piece but must be made up of a large number of small pieces. The rate at which the weight must be diminished can afford a measure of the rate at which the strain energy in the substance is dissipated and converted into heat, provided there is a linear relation between stress and strain.

Fig. 5 gives the curve of dissipation obtained from a cylinder of pitch. The experiment was not carried to completion, and at the end of one hour, on removal of the whole weight still on the apparatus, the pitch was found to be able to recover several degrees. The curve is not a logarithmic one as might have been very well expected; the lower part indeed fits well with a rectangular hyperbola.

Fig. 5.



Dissipation Curve for Pitch.

This gradual dissipation of strain energy bears an analogy to the dissipation of the energy which takes place in an electrically strained dielectric which is partly conducting.

We may look upon the steady state of flow as one in which there is a continuous transformation of the elastic strain energy into heat, and an equal continuous replenishment of the strain energy from the work done by external forces—a kind of dynamic equilibrium.

This point of view is suggestive as giving an insight into processes going on in substances in the viscous flow; namely, that the sequence of events is primarily a production of elastic strain which is rapidly dissipated by the breaking down, so to speak, of the strained material. The fact that the strain energy can disappear without any deformation occurring, enables us to draw a distinction between plastic and viscous substances. In a plastic substance, when held under constant deformation beyond its elastic limit, the strain energy it then contains does not sensibly lower with time; whereas we have seen that the contrary is the case with viscous substances.

The rate at which the dissipation of strain energy takes place may be different in different substances. We have thus to consider a new property of the substance: the rate at which the strain energy is dissipated under specified conditions.

The Viscosity of Soda-Glass at different Temperatures.

A tube of glass was used in these experiments which admitted the insertion down it of a thermoelectric junction, thus obtaining a ready means of measuring temperature. The whole was surrounded by a thick-walled iron tube which could be raised in temperature by a row of small gas-jets placed beneath. The bore of the iron tube was a little larger than the glass in order to allow the latter to turn freely.

In the case of a tube the formula for the viscosity becomes

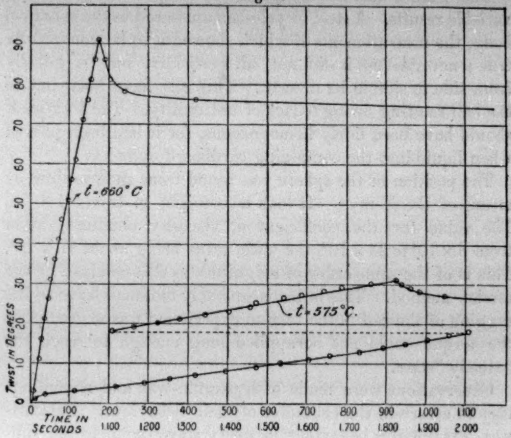
$$\mu = \frac{2T}{\pi U(R_1^4 - R_2^4)},$$

where R_1 and R_2 are the external and internal radii respectively.

The results are given in a table at the end, and in fig. 6 are plotted the results for two experiments.

The authors entertained at one time the idea that the elastic viscous recovery might only be found to occur with substances which were mixtures; one constituent of which might be elastic and the other viscous. With the object of testing the matter some cylinders of sodium stearate were prepared. It was found that elastic viscous recovery occurred in this substance similar in every respect to that observed with such mixtures as glass or pitch. The viscosity of the substance is given in the general table.

Fig. 6.



The Viscosity of Glass.
(Different values of the torque were used in the two cases.)

Shoemaker's Wax.

It was desirable that determinations of the viscosity of the same substance should be made by the method here described and also some other. It was found that shoemaker's wax was, on the one hand, just sufficiently viscous to allow cylinders to be made from it for determination with the torsion method, and just sufficiently fluid to admit of its viscosity being determined by allowing a spherical body to drop through it.

The mean value obtained by the torsional method working with two different-sized cylinders was $\mu = 4{\cdot}7 \times 10^6$. This value is open to considerable doubt, for cylinders of shoemaker's wax sag in the centre rather too quickly to give really reliable results, and would have to be supported by a fluid of the same density in the manner explained earlier in the paper.

The Stokes method adopted for comparison gave very variable results. A steel bicycle-ball answered as the spherical body, the measurements of which showed it to be wonderfully true; nevertheless it did not fall vertically, but irregularly from side to side in its descent. This may have been due to the ball rotating owing to lack of uniformity. The wax itself should have been fairly homogeneous, for it had been poured when liquid into the containing cardboard cylinder.

The position of the sphere was found from time to time by means of the X-rays. It took a fortnight to travel 1·8 cm. The value for the coefficient of viscosity obtained varied from 6×10^6 to 23×10^6, the mean value being about 10×10^6. This is of the same order of magnitude as that obtained by the torsion method. This latter is probably too small because the sagging of the rod in the torsion experiment was so great that the torque could not be applied long enough to reach the "steady" state.

Observations were made with paraffin-wax and modelling-clay to ascertain the character of their behaviour. Paraffin-wax exhibited a behaviour in every way similar to the substances already mentioned, but the modelling-clay acted quite differently. When subjected to a given torque it moved slowly up to a given position and stopped there. On removing the stress, it made an immediate partial recovery to a point where it permanently remained.

The following list contains the results obtained with the several substances experimented with.

Substance.	Temperature.	Coefficient of Viscosity.
	° C.	
Pitch	0	$5{\cdot}1 \times 10^{11}$
"	8	$9{\cdot}9 \times 10^{10}$
"	15	$1{\cdot}3 \times 10^{10}$
Glass (Soda)	575	$1{\cdot}1 \times 10^{13}$
"	660	$2{\cdot}3 \times 10^{11}$
"	710	$4{\cdot}5 \times 10^{10}$
Sodium stearate	8	$5{\cdot}0 \times 10^{11}$
Shoemaker's wax	8	$4{\cdot}7 \times 10^{6}$

DISCUSSION.

Mr. ROLLO APPLEYARD said that with materials such as Prof. Trouton had investigated, it was necessary to exercise great care in eliminating errors due to slip and other disturbances at the ends of the bars. For this reason it was, as a rule, better to attach the rotating-index pointer not to the clamp which applies the twist, but at a place at a little distance along the bar itself. A convenient way of doing this with these comparatively weak materials was to slip an indiarubber ring over the bar. Between this ring and the bar could then be held a piece of cork carrying a pointer. Instead of applying separate weights to a scale-pan in such tests, it led to smoother results if the weights were increased by the flow of water into a vessel. Mr. Appleyard thought that it was almost hopeless to seek for mechanical "constants" in respect to such highly extensible and complex substances as pitch, and suggested that for such materials, and perhaps for materials in general, it might be better to assume that the ratio of strain to stress is a function, possibly a logarithmic function, of the stress. The investigation of this function from small displacements right up to the breaking strain, would be more useful than figures based upon the assumed constancy of Young's modulus and Poisson's ratio.

Prof. TROUTON, in reply to Mr. Appleyard, said that the rapid flow on first applying the torque was not due to any end effect, but was a property of the material.

VII. *On a New Form of Sensitive Hot-Wire Voltmeter.*

By R. Threlfall, *F.R.S.*[*]

The practical need for a sensitive alternate-current voltmeter arises in connexion with the measurement of large alternating currents. The instruments at present employed for the purpose of measuring alternating currents are substantially of two types,—the ampere gauge of Lord Kelvin forming the standard and almost only representative of one class ; and instruments based on transformers forming the other. Both classes of instruments require calibration in manufacture, and from time to time, and it is then that the want of a sensitive voltmeter is felt; for the obviously most direct method is to measure the potential drop across a standard resistance traversed by the whole current in question. It may seem curious that the larger the current to be measured, the more sensitive must be the voltmeter employed ; but a little consideration will show that this is the case because it is not practicable to go on increasing the weight of a standard resistance without limit. In the other alternative the heating becomes excessive, and there is a risk of damaging the standard. For instance, suppose that it is a question of measuring 2000 amperes by means of a resistance of ·0002 ohm, the P.D. drop is ·4 volt, and the power expended in heat is ·8 kilowatt—quite a consideration. The practical standard alternating-current voltmeter must therefore be sensitive and adjusted to work across an external resistance which may be considered negligible in comparison with its own resistance. If the hot-wire form be adopted, it is seen that the electrical considerations point to the wire being as short as possible, and also as fine as possible ; for it has often been shown that the rise of temperature for a given current-density increases as the diameter of the wire decreases.

Taking everything into account, the most suitable material appears to be pure silver. As this can be obtained commercially nicely gilded and of such a thickness that two miles go to the troy ounce, it was not worth while trying to improve upon it in the laboratory, though no doubt finer stuff could be produced by special artifice. For instance, by making a silver wire or strip the anode in a dilute solution of potassium iodide, it can be very materially reduced in thickness, but the mechanical properties are impaired to some extent in spite of the equalizing action of the coating of silver iodide which is formed.

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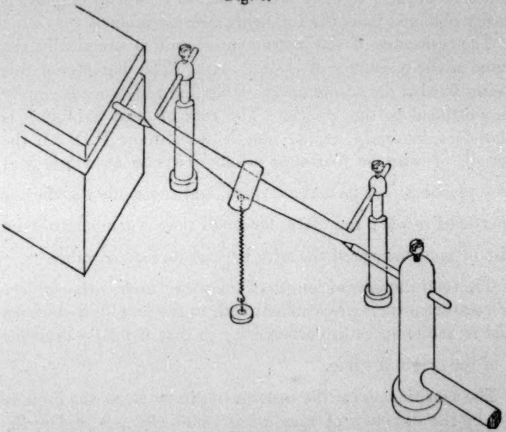
[*] Read November 27, 1903.

In order to make use of the gilt silver wire as a hot-wire voltmeter, it is necessary to be certain that the wire is always stretched by the same force, and then the small changes in length consequent on the passing of a current can be measured.

The chief peculiarity of the instrument in question is in regard to the means adopted for securing uniformity of tension of the fine wire.

Fig. 1.



Referring to fig. 1. The fine wire is seen stretched between two supports, and pulled downwards at the centre by a microscopic hook and spiral spring insulated from the base; a small mirror hinged on a wire stretched alongside rests on the head of the hook. The deflexion of the wire is exaggerated in the diagram, in reality the interior angle is about 175°, or from that to 178°.

In order to get an idea of the relation of the sag of the wire to its increment of length, suppose that it is stretched quite straight to begin with; that its length is $2l$, and that we require to find x the contraction of the spiral spring for a given increment of length a in the wire. This is seen to be given by $x^2=la$, when a is small.

To form an idea of the multiplication obtainable in practice, suppose that the distance between supports $=5$ cms., and that the wire is heated through 1° C., the initial interior angle being 175°, and the coefficient of expansion of the wire per degree ·00002. Calculation shows that in this case the magnification is about 22; and of course it increases rapidly the more nearly the initial angle approaches 180°. The condition as to the length of wire which it is best to employ, is easily obtained from the following considerations.

The resistance of the instrument-circuit is practically the same as the resistance of the fine wire. The diameter of this being fixed at the minimum available, the resistance is simply proportional to the length. The rate of heat evolution is therefore inversely as the length at constant P.D. If the length of wire be increased n-fold, the rate at which heat is supplied is $\frac{1}{n}$ of its former value, and the cooling surface is increased n-fold; therefore, for small rises of temperature the rise of temperature of the wire is $\frac{1}{n^2}$ of its former value.

The total increase of length of the wire which is the subject of measurement is proportional both to the length of the wire and to the temperature difference; so that it finally becomes $\frac{1}{n}$ of the original value.

The sensitiveness of the instrument, in so far as the limit is set by the difficulty of measuring small changes of length, is therefore inversely as the length of the wire. If the sensitiveness be regarded as limited by the least amount of sag that can be perceived due to heating, we have to consider what further condition is imposed. The practical condition is that enough sag must be allowed initially to prevent the wire getting broken if the current is accidentally taken off while the measuring apparatus is in the position necessary to make the sag constant in spite of the heating. In this case the magnification is constant, and x depends only on the length—giving the same condition for sensitiveness as obtains when we consider the measuring apparatus as imposing the limit.

The above principles are put into practice in the following manner. The active wire is carried between two adjustable supports, one of which must be insulated from the rest of the apparatus. The other support can be moved to and fro in the direction of the wire by means of a micrometer-screw, and the motion may be further reduced by means of a lever, if so desired. The mirror may be a centimetre long and 4 mms. wide, and the hinge-wire may be about 2 mms. from the vertical plane of the active wire. The mirror is backed by a scrap of mica, and is supported by the head of the hook which pulls the active wire down. As the active wire heats up, the hook descends and the mirror is tilted downwards. There is a small incandescent lamp in the lid of the box enclosing the instrument, and a ground-glass or celluloid scale opposite a slot in the front of the case. By means of a prism and lens an image of the filament is thrown on the screen by the mirror, and by turning the micrometer-screw this image can be brought to any desired point. The mirror is not affected as to position by swinging the box from the end of a rope; but it is sensitive to vibrations of short period, and it is best to observe when the instrument is suspended rather than when it is resting on a table—especially in an engine-house. Small voltages can be read off at once on the scale, but larger ones are compensated by working the screw. The whole apparatus is enclosed in an aluminium box measuring 1 foot × 1 foot × 8 inches high; and these dimensions can be reduced if desirable.

The following data refer to an instrument designed to work with potential-differences up to about 0·5 volt:—

Distance between supports	6·5 cms.
Initial sag of wire..................	·127 ,,
Interior angle	175° 30′

Resistance cold with leads about 1·2 ohms.

Effective distance, mirror to screen in cover of instrument, 28 cms.

With this instrument ·03 volt gave a deflexion which could just be easily seen without the assistance of a lens, and required $\frac{1}{25}$ turn of the micrometer-head to compensate it.

As it was intended to employ the instrument in measuring large alternating currents, it was thought well to calibrate it by sending a unidirectional current through the standard manganin resistance of ·0002 ohm which it was proposed to employ; and at the same time measuring the P.D. across this by a potentiometer and cadmium-cell.

Four settings of the micrometer-head were made for each observation, and the mean taken. It was found that the zero reading varied by two or three twenty-fifths of a turn as the temperature changed; also it required about half a minute for the wire to reach its steady state after the voltage was applied. In all cases the micrometer-screw was brought to near its zero reading before the current was interrupted, though a special experiment had shown that no damage appeared to result from the neglect of this precaution.

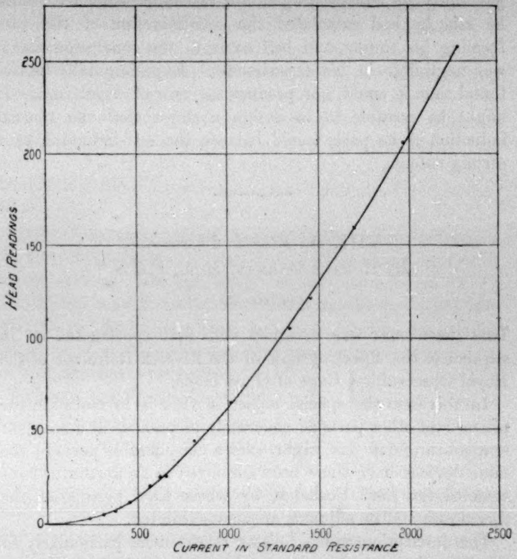
The micrometer-head was divided into 25 large divisions, and each large division was subdivided into ten small divisions. The unit in the column "Head Readings" is one large division of the head.

The source of light was a 4-volt ½ C.P. lamp in the top of the case fed by two pocket storage-cells.

Table of Calibration of Instrument.

Head Readings.	Volts on Instrument.	Current in Standard Resistance.
		Amperes.
2·57	·04767	238
4·38	·0628	314
6·48	·07710	384
11·5	·09326	466
20·2	·1143	571
25·3	·1264	632
25·9	·1320	660
45·0	·1630	815
65·82	·2030	1015
106·1	·2662	1331
122·7	·2890	1445
160·2	·3384	1692
206·3	·3936	1968
258·1	·4545	2272

Curve of Calibration of Hot-Wire Instrument.



DISCUSSION.

Mr. **DUDDELL**, referring to the voltmeter exhibited, asked if there was any trouble with regard to the temperature variation of the resistance of the wire. In using the instrument with large alternating currents it was necessary to use a standard shunt, and difficulties then arose because of the fact that the inductance of a shunt, formed of straight bars and capable of carrying the necessary current, was comparable with its resistance. It might be possible to get a satisfactory shunt by doubling the bars upon themselves to reduce the self-induction.

Prof. **THRELFALL** said that there was no difficulty with regard to changes in temperature of the room because they only altered the zero of the instrument. Referring to the possibility of constructing a low-resistance standard shunt, he said he had calculated the self-induction of the bars forming his shunts, and had come to the conclusion that it was negligible in his experiments. Experimentally he had found that it could not produce an error of 1 per cent. It might be possible to so design a shunt that the mutual induction of its parts would balance the self-induction to a certain extent.

VIII. *An Electric Thermostat.*

By Horace Darwin, *M.A., F.R.S.*[*]

[Plate V.]

This thermostat was designed and first constructed as an adjunct to the Spectrograph of the 24-inch Refractor of the Royal Observatory, Cape of Good Hope.

In this case the special object in view is to maintain the prisms and other parts as accurately as possible at a constant temperature, day and night, over a considerable period; the same devices have since been employed in an apparatus constructed for Lord Berkeley, by whose kind permission the thermostat and its adjuncts are now exhibited.

The description which follows relates more particularly to this latter instrument.

The general arrangement of the various parts is indicated diagrammatically in fig. 1 (Pl. V.). A is an oil-bath, well lagged outside, whose temperature has to be maintained as accurately as possible constant at any desired temperature between 10° C. and 80° C. Three vertical tubes A_1 A_2 A_3 are so connected to the bath A that by means of a fan kept rotating within A_2, a constant circulation of oil is maintained. The oil from the bath enters the tube A_1 near the top, and passes down A_1, up A_2, down A_3, and back into the bath.

Two heating-coils are provided, and one of these, which is placed in the tube A_1, has a current uninterruptedly flowing through it from the battery of accumulators Q.

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[*] Read November 27, 1903.

This may conveniently be termed the permanent heating-coil, the circuit which includes it being called the main circuit. The other heating-coil forms a shunt across the terminals of the battery ; it is only traversed intermittently by a current, and is called the intermittent heating-coil, the circuit including it being referred to as the shunt-circuit. By suitably controlling the current passing through the heating-coils, the oil-bath may be maintained at a very nearly constant temperature. For the sake of simplicity we will assume henceforward that the temperature is to be maintained constant at 25° C.

There is a further set of four resistances, two of copper and two of manganin, arranged as a Wheatstone bridge B, and so adjusted that the bridge is only balanced at 25° C. This bridge may be called the controlling-bridge, since it is through its deviations from balance and the consequent deflexions of a galvanometer, that the supply of current to the heating-coils is controlled.

The main circuit includes " series-coils " which are situated outside the oil-bath, and whose function is to modify the supply of current to the permanent heating-coil. Intermittently the shunt-circuit is closed, the total heating effect being thus augmented so long as the shunt-circuit is complete. There are three respects in which the operation of the heating-coils may be automatically adjusted. Thus an increase in the heating effect is produced by a more frequent closing of the shunt-circuit, by increasing the duration of each such closing, and by diminishing the resistance which the series-coils interpose in the main circuit.

Suppose now, as a particular case, that the thermostat has been running continuously for a considerable time, the temperature of the atmosphere having been fairly constant meanwhile, so that something approximating to a steady state has been reached: that is to say, though the actual rate of heat production varies abruptly, as often as the shunt-circuit is closed or opened, yet the mean heating effect of the current remains approximately constant from one half hour to another. If a fall now takes place in the temperature of the surroundings, so that there is a greater demand for heat, the necessary readjustment will be effected as follows :—

The increased demand will be immediately met by a more frequent closing of the shunt-circuit (containing the intermittent heating-coil), and the same mechanical movement which produces this more frequent closing of the shunt-circuit will cause the closing to become progressively of longer duration, and will also in time decrease the resistance of the series coils, so that by degrees the closing of the shunt-circuit ceases to be required so frequently.

The controlling-bridge B has one pair of opposite arms of copper, these two arms being immersed in the oil-bath. The remaining arms are of manganin, the connexion to battery and galvanometer being made in the usual way. It is thus evident that if the bridge is balanced at some definite temperature, a change of temperature will disturb the balance; for the product of the resistances of the two copper arms changes rapidly with change of temperature, while the corresponding product for the two manganin arms is practically constant, so that for any determinate adjustment of the bridge, equality of these two products can only obtain when the copper arms are at one particular temperature. By adding to the resistance of one of the manganin arms the temperature of balance is raised; while by adding to the resistance of one of the copper arms it is lowered.

One manganin arm consists of two coils in series, and one of these can be cut out at will by means of a mercury key. By setting over this key, the temperature at which the bridge is balanced is lowered 40° C. By the action of a similar key cutting out a coil in one of the copper arms, the temperature of balance is raised 20°. The four combinations of the positions of these two keys give a range of 60° by steps of 20°. Intermediate settings are made by means of a compound slide-wire whose total effective range corresponds to about 22°. This compound slide-wire consists of the slide-wire proper with a range of about 7° and three coils with a range of 5° each. If these three coils are called *a*, *b*, *c*, respectively, while the slide-wire is called *s*, then we have *a*, *b*, *c* and *s*, always joined in series between the points B_1 B_2 in fig. 1, while a special switch, which can be set in four positions, enables the order of the resistances between these two points to be made *a*, *b*, *c*, *s* ; *b*, *c*, *s*, *a* ; *c*, *s*, *a*, *b* ; or *s*, *a*, *b*, *c*, at pleasure.

The main circuit includes in series with the permanent heating-coil, some or all of the series-coils F. The current is led to the rail N, from which it passes, by way of the slider W, to one or other of the blocks L, according to the position of W. Thus it is evident from fig. 1 that the resistance offered by the series-coils included in the main circuit may vary, and the amount it varies is from 0 to 30 ohms.

The galvanometer G is of the suspended coil type, a long boom G_1 being secured at right angles to the axis of the coil, so as to swing in a horizontal plane whenever the deflexion of the galvanometer varies. The connexions are so arranged, that when the temperature of the copper arms of the controlling-bridge B rises above its normal value (assumed as above to be 25° C.), the galvanometer-boom is deflected "upwards" in the diagram, a fall below the normal temperature causing a deflexion "downwards." The sleeve X is kept steadily rotating on a fixed shaft S by means of a small worm-gear M driven from shafting. The direction of rotation is such that points above the axis are carried towards the reader in fig. 1. The cam C performs a variety of functions ; it is rigidly connected to the sleeve X, and is thus capable of displacement endwise with respect to the fixed shaft S. A helically bounded cylindrical segment D (called the shunt-circuit drum) is rigidly connected with the cam C, so that these two pieces rotate with the sleeve X, while a contact-piece W (called the slider) agrees with it in endwise displacement. The rocking-shaft R carries the hit-or-miss arm H, and the shunt-circuit brush J, both of which are made fast upon it, so that they always move together about the axis of the rocking-shaft R, while they are incapable of endwise displacement. The hit-or-miss arm H has a comb K attached to it crosswise, the teeth of the comb being disposed downwards, and the throats between them serving to engage with the cam C, as occasion arises. The outer end of the hit-or-miss arm H carries an ebonite sector H_1 (fig. 1).

In considering the action of the cam we will confine our attention for a moment to one of its functions, that, namely, of causing the hit-or-miss arm H to rise and fall again for each revolution of the sleeve X. This action of the cam is of course due to its radius (reckoned from the axis of the shaft S) being variable (fig. 2). The sector of larger radius is called C_1, being uppermost at the instant when our observation commences, the hit-or-miss arm H is raised to its fullest extent, and the ebonite sector H_1 is lifted clear of the galvanometer-boom; after some seconds, the rotation of the sleeve X brings the sector C_2 uppermost, so that the hit-or-miss arm H is left free to fall until its descent is arrested. The extent of the fall permitted to the arm H depends upon the position of the galvanometer-boom G_1 at the time being.

(*a*) If the oil immediately surrounding the copper arms of the bridge B is slightly too cold, the boom is deflected in the direction marked "cold" in the diagram, so that the ebonite sector H_1 clears the boom, and allows the arm H to come down as far as a stop-screw, thus attaining its lowest position. At the same time the shunt-circuit brush J, in virtue of its attachment to the same rocking-shaft R, as the arm H, has likewise reached its lowest position, in which it can make contact with the drum D, as soon as the latter has turned a little further. The contact thus established causes a current to flow in the intermittent heating-coil until the cylindrical sector D has passed the shunt-circuit brush J, shortly after which the sector C_1 of the cam comes under the comb K, and once more lifts the arm H and the brush J to their highest position.

(*b*) On the other hand, if the temperature of the oil surrounding the copper arms of the bridge B is slightly too high, the boom G_1 is deflected in the direction marked "hot," and as the cam C leaves the arm H free to descend, the ebonite sector H_1 is arrested upon the boom G_1, so that the arm H and the shunt-circuit brush J do not reach so low a position as in the alternative case just considered; it results from this that no contact is established between the brush J and the drum D, no current being allowed to flow through the intermittent heating-coil.

The sequence of operations just described would of itself be sufficient to ensure a fairly constant temperature in the oil-bath ; any one revolution of the sleeve X being accompanied or unaccompanied by a closing of the shunt-circuit according as the temperature of the copper arms of the bridge B was too low or too high.

In order to make the regulation less spasmodic than it would be under such conditions, two further adjustments are introduced. In the first place, the series-coils included in the main circuit are automatically * diminished in resistance as the demand for heating becomes heavier, and *vice versâ* ; and in the second place, the time during which the circuit is closed is made to bear a greater or a less proportion to the period of revolution of the sleeve X, as the case may be. If diminished resistance in the series-coils and increased duration of short-circuiting are required, it is evident that both of these results may be accomplished by displacing the drum D and slider W along the spindle towards the right in fig. 1. This, of course, implies an equal displacement of the cam C, since C, D, and W are all connected axially. It is, in fact, by means of the cam C that the displacement is effected. The cam acts both radially and axially, its radial action causing the rise and fall of the comb K, and consequently of the hit-or-miss arm H ; while by its axial action, the cam itself, together with the drum D and slider W, is displaced endwise along the spindle S. The general form of the cam C is best seen in fig. 2 ; the view being that which is obtained by looking along the shaft S from right-hand side in fig. 1. In this figure the measurements indicated in millimetres are the distances of the corresponding points of the periphery from the principal plane of the cam, that distance being marked + which is towards the observer, and *vice versâ*. To make the action of the cam more readily comprehensible, the distinctive features of the two sectors are here set out in parallel columns.

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* The device for automatically varying the external resistance of the heating circuit, namely, the slider W and resistance-blocks L, was suggested by Mr. Lunt, of the Royal Observatory, Cape of Good Hope.

Sector C_1.	*Sector* C_2.
Has a larger radius.	Has a smaller radius.
Is left-handedly helical, and so traverses the cam, drum and slider towards the left in fig. 1.	Is right-handedly helical, and so traverses the cam, drum and slider towards the right in fig. 1.
Has a throw of 4 mm.	Has a throw of 8 mm.
Operates at every revolution of the sleeve X.	Operates only when the hit-or-miss arm has missed the boom G_1.

It should further be observed that when the sector C_1 comes under the comb K, whether the arm H is up to that moment resting at the lower level upon the stop-screw, or at the higher level upon the boom G_1, the effect of the sector C^1 will be to raise the arm H clear of the boom G_1, leaving the latter free to take up its equilibrium position under the influence of such small current as may then be traversing the galvanometer-coil, owing to want of balance of the controlling-bridge; thus determining the greater or less descent of the arm H, after the sector C_1 has passed from under the comb K, as already explained. As regards the displacement of the cam C, together with the drum D and slider W, along the axis of the shaft S; it is apparent from the foregoing paragraph, that for *every* revolution of the spindle a displacement of 4 mm. to the left occurs, while a displacement of 8 mm. to the right occurs only for those revolutions in which the arm H misses the boom G_1. Thus it follows that when the hit-or-miss arm H just hits the boom G_1 as often as it misses, the cam, drum and slider will only suffer small axial displacements on either side of a mean position, but on the whole will be without progressive axial motion. On the other hand, if the demand for heating current is greater than can be met by a steady average of one hit to one miss (that is to say, by one short-circuiting of the series-coils for two complete revolutions of the sleeve X), the galvanometer-boom G_1 will be more often deflected to the "cold" side than to the "hot" side, and the misses will preponderate over the hits, In consequence of this, the sector C_2 of the cam will operate more often than an average of once in every two revolutions of the sleeve X, and so the displacement of 8 mm. to the right will occur on the whole more than half as often as the displacement of 4 mm. to the left. The result will be a progressive displacement to the right, causing a wider part of the brass sector D to come under the brush J, so that each short-circuiting is of longer duration, and perhaps also causing the slider W to move from one of the blocks L to another, thus reducing the resistance interposed by the series-coils F in the main circuit. These effects will diminish the demand for such frequent short-circuiting of the series-coils; and finally, when the displacement of cam, drum and slider has been carried to such a point, that on an average the arm H hits the boom G_1 as frequently as it misses, there will be no further displacement of the cam, drum and slider, which will now execute small excursions on either side of a mean position as before. Quite similarly, a reduced demand for heating current will cause the hits to become more frequent than the misses and to remain so until a sufficient displacement of the cam, drum, and slider in the negative direction has been effected, when the hits and misses will once more occur with equal frequency on the whole.

A special feature in the design of the cam C relates to those cases in which the temperature of the controlling-bridge is from any cause persistently too high or (as in starting "all cold") persistently too low. It is essential that in such circumstances the cam should neither become jambed at the end of its run, thus stopping the whole mechanism, nor lose touch with the comb K, so as to fail in operation when the adjustment of temperature has improved. The form given to the cam, and shown in fig. 2, secures this result. It should be mentioned that the comb K on either side of the toothing is cleared away to the level of the throats between the teeth.

I wish to thank Mr. C. Young for the great help he gave me in the design of the instruments. Dr. C. V. Burton has also given me great assistance in writing this paper.

DISCUSSION.

Mr. W. C. D. WHETHAM asked what accuracy was aimed at in the instrument.

Mr. A. P. TROTTER said that the use of thermostats would become more and more important with the perfection of thermometers. It was possible to keep a temperature constant within $\frac{1}{100}$° C., but it was very difficult to determine the temperature to that degree of accuracy. In high-class resistance measurements with a Carey Foster bridge one of the greatest difficulties was with regard to the temperature, which must not only be constant but also capable of exact determination. In the constant-temperature room at the Board of Trade Standardizing Laboratory the temperature varied by less than $\frac{1}{5}$° C. from day to day, and he would like to know the greatest accuracy obtainable with ordinary devices in such a room before considering the introduction of the complicated instrument shown to the meeting.

Dr. R. T. **Glazebrook** expressed his interest in the Author's communication, and said he had recently been trying to solve a somewhat similar problem by making use of a horizontal boom, attached to the suspended coil of a galvanometer, for making and breaking a current. His object was to regulate a current of the order of 50 amperes. Dr. Glazebrook asked if it was desirable or important to utilize all the adjustments in the instrument.

The **Author**, replying to Mr. Whetham, said the instrument had been made as sensitive as possible, but he was not aware within what limits it would maintain the temperature of the bath. The thermostat supplied to the Cape Observatory was capable of keeping the temperature within $\frac{1}{100}$° C. for a period of 8 hours. In reply to Dr. Glazebrook, he said that when working with small variations of the external temperature it might be possible to dispense with one or other of the adjustments; but when the external variations were large it was necessary at times to supply a large quantity of heat to the oil-bath, and in order to do this effectually he thought all the adjustments of the instrument were desirable.

IX. *On the Occurrence of Cavitation in Lubrication. By* S. Skinner, *M.A., University Demonstrator of Experimental Physics, Cambridge*[*].

[Plates VI.-VIII.]

§ 1. The following experiments[†] arose from an observation made when determining the refractive index of a liquid by means of Newton's rings. As Newton showed, the rings can be obtained when a liquid is run into the space between the lenses (Opticks, Obs. 10); and by comparing the radii of the rings with those obtained with the same lenses and air we have a means of measuring the refractive index between air and the liquid. The only difficulty is to arrange the illumination suitably as the rings are far fainter with the liquid than with air.

If when the liquid has been introduced the upper lens be rolled on the lower the observer sees following the central dark spot, viewed by reflected light, a crescent-shaped space, very bright provided the illumination be sufficiently oblique. This is a vacuous or vapour-filled space, for when the motion of rolling ceases the liquid flows into the space and completely fills it. Starting the rolling in the opposite direction the same appearances are seen in the reverse order. It was the observation of this space which led me to make the following experiments, and at the time I was unaware that Newton had observed its formation, for the paragraph in which he mentions it is omitted from the quotations which have found a place in modern text-books. In Observation 11 Newton writes:—" When the Water was between the Glasses, if I pressed the upper Glass variously at its edges to make the Rings move nimbly from one place to another, a little white Spot would immediately follow the center of them, which upon creeping in of the ambient Water into that place would presently vanish. Its appearance was such as interjacent Air would have caused, and it exhibited the same

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[*] Read November 27, 1903.

[†] These experiments formed the subject of a Demonstration before the Cambridge Philosophical Society, November 10, 1902.

Colours. But it was not Air, for where any bubbles of Air were in the Water they would not vanish. The Reflexion must have been caused by a subtiler Medium, which could recede through the Glasses at the creeping in of the Water."

§ 2. The inflow of the liquid to fill the vacuous space must depend in some way on its viscosity. When a more viscous liquid is used all the effects are more pronounced, and with glycerine or lubricating oil quite large vacuous spaces, frequently broken up into a number of small bubbles, are obtained. The method of observation with oblique light takes advantage of total internal reflexion, and consequently necessitates a certain angle, depending on the refractive index of the glass, and it may be inconvenient. Another mode of observing the space is to use sodium light, when the relatively bright Newtonian rings in the space show up well in contrast with the faint rings in the liquid.

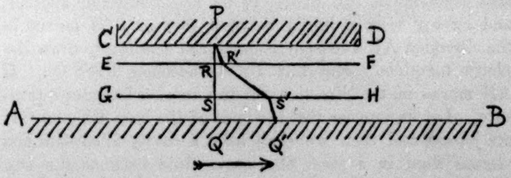
§ 3. A third mode of observation, which is much the most convenient, is to use a deeply coloured liquid and to look at the space by transmitted light. I have found that a very convenient liquid is a strong solution of fuchsin in glycerine. This red solution is so deeply coloured that even up to the point where the lenses are nearest some colour shows, and that light is only brightly transmitted at the place where there is a break in the liquid. Placing this liquid between the lenses, and using daylight reflected from a sheet of white paper and transmitted through the system, all the appearances which are here described may be seen. It is also very convenient for photographing effects.

§ 4. When the rolling ceases it has been remarked that the cavity fills up quickly with mobile, and slowly with viscous liquids, and this filling of the cavity is quite complete unless some gas or air has found its way in. The liquid is forced back to fill the vacuous space by the atmospheric pressure, and the rate of filling is decided by the viscosity of the liquid. That it is so is seen by the slowness with which oil or glycerine fills the cavity made in it, and is also evident from the relatively large size of the cavity formed in these liquids. The cavity which is formed must be produced either by splitting the liquid itself or by tearing the liquid from the glass surface. The effect may be described as a case of "cavitation." This word has been used by the Hon. C. A. Parsons ('Nature,' May 1898) to describe the production of a cavity in water by a very rapidly rotating screw-propeller. In his experiment the atmospheric pressure was removed from the surface of the water by an air-pump.

§ 5. Before describing the actual effects some account of the stresses in the liquid will help to make the conditions of the cavitation more clear; and in this we shall follow the graphical method used by Osborne Reynolds in his paper on the "Theory of Lubrication" (Phil. Trans. A. 1886). In this paper the origin of the force resisting the motion of two surfaces separated by a continuous and copious supply of lubricant is discussed, and it is shown that it is wholly accounted for by the viscosity in the lubricant.

Let AB and CD represent the sections of two parallel solid surfaces, extended infinitely at right-angles to the

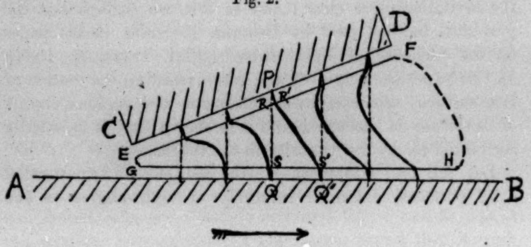
Fig. 1.



paper, and between them let us suppose three fluid layers lie, bounded by the lines EF and GH. Let us suppose the viscosity of the layers in contact with the solid walls is the same, and that the viscosity of the central layer is very much less. Let PQ represent a line in the layers when AB and CD are at rest. If now AB moves with a velocity QQ′, then the relative velocities of the particles in parallel layers will be represented by the displacement of PRSQ to PR′S′Q′, RR′ and SS′ representing the velocities at the boundaries of the central layer. If, as we have supposed, the viscosity of the central layer is very small compared with that of the other layers, the inclination of R′S′ will be great, whilst PR′ and S′Q′ will be nearly parallel to RQ. In words, the tangential force opposing the motion of AB will be almost entirely due to the thin central layer of very slightly viscous fluid.

Next we consider the case where CD makes an angle with AB, and EF and GH represent as before the boundaries of the three liquid layers. It is shown by O. Reynolds (*loc. cit.*)

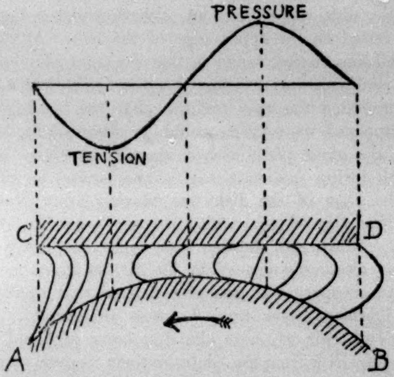
Fig. 2.



that when one of two plates, at an angle with one another, and having viscous liquid between them, AB is moved in the direction AB a tension is produced tending to draw the plates together. The line PRSQ becomes PR′S′Q′. If AB moves in the direction BA this tension becomes a pressure. Let us suppose that the ends of the lines EF and GH are joined, and then we shall have a cavity filled with less viscous fluid in a mass of viscous fluid between the two plates. Cases similar to this are treated in this paper.

§ 6. In figs. 3 and 4 (Pl. VI.) is represented a cylindrical lens lying on a plane surface with some fuchsin-glycerine solution between them. In fig. 3 everything is at rest, and the line of nearest approach of the surfaces is shown by the absence of colour, *i. e.* the bright band down the centre. In fig. 4 the cylindrical surface is being rolled by the fingers from left to right, and the transparent line of nearest approach is seen to have moved slightly to the right hand. Following it is seen a dark line behind which is a broader light space traversed horizontally by small streaks of dark liquid. This broader light space is filled only with the vapour of the liquid, for, as we have already stated, it is completely filled with dark liquid as soon as the motion ceases. This cavity has been formed by the splitting of the liquid, and it is of special interest to notice that it is formed not immediately at the line of nearest approach but some short distance behind this line. That the greatest tension in a lubricating liquid is not at the point of nearest approach of the surfaces was shown theoretically by Reynolds, who calculated the pressures and tensions in a layer such as this. Fig. 5 is

Fig. 5.



taken from Reynolds' paper, and from it we see that the point of maximum tension is behind the point of nearest approach.

§ 7. In figs. 6 and 7 (Pl. VII.) is represented a lens lying on a plane surface. In fig. 6 everything is at rest, and the light spot at the centre shows the point of nearest approach of the lens to the plane. In fig. 7 the lens is being rolled from left to right by the fingers and a collection of cavities has been formed behind the point of nearest approach. As with the cylindrical surface, on stopping the motion the cavities fill completely.

§ 8. In fig. 8 (Pl. VIII.) a lens is being drawn along a plane, and the cavity is visible following the motion, which is from left to right. In fig. 9 (Pl. VIII.) a biprism is being pushed from left to right, and a large cavity is visible behind the edge of the prism which almost touches the plane. The biprism was held so that the angle between the left-hand face and the plane was greater than that between the right-hand face and the plane.

§ 9. Some experiments were made to imitate the actual case of a fully lubricated axle rotating under a bearing. For this purpose a thick disk with its edge worked to a spherical surface of curvature equal to the radius of the disk, mounted on an axle, was arranged so that the lower part of the disk dipped into an oil-bath whilst a flat plate of glass rested on the upper edge of the disk. As the disk rotated oil was carried round so that the point of contact of the disk and plate was maintained copiously lubricated. The disk represented the axle and the plate the bearing on it. What happened on rotation could be observed by looking through the glass plate with a magnifying-glass. It was seen that during motion a cavity was formed on the side where the edge of the disk was moving away from the plate. The size of the cavity depended on the rate of rotation. When the rotation was rapid the cavity grew large and ultimately opened to the air, so that the cavity became an air-space. It was also noticed that small drops of the lubricant were carried across the cavity. These probably had their origin in the thin layer of liquid which separated the disk from the plate at their nearest approach. That there was a pressure on the one side and a tension on the other was shown by drilling a hole through the glass plate and attaching a tube to this hole. When the hole was on the approaching side of the point of nearest approach oil was forced up the tube, whilst when it was on the side where the disk's edge receded air was sucked in through the tube. In fact the latter arrangement formed a small suction-pump. The high pressure in a thoroughly well lubricated bearing was observed by Tower in his experiments on friction (Proc. Inst. Mechanical Eng. 1883–1884).

§ 10. It has been shown by Berthelot *, by O. Reynolds †,

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* *Ann. de Chimie et Phys.* vol. xxx. p. 232 (1850).

† Reynolds, 'Collected Papers,' vol. i.

by Worthington* that a liquid can sustain a large tension or negative pressure. The ascent of sap † in tall trees has been explained by a theory in which the liquid sap is under tension. In general the effect has been studied in liquids at rest. It may be supposed that if the movement of the cylinder on the plane in the experiment described in § 6 were sufficiently slow, the negative tension would not be great enough to break the liquid but would help in drawing the liquid forward to fill the space left behind the moving surface. Experiments in which the motion is slow do not show the formation of a cavity, and may be regarded as supporting this view of the action of the tensional strength of the liquid.

§ 11. The slipperiness of ice ‡ has been attributed to the presence of a layer of lubricating water under the body pressing on the ice. The water is produced by the lowering of the freezing-point where the pressure is experienced. On this view the object glides on a liquid layer and consequently viscous friction in water takes the place of the rubbing friction between the solids. Joly shows by calculation that the weight of a man concentrated on the blade of a skate is sufficient to lower the freezing-point very considerably. Reynolds, arguing from the difficulty of slipping on very cold ice, comes to a similar conclusion. I wish to point out that sliding on a liquid layer is a condition under which cavitation will occur in the liquid, and that this will aid the slipping. I find that a cavity can be seen when a convex lens is pressed strongly on ice and pushed along. A cavity of this kind may be formed behind the sliding contact of a hog-back skate, and behind that of a curling-stone.

§ 12. We may now point out the influence which the formation of a cavity has on lubrication. In ball-bearings completely immersed in oil the experiments show that there must be a small cavity near the point of nearest approach of each ball to its neighbours and also to the surface on which it is running. As the friction of the bearing is the viscous friction of the oil, from the considerations in § 5 it follows that it must be reduced by the formation of these cavities which are filled with relatively non-viscous vapour. In fact, if a steel ball thoroughly oiled be rolled against an oiled glass plate the cavity may be directly observed. The high lubricating property of oils owes its origin not only to their superior viscosity but also possibly to the facility with which cavities may be formed in them.

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* Phil. Trans. 1892, A. p. 355.

† Dixon & Joly, Phil. Trans. 1895, B.; Askenasy, *Verhandl. d. naturh. med. Vereins Heidelberg*, N. F. v. (1895).

‡ Joly, Proc. Roy. Dublin Soc. p. 453 (1886); O. Reynolds, Mem. and Proc. of the Lit. and Phil. Soc. of Manchester, xliii. (1899).

DISCUSSION.

Prof. J. D. **EVERETT** said it was interesting to note that cavitation had been observed and correctly explained by Newton. A similar phenomenon was exhibited by a steam-roller working on mud. It pushes up the mud in front, and produces cavities behind.

Mr. W. A. **PRICE** said that the observations of the Author could not fail to be of interest to engineers. He asked to what extent cavitation was going on in an ordinary round bearing or in the slide-valve of a locomotive, and suggested that the views of engineers upon the subject of lubrication might require considerable modification in the light of the present paper.

Mr. C. V. **BOYS** expressed his interest in the paper and the experiments shown by the Author. He said that, like Mr. Price, he had been thinking of the influence of cavitation upon machine bearings, but unlike him he had not come to the conclusion that our ideas on the subject should be changed. In the Author's experiments cavitation was produced by the approach of two surfaces on one side of their line of contact and their recession on the other side. In the case of a cylindrical bearing the space between the shaft and journal might be completely filled with oil, and if the shaft were truly concentric there would be neither approach nor recession. He thought that in the case of any cylindrical bearing the eccentricity could not be enough to make the rate of approach and recession great enough to produce cavitation. The case of ball-bearings, when using a thick lubricant, was different, and cavitation might occur. If it did it would occur behind the point of contact, and although the production of cavitation involved, in an infinitesimal degree, a loss of energy, he questioned whether it did any appreciable harm. Mr. Boys referred to the dendritic patterns formed, when mixing paints, by lifting the muller from the plate, and suggested that their formation was a phenomenon of the same kind as the cavitation exhibited by the Author.

Mr. **Appleyard** said that in reference to Mr. Skinner's suggestion that cavitation reduces friction, this might be the case in some instances where the rolling surfaces are perfectly smooth; but with imperfectly smooth surfaces it would be detrimental, for it would result in direct contact. This might explain some of the mysterious examples that occurred of sudden heating of apparently well lubricated bearings. In the case of a wheel running on a plane wet road, cavitation probably would not come into the problem of slip because, as Mr. Boys mentioned, the cavitation was always left behind. But in the case of a shaft revolving in a bearing, the cavitation left behind by one small portion of the revolving surface provided an oil-free area for the next portion; and there might be contact or there might not according to the degree of smoothness of the surfaces.

Dr. A. **Griffiths** asked the Author if cavitation in lubrication was supposed to be an advantage or a disadvantage.

Mr. **Skinner**, replying to Mr. Boys, suggested the construction of glass bearings to determine whether cavitation occurred in cylindrical shafts. He thought that cavitation in lubrication was an advantage as a rule, because spaces formerly filled with viscous material were then filled with something with small viscosity.

X. (1) *A Lever Gauge.*

(2) *Note on the illumination of Mirror-instruments.*

By R. Appleyard.

[Read March 8, 1901.]

XI. *The Photographic Action of Radium Rays.* *By* S. Skinner, *M.A., University Demonstrator in Experimental Physics at the Cavendish Laboratory, Cambridge* *.

It is well known that a photographic plate by exposure to Radium rays is affected in such a way that the plate develops similarly to its development after exposure to light. The following experiments are an attempt to answer the question : Are the actions the same? As far as can be seen by eye, plates exposed to the two agents appear on development to be darkened in a similar way, and we may conclude that the final result of the actions and of the development is the same. I have attempted to obtain information of the intermediate stages, and some of the experiments described below appear to indicate that only slight differences occur in these early stages. I shall describe the experiments in the order in which they may be best understood, although this was not the order in which they were made.

A plate, inclosed in two envelopes of paper, one red and the other black, was placed under a capsule containing 10 milligrammes of radium bromide at a distance of 1 centimetre from the plate for periods of time varying from half a minute to 48 hours. A different part of the plate was presented to the radium for each period of time. It was then developed in alkaline hydroquinone for three minutes, and the intensities of the developed images were compared by means of a commercial densitometer. A similar experiment was also made with 50 milligrammes of radium bromide. For most of the experiments a red label rapid Ilford plate was used, and in a few a 'Lumière' plate.

The general result of these experiments showed that the intensity of the developed image increased rapidly to a maximum value, then decreased rapidly, and finally the intensity decreased very slowly until a stage was reached in which there was practically no dark image formed on development. The time of exposure required to reach the maximum depended on the mass of the radium bromide used, that with 50 mgr. being earlier than with 10 mgr. In fact, after 40 hours' exposure the effect on the plate might be compared to the complete reversal of image obtained when a plate is very much over-exposed to a bright light. The following curves refer to Ilford red label rapid plates, I. exposed respectively to 50 mgr. and II. to 10 mgr. (fig. 1). With this class of plate the maximum was reached

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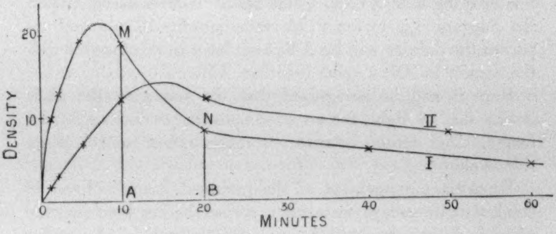
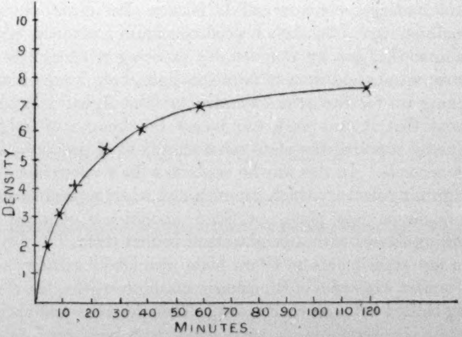
* Read January 22, 1904.

Fig. 1.



in 15 minutes with 10 mgr. of radium bromide, and in 7 or 8 minutes with the 50 mgr. With Lumière plates the maximum was reached in about 2 hours, see fig. 2.

Fig. 2.



To verify the result that an exposure greater than a certain critical value will produce a less dense image, the following experiment was made. A slit was cut in a thick sheet of lead and a covered photographic plate was placed below it. It was then exposed to radium rays for 10 minutes. Then the lead plate was removed, and the plate was exposed for a further period of 10 minutes. In this way the parts opposite the open slit received altogether 20 minutes' exposure, whilst the protected parts were exposed only for half that time. On development the slit appeared as a light line crossing a dark field. This result follows at once from the diagram (fig. 1), for if OA represents the 10 minutes' exposure the density will be AM, and for a time exposure OB the density is BN, a value less than AM.

Here it will be recognized that the action on the plate is like that of light, for an overexposure produces a fainter image. The action, however, is more under control as its rate is slower.

From the arrangement of the radium * I am inclined to think that the α-rays were all absorbed before reaching the plate, and that we have to consider the action as coming from the β- and γ-rays alone, these two sets of rays having much more penetrating properties than those of the α group.

The next experiments were made with the view of finding out whether a plate exposed to the light of an electric spark would undergo any reversal if it was afterwards exposed to radium rays. In fact, I tried to obtain a reversal effect similar to that got by Clayden, by exposing a plate first to electric sparks and then to faint gas-light, only I substituted radium rays for the after-exposure to faint light. Clayden showed that it was easy to reverse the images of bright sparks by exposing the plate subsequently to a gas-flame for a few seconds. In this way he explained the reversed images of lightning-flashes which are obtained when several images are taken on one plate, or when images are obtained of lightning-flashes with a bright cloud behind them.

In my experiments an Ilford plate (red label) was exposed in a camera to a series of Wimshurst machine sparks, the plate being shifted between each spark. Then the plate was placed

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* The radium bromide was contained in an ebonite capsule covered with a thin mica plate. The rays passed through the mica.

in its envelopes, and a part of it was protected by a thick leaden screen. The rest of the plate was exposed to the radium. This was repeated for different periods of radium exposure. On developing these plates, it was found that in those which had been exposed to the radium for only short periods, gradually increasing in length, there was a progressive elimination of the spark image, but the density of the parts corresponding to the radium exposure was not so great as that of the parts of the plate only exposed to the spark, although the radium action had been able to obliterate completely the spark effect. On the other hand, when those plates which had been exposed longer, or had been exposed to the greater mass of radium, were developed, I found that reversed images of the sparks appeared. These images were not so wide as the spark images. In another case, with still further exposure, I observed the radium reversing its own image, and across the radium reversed part of the plate there appeared a faint dark image of the spark. This might be described as a re-reversal of the spark.

The conditions of these experiments were such that probably only the β- and γ-rays were acting. In the Philosophical Magazine for November, 1903, a paper appears by R. W. Wood, on the subject of photographic reversal, and in it he says that he has been unable to reverse a spark-image by Röntgen rays, which are supposed to be similar, if not identical with the γ radium rays. If this is the case we must ascribe the reversal, which I have described as produced by radium, to the action of the β-rays alone. This point, however, needs further investigation with screens capable of cutting off each class of rays.

In Wood's paper he gives what may be called a reversal order. It is :—Pressure-marks, x-rays, Light-shock, Lamp-light. In this order each successive influence is able to reverse those before it. He also states that Becquerel rays will reverse pressure-marks, and that lamplight will reverse the effect of Becquerel rays. The Becquerel rays which he used were probably those from a compound of uranium. My experiments show that Light-shock (from an electric spark) may be reversed by Radium rays, and consequently

Radium rays may be inserted provisionally between Lightshock and Lamplight in the above order.

Conclusions.

1. The density of the image produced on a plate by exposure to radium rays (β and γ) increases to a critical value and then decreases, at first rapidly and afterwards very slowly, until a time is reached when the image is totally reversed.
2. Spark-images are at first obliterated by radium rays which do not cause such a great density as that of the spark-images obliterated.
3. With prolonged exposure radium rays reverse spark-images.

I am greatly indebted to Mr. Hayles, of the Cavendish Laboratory, for carrying out many of the experiments described in this paper.

XII. *Some New Cases of Interference and Diffraction.* By R. W. Wood, *Professor of Experimental Physics, Johns Hopkins University*[*].

[Plate IX.]

In the following paper I propose to discuss certain types of the interference of light which have been known for many years, as well as some cases which, so far as I know, are quite new; the colours of mixed plates and the phenomena of interference in transparent films deposited on metallic reflectors being the cases chiefly considered. Several years ago it occurred to me that it would be worth while to try to devise some permanent film which would show Newton's colours under the most favourable conditions, and the cases cited in the present paper are, for the most part, the outgrowth of experiments made in this direction. The facts which have been brought out may be summed up as follows.

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[*] Read January 22, 1904.

The colours of mixed plates are due to diffraction, and should not be classed with interferences in thin films. The explanation originally given by Young, and the treatments given by Verdet and others, are unsatisfactory, in that they do not indicate what becomes of the energy.

In the cases of films deposited on perfectly reflecting surfaces which, according to the elementary theory, should exhibit no interference-colours, we may, under certain conditions, have colours far more brilliant and quite as saturated as any shown by the soap-bubble. In other cases, where at first sight no interference appears to have taken place, we may, by employing polarized monochromatic light, obtain fringes of a very curious nature, which are the result of the interference between the elliptical vibration coming from the metal surface and the plane-polarized vibration reflected from the surface of the transparent film. We will go back now to the earlier experiments.

Preparation of Films for the exhibition of Newton's Colours.

In the case of a thin transparent lamina, such as a soap-film, the amplitudes of the disturbances reflected from the two surfaces are equal, and consequently completely destroy each other when the phase-difference is 180°. Inasmuch, however, as only a small percentage of light is reflected from each surface, the colours, though saturated, are not as intense as is desirable. In the course of some experiments with selenium I found, if a plate of mica is pressed against a pool of the molten substance on a glass plate, and the whole allowed to cool under pressure, that on stripping off the mica, films of mica of variable thickness were left upon the surface of the selenium, which showed Newton's colours of great beauty, arranged in mosaics. The patches of equal thickness being sharply bounded by straight lines, present an appearance similar to that of selenite-films under the polariscope. The selenium has a much higher refractive index than the mica, consequently the reflexion at each surface is the reflexion of rays incident from a rare to a denser medium, and the difference of phase is given by the difference of path alone; *i.e.*, we do not have the loss of half a wave-length due to reflexion under opposite conditions, as would be the case if the mica films were in air.

If the mica is cemented to the plate with sealing-wax or any of the common resinous cements, very little trace of the colours is to be seen, owing to the fact that the refractive indices of the two media being so nearly the same, practically no energy is reflected from the boundary. The use of selenium can be avoided by very lightly silvering the surface of the mica, which may then be cemented to the glass with any good laboratory cement, the metallic layer taking the place of the medium of high refractive index. This latter method is the best for the preparation of large mosaics suitable for lantern-projection. In the patches which show no colour by reflected light, the interference may be detected with a small spectroscope, the spectrum appearing crossed by black bands, corresponding in position to the wave-lengths absent in the reflected light. Still more brilliant films can be prepared by first thickly silvering the mica, cementing the silvered side to the glass plate, and then stripping the mica off. The colours are scarcely visible, owing to the disproportionality between the amplitudes of the two interfering streams of light, but appear as soon as the upper surface of the mica is half-silvered, which can be done by immersing the plate in a silvering solution until the colours reach their maximum brilliancy. To obtain films which show the maximum brilliancy, it is clear that the amplitude of the stream reflected from the first surface must have the largest possible value consistent with the fulfilling of the condition that it be completely destroyed by the disturbance coming from the second surface when the phase-difference is 180°.

In the case of transparent films, the wave-lengths absent in the reflected light appear in excess in the transmitted light, there being no destruction of energy. If the second surface is a perfect reflector, the energy thrown down upon it by interference at the first surface will all be returned. If we consider the upper metallic surface as nonabsorbing, and work out the case by the method of multiple reflexions, we find that no colour will be produced, light of all wave-lengths being reflected with equal facility. The fact that brilliant colours appear, means that light is absorbed at one or both of the silvered surfaces, since this is the only way in which the energy of the absent wave-lengths can disappear. If we assume a certain percentage reflected and a certain percentage absorbed by the thin silver film, we find that the waves for which the phase-difference is 180° are compelled to make more transits through the film than those for which the phase-difference is 360°. It is possible to obtain experimentally a condition in which the former are almost completely absent in the reflected light, while the latter are reflected with scarcely any loss of intensity. To calculate the most favourable conditions, we should require data regarding the percentages reflected and transmitted by films of various thicknesses.

It next occurred to me to substitute a thin film of collodion for the mica, half-silvering the film as before.

A sheet of glass can be silvered chemically, or procured by removing the varnish from the back of a piece of modern mirror-glass with alcohol. The silver film is then flowed with collodion diluted with three or four parts of ether. As soon as the film dries colours appear, contrary to theory. These colours may be quite brilliant, and are due to diffraction, as I shall show presently. If the plate be now immersed in Brashear's silvering-bath, the colours will instantly disappear, owing to the fact that the collodion-film and the solution have nearly the same refractive index. As soon as the silver begins to deposit, the colours reappear and increase rapidly in intensity. The bath should be rocked, the process being similar to the development of a negative. A little experience will enable the moment of maximum brilliancy to be correctly judged, when the plate should be immediately removed from the solution, washed, and dried. It is well to provide the plate with a cover of glass mounted over it at an angle of 20°, the whole forming a prismatic box. The object of inclining the cover is to get rid of the light reflected from it, which would otherwise dilute the interference colours. Plates prepared in this way show a wonderful blaze of colour and make excellent preparations for the lantern.

The Colours of Mixed Plates.

Interference colours of this type were discovered by Young, and described in the 'Philosophical Transactions' for 1802. They were subsequently studied by Sir David Brewster. Verdet and other writers on Optics have classified them with Newton's thin-film colours, and have given treatments which are not very rigorous and fail to show where the energy goes to.

The colours are very easily obtained by pressing a little white of egg between two pieces of plate-glass, separating the plates and squeezing them together a number of times so as to form a froth. The plates are to be pressed firmly together with a rotary sliding motion just before the froth becomes sticky, enclosing a film made up of air and albumen in the form of a mosaic. The colours are best seen by holding the plate towards a window or other bright source of light on a dark field. Certain wave-lengths will be found to be absent in the directly transmitted light. Young's explanation was that the path-difference between a ray passing through an air-space and one passing through the albumen was an odd number of half wave-lengths for such colours as failed to appear in the transmitted light. Neither Young nor subsequent writers, so far as I have been able to find, show what becomes of these absent colours, though both Young and Brewster observed the coloured fringes which appeared in the dark background to one side of the source of light. Brewster published a paper in the 'Philosophical Transactions' for 1837 in which he referred the colours to diffraction, though his treatment was not very complete, and concerned chiefly the case of diffraction by a transparent lamina bounded by a straight edge. Verdet objected to this explanation on the ground that the colours are independent of the size and arrangement of the air-bubbles, depending only on the thickness of the albumen-film and the angle of incidence. The interference phenomena of mixed plates are easily explained by the elementary theory of diffraction, and they should be classed with laminary diffraction effects, and not with thin film interferences, as is usually the case.

The following treatment may not be new, but I do not remember to have seen it anywhere:—In fig. 1 (Pl. IX.) let A A represent the glass plates with the albumen and air elements between them. We will assume the thickness of the albumen such that green light suffers a retardation of $\frac{\lambda}{2}$ in traversing it. If B is the lens of the eye, and parallel rays traverse the plate, the secondary disturbances represented by the dotted lines (normally diffracted rays) will be brought to a focus at E; that is, the reduced paths of all these rays are equal and the disturbances arrive at E in the same phase, if there be no retardation. The disturbances coming from the albumen elements are retarded, however, and reach E half a wave-length behind the disturbances coming from the air elements. The two sets destroy each other at this point, and green light will not be represented here. In general, light will be absent at this point if the retardation of a ray passing through albumen with respect to one passing through the adjacent air-space is $(2n+1)\frac{\lambda}{2}$. If the film is fairly thick, this condition may hold for a number of colours in the spectrum, which will consequently be absent in the image of any source of light seen through the plate. The question now is: What becomes of this energy? In the case of thin film interferences, the wave-lengths absent in the transmitted light appear in the reflected. This is not the case with mixed plates, which show no colour by reflexion. If we refer to fig. 1, it is clear that if we take parallel rays diffracted in an oblique direction, the phase-difference introduced by the retardations in the mosaic may be compensated by the obliquity, the agreement of phase being more or less complete for green light in the point F. The case is analogous to a laminary grating, which yields coloured central images, the absent wave-lengths appearing in the spectra. Mixed plates throw the light absent in the direct image into a halo or ring, which is seen to surround the source of light. Laminary diffraction phenomena produced by straight edges, two parallel slits, and gratings are very fully treated in Verdet's 'Optics.'

with a rather dilute solution of gelatine, and allowing it to dry in a slightly inclined position.

The easiest way to get them, however, is to blow out the end of a rather large glass tube into a large thin balloon of tissue glass, picking out a portion, by the light of a sodium flame, which shows fairly straight interference-fringes two or three millimetres apart. A small piece of the thin glass is laid, with its slightly convex side down, upon a clean mercury surface, and sodium light, polarized in azimuth 45°, reflected from the surface at an angle of about 60°. On viewing the reflected light through a nicol, the curious double fringes can be found easily by slowly turning the nicol. The light will be found to be plane-polarized along the lines 1, 3, 5, 7, &c. of fig. 4, though in general the planes of polarization along one set of lines is inclined to the plane of polarization along the alternate lines, as is indicated below the figure, the arrows representing the direction of the vibration (electric vector). Between the lines of plane-polarized light, which appear as dark fringes when the nicol is so oriented as to extinguish the light, we have either elliptically or circularly polarized light, as can at once be shown by the introduction of a quarter-wave plate, which enables us to extinguish the light along the lines 2, 4, 6, and 8 by suitable adjustment of the mica plate and analyser. The direction of revolution of the vibration along lines 2 and 4 is opposite to that along lines 6 and 8.

I at first attempted to work out the problem by compounding a plane-polarized vibration, coming from the air-glass surface, with a nearly circular vibration coming from the metal. This was a failure, but it at once occurred to me that owing to the refraction by the glass or gelatine, the incidence-angle of the light falling upon the metal would be too small to produce circular polarization.

It seemed best, upon the whole, to determine experimentally the exact nature of the vibration coming from each surface, and then to try whether, by compounding them with various phase-differences, the observed results could be accounted for.

The incident light vibrated in a plane indicated by the arrow at the top of fig. 4. The light reflected from the glass surface is of course plane-polarized, vibrating parallel to the surface when the angle of incidence is equal to the polarizing angle. For larger angles of incidence, the plane of the reflected vibration makes an angle with the surface, depending on the magnitude of the reflected component, which lies in the plane of incidence. To determine the nature of the vibration coming from the glass-metal surface, it was necessary to get rid of the light reflected from the upper surface of the glass. This was done by laying a small piece of rather thick plate-glass on a plate of speculum metal with a film of benzole between, and allowing a narrow beam of light to fall upon the surface of the glass. The images reflected from the two surfaces appeared separated, and could be independently examined with a nicol. The benzole film practically brought the metal surface into optical contact with the glass. The reflected beams were found to be plane-polarized, the vibrations being in the directions shown in fig. 5 *a*, for an angle of incidence near the polarizing angle. As the angle of incidence increased, the planes of the two vibrations both turned towards the vertical, and finally made an angle of 90° with each other, *i. e.* 60° and 30° with the reflecting surface.

If we compound the two components shown in fig. 5 *a* with varying phase-differences, we can account easily for the polarized fringes. In fig. 5 *b* let BC be the vibration from the glass and BA that from the metal. When the path is zero or a whole number of waves, we have the plane-polarized resultant BE. If the path-difference is an odd number of half-waves, we have the plane-polarized resultant BF. These two states occur along the lines 1, 5 and 3, 7 respectively. The angles which the planes of vibration make with the reflecting surface should be respectively greater and less than the angle made by the component coming from the metal, as was found to be the case. As we increase the angle of incidence the component BC (from glass) increases in magnitude and turns up towards the vertical, the inclinations of the planes of polarization of the two sets of fringes to each other becoming greater.

Between the lines along which the light reflected from the film is plane-polarized, we have lines of elliptically (or in some cases circularly) polarized light. The directions of revolution were determined with the quarter wave-plate, and are as shown in fig. 4. Geometrical computations of the elliptic vibrations, resulting from two components such as are shown in fig. 5 *b*, agreed perfectly with the experimentally determined orbits, both in respect to the directions of the major axes and the directions of rotation. The method used was similar to the one given in Müller-Pouillet's text-book (vol. ii. 1. p. 1135), the only modification necessary being the rotation of one of the sets of parallel lines, which represent displacements, through a certain angle, since, in the case with which we are dealing, the two components are not at right angles. By sufficiently increasing the incidence-angle the components AB and BC can, however, be brought to very nearly a right angle. At the same time the intensity of the component from the glass surface has increased to such a degree as to be about equal to the one from the metal, and we have practically circular polarization along certain lines.

A careful study of these fringes makes a most excellent exercise for advanced students in Optics. The conditions under which they are obtained are very simple, and I consider their study far more instructive than the mere inspection of a collection of crystal sections in a polariscope arranged for convergent light.

Colours of Frilled Transparent Films on Metallic Surfaces.

We will now consider a remarkable case of interference which appears to be essentially different from any of the cases which have been previously studied. The theory of thin films shows, as Lord Rayleigh points out in his article on "Wave-theory of Light," that a transparent film on a perfectly reflecting surface shows no interference-colours. As I have already pointed out, a thin film of collodion deposited on a bright surface of silver shows brilliant colours in reflected light. It moreover *scatters* light of a colour complementary to the colour of the directly reflected light. This I find is due to the fact that the collodion film "frills," the mesh, however, being so small that it can only be detected with the highest powers of the microscope. Commercial ether and collodion should be used. If chemically pure ether obtained by distillation is used, the film does not frill, and no trace of colour is exhibited.

In the cases of the transparent films with the first surface lightly silvered, the second heavily coated, the waves absent in the reflected light are absorbed by the metal, as I have already shown. In the present case these waves are scattered by the granular surface. If a spot on the film which appears purple by reflected light is illuminated with sunlight, it will be found that green light is scattered, not in all directions, but through a range corresponding to the size of the granulation, as in the case of mixed plates.

If the light is incident normally, the scattered light comes off through an angular range included between ten and thirty degrees, and again at an angle of nearly 90°, the latter being strongly polarized. Conversely, if the sunlight be incident at nearly 90°, strongly polarized light is scattered normally. Considerable difficulty has been found in explaining these colours satisfactorily. They appear to be saturated, *i. e.* certain wave-lengths are completely absent in the reflected light, and until the granulation was detected with the microscope it was impossible to make even a satisfactory hypothesis. Even now the polarization effects are difficult to account for.

At first sight it may seem as if the colours could be classed with the phenomena of mixed plates, their brilliancy and saturation reminding one of the appearances produced by laminary retardation. The films, however, show no colour by transmitted light when deposited on glass, and the effective doubling of the retardation, by the reflexion back through the film by the metal surface, can hardly account for the observed effects. Moreover, the energy stream reflected from the surface of the collodion appears to be essential, for if we employ light polarized perpendicular to the plane of incidence, and set the plate at the polarizing angle of collodion, so that no reflexion occurs except at the metal surface, all trace of colour disappears. If the angle of incidence is larger than the polarizing angle, the colour of the reflected light changes to its complementary tint when the plane of polarization is made parallel to the plane of incidence. As I have shown in the preceding part of this paper, the effects at large angles of incidence involve the interference of two streams of light polarized in planes inclined at 90° to each other, which are complicated enough with monochromatic light and structureless films. For the present only normal incidence will be considered. Though I know of no direct way of proving that, in this case, the light reflected from the collodion surface is an essential factor, there is strong indirect evidence.

If the film is wedge-shaped and sodium light is employed, the dark fringes seen at normal incidence move towards the thick edge of the wedge as the angle of incidence is increased, exactly as they do with thin films of the ordinary type. If the incident light is polarized perpendicularly to the plane of incidence, the fringes gradually fade out, disappearing at the polarizing angle. This indicates that they are produced in the same way at normal incidence as at the polarizing angle, namely, through the agency of light reflected from the surface of the collodion.

If we consider some value of λ, for which the path-difference between the rays reflected from the collodion and metal surfaces amounts to an odd number of half-waves, the colour corresponding to this wave-length will be weakened in the reflected beam owing to interference. In the case of transparent thin films the absent colour appears in excess in the transmitted light, while in the present case it is thrown back through the film by the metal surface. It is thus clear that the colours which are weakened in the reflected light are made to traverse the frilled film a greater number of times than the colours for which the path-difference is an even number of half-waves.

This accounts for the fact that these colours are more strongly scattered by the granulations of the films.

A collodion surface only reflects about 5 per cent. of the incident energy; and it was found impossible to account for the strong colours seen in the reflected light, by compounding the feeble stream of light from the collodion with the powerful stream coming from the metal.

It appeared, however, that the observed effects could be accounted for, if the somewhat arbitrary assumption were made that the granulated surface reflected more strongly than a smooth surface. As I have already said, the granulations are too small to interfere with the regular reflexion of light, the scattering being selective, so to speak, *i. e.* confined to the waves which, owing to interference, are compelled to traverse the film a number of times.

The assumption above referred to appeared to be too arbitrary to make without some experimental evidence, and experiments were therefore made to determine the effect of the "frilling" of the film on its reflecting power. One of the faces of a 60° prism of crown glass was flowed with collodion of the same dilution as that used in the preparation of the coloured films. It showed in reflected light interference-colours, which, however, were very much diluted with white light, owing to the small difference between the refractive indices of the two media. In working with the film on silver it was found that, if the colours did not appear at once, as soon as the film dried, they could be brought out by breathing on the film, the deposit of moisture being advantageous to the formation of the granulations. It was always possible to intensify the colours in this way. The film deposited on the surface of the prism was treated in this way, one half of it being screened from the deposit of moisture by a plate of glass. As soon as the moisture had evaporated, it was found that the reflecting power of the surface had been greatly increased, the film appearing almost as bright as a half-silvered surface.

The increase in the brilliancy of the reflected light was about three-fold, as was shown by covering the unfrilled portion with a sheet of thin glass, which about equalized the intensities. In other words, the frilled collodion-surface regularly reflects white light, of an intensity very nearly equal to that of light reflected from thin glass surfaces.

On examining the granular surface with polarized light, it was found that the angle of maximum polarization was in the neighbourhood of 63°, which would make its refractive index about 1·96. The polarizing angle of the smooth collodion was about 56°, the corresponding refractive index being 1·48.

An attempt was made to determine whether the granulation gave rise to elliptical polarization, the abnormal value of the refractive index suggesting the properties of the surface-films, which play such an important part in the theory of elliptical polarization. No decisive results were obtained, for though the phenomenon was found, it seemed impossible to eliminate the component reflected from the collodion-glass surface, which, as I have shown, may, by interference with the component reflected from the air-collodion surface, give rise to an elliptical vibration.

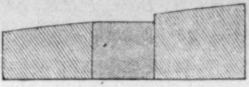
The interferometer failed to show any change in the refractive index as the result of frilling, which indicates that the effect is confined to the surface. A film deposited on glass of such thickness as to produce a shift of half a fringe width (sodium light) was frilled by moisture, one half being protected by a glass plate. No shift was found at the line of demarcation, as would have been the case if the refractive index of the film had been raised from 1·48 to 1·96 throughout its entire thickness.

It is my plan to make a further study of the apparent effect of the granulation on the refractive index of the surface, by the method of total reflexion.

DISCUSSION.

Mr. JULIUS RHEINBERG said Prof. Wood's paper had been of special interest to him, in particular as he found in it confirmation of the results afforded by some experiments he had made some time ago. He thought it was not generally known that some striking examples of the colours of mixed plates were to be found in Nature. A few years ago there had been some discussion in the Quekett Microscopical Club as to the origin of certain brilliant colours shown by the diatom *Actinocyclus Ralfsii* when seen with an objective of small numerical aperture, these colours being altogether absent when an objective of large numerical aperture was used. From an optical point of view, the diatom-valves in question might be looked upon as plates of colourless siliceous material perforated by numerous tiny holes.

Searching for the cause of the colour phenomena, it was found, on viewing the diffraction effect which the object produced in the upper focal plane of the objective when a narrow cone of light from the condenser was used, that the central dioptric beam (or, as Prof. Wood termed it, the normally diffracted beam) was *coloured* instead of white, whilst the laterally diffracted rays, instead of appearing in the usual spectrum colours, were differently tinted. The dioptric beam was always of the same colour as the diatom when viewed with the narrow-angled objective, and it occurred to him, that therefore the image formed solely by the beams diffracted laterally ought to be of the complementary colour. To test the matter a circular glass disc was made, consisting of a tiny circular disc with plane parallel surfaces, inserted in the centre of a larger disc slightly prism-shaped in section. When this was placed over the wide-angled objective, it had the effect of displacing the image formed by the rays which passed through the peripheral part of the disc, with regard to the image formed by rays passing through the central part, so that two images of each diatom were formed side by side. The one image was due to the dioptric beam only, the other to the laterally diffracted rays only. It was, in fact, just the same as if two objectives—one a narrow-angled one, the other a wide-angled one—were being used on the same object at the same time. He had at the time explained this matter as being a natural case of the colours of mixed plates,*, and was pleased to find the further confirmation of that view afforded by Prof. Wood's interesting paper.



Section of Double-Image Disc.

A number of specimens of *Actinocyclus Ralfsii* of various colours were on view under the microscope—some blue, some green, and some yellow. In all cases, the second image, formed by the laterally diffracted beams, which was seen at the side of the first one, was of the complementary tint. By closing an iris diaphragm which was fixed just over the objective, these second images could be cut out, leaving only the first ones visible. The objective used was an 8 mm. apochromatic of ·65 N.A. An auxiliary lens was arranged so that it could be brought rapidly in position over the eyepiece of the microscope; by this means, the diffraction effects produced by the object in the upper focal plane of the objective (which are repeated in the Ramsden circle of the eyepiece) could be conveniently examined alternately with the object itself.

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* *Vide*: "The Origin of Certain Colour Phenomena typically shown by *Actinocyclus Ralfsii*," Journal of the Quekett Microscopical Club, 1901, pp. 13–24, and "Note on Double-Image Discs and Complementary Interference Colours," id. 1901, pp. 151–152.

XIII. *A Quartz-Thread Vertical Force Magnetograph. By* W. Watson, *A.R.C.S., D.Sc., F.R.S., Assistant Professor of Physics at the Royal College of Science, London* *.

[Plates X. & XI.]

The late Dr. Eschenhagen was the first to show that when the moment of inertia of the suspended system in a horizontal-force magnetograph is made very much smaller than that employed in the ordinary form, it is possible to detect variations in the earth's field of quick period. Since the study of these short-period variations promises to be of considerable interest, the design of self-recording instruments capable of giving satisfactory records becomes of importance. Both Dr. Eschenhagen † and the author ‡ have described forms of apparatus suitable for recording the variations of the declination and horizontal component. The design of a satisfactory form of instrument to record the rapid variations of the vertical component presents, however, some very considerable difficulties. Thus, if the attempt is made to reduce to any great extent the moment of inertia, and hence the mass, of the balanced magnet in the ordinary form of Lloyd's balance, it is found that irregularities are immediately introduced owing to imperfections or dirt interfering with the free movements of the knife-edge. Further, the slight tremors to which the base of the instrument is almost always subject causes the knife-edge to slip about on the plane, and thus the azimuth of the magnet varies, and there is difficulty in obtaining a satisfactory record.

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* Read February 26, 1904.

† 'Terrestrial Magnetism,' v. p. 59 (1900).

‡ Ibid. vi. p. 187 (1901).

The above considerations led the writer, about two years ago, to try various other forms of Lloyd's balance; and the following paper contains a description of the instrument finally arrived at, and which has been actually run for some considerable time.

In principle the instrument resembles the quartz-thread gravity-balance designed by Prof. Threlfall [*]. In addition to the advantages to be expected from this form of balance due to the suppression of the knife-edge, it promised to allow of a very simple method of compensating for the effects of changes of temperature. This is of considerable importance, for many vertical-force magnetographs might almost be used more efficiently as thermographs than as instruments for recording changes in the vertical component.

The principle of the instrument is to have a magnet NS (fig. 5, Pl. X.) suspended on a horizontal quartz fibre AB, the fibre being kept stretched by means of a spring. The centre of gravity of the magnet and the torsion of the fibre are so adjusted that the axis of the magnet is horizontal. In order to see how the temperature compensation can be effected, let us suppose that the axis of the supporting fibre passes through the centre of gravity of the magnet, so that in the northern hemisphere the fibre has to be twisted, say by turning the end B, in the clockwise direction as seen when looking along the fibre from B towards A. If now the temperature rises two effects will be produced. In the first place, the magnetic moment of the magnet will decrease, and hence the couple acting on the magnet, due to the vertical component of the earth's magnetic field, will decrease. The result will be that the north end of the magnet will rise. Secondly, owing to the fact that the torsional rigidity of fused silica increases with rise of temperature, the couple due to the torsion of the fibre will increase. On this account also the north end of the magnet will rise. Next let us adjust the balance of the magnet so that the centre of gravity lies on the same side of the axis of the fibre as the south pole, the displacement being such that to make the magnet lie with its axis horizontal, the fibre has to be twisted in the anti-clockwise direction. In this case, when the temperature rises the north end of the magnet will tend to rise owing to the decrease in magnetic moment, but will tend to fall owing to the increase in the stiffness of the fibre. Thus, by suitably adjusting the horizontal displacement of the centre of gravity of the magnet, that is the initial torsion of the fibre, we can so arrange matters that the decrease in the couple due to the one effect is exactly equal to the increase due to the other; and so changes of temperature do not affect the position of the magnet. In the instrument described, the horizontal displacement of the centre of gravity is effected by moving a small weight along the magnet; and the possibility of obtaining complete compensation depends on the fact that the coefficient of increase of the rigidity of the fibre is much greater than the coefficient of linear expansion of steel*.

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[*] Trans. Royal Society, cxciii. p. 215 (1899).

The construction of the instrument will be evident from figs. 1, 2, 3, 4 (Pl. X.). The base of the instrument consists of a strong metal casting having uprights at the ends which carry the attachments for the ends of the fibre. Two uprights screwed to the middle of the base serve to support a circular plate U which carries the plano-convex lens L, used to form an image of the slit on the recording-drum. The circular disk V can turn through a small angle, its position being determined by two adjusting screws, and it carries the right-angled reflecting prism P. One end of the quartz fibre is fused to a quartz spring G, while the other end is fused to a small rod of quartz which is soldered into a small metal rod, and this rod is clamped by means of a screw to the screw H. Both the screws H and I have a key-way cut along them, so that by means of the nuts O they can be moved parallel to their axis but without rotation. These screws pass through holes in the two collars *m* and M, which themselves fit in the uprights carried by the base. The collar M carries a divided head, and is fitted with a fine adjustment Z, shown in fig. 3. Since only the portion FB of the quartz thread is twisted, the fine adjustment and divided head allow of the twist put into the thread being accurately adjusted and read off. A knowledge of the amount of twist is of assistance when adjusting the temperature compensation.

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* The simple rigidity of fused silica in the form of fibres of moderate diameter increases by ·00013 of its value for 1° C., while the coefficient of linear expansion of steel is ·000011.

The fixed mirror D is supported on an L-shaped piece of brass, which is held in place by three studs fixed to the base which pass through loose-fitting holes. The L-shaped piece is held up by small spiral springs against three nuts S, R, Q, which screw on these studs. By adjusting these three nuts the position of the spot of light reflected from the fixed mirror can be adjusted.

The suspended system consists of two magnets 8 cms. long and 1 mm. in diameter attached by means of small platinum straps, *p* (fig. 4), to two rods of fused silica which form part of the plate of fused silica C forming the mirror. These rods are bent as shown, and are fused to the ends E and F of the suspension fibre. There is also a small vertical rod of fused silica, which serves to carry a small weight *q* used when adjusting the sensitiveness. The reason why this gravity-bob is carried by a rod of fused silica is that, owing to the exceedingly small coefficient of expansion of silica, the centre of gravity, and hence the sensitiveness, is not appreciably altered by changes of temperature. Since the weight *q* cannot be moved vertically, the sensitiveness is adjusted by filing the weight till the required sensitiveness is nearly attained. The final adjustment is made by means of a very small weight *r* which can be screwed up and down. The upper surface of the mirror is platinized *. It will be noticed that the movable system consists of very few parts, while the mirror being attached to the fibres by fusion, we are in no way dependent on cements.

Since the tension put into the thread is such as to stretch the spring G through two or three millimetres, the variations in tension produced by changes in the length of the metal base are insignificant, and will certainly, over the comparatively small temperature-range likely to occur, be proportional to the first power of the temperature, and hence can be eliminated when making the temperature adjustment. A photograph of one of the suspensions is shown in fig. 6 (Pl. XI.) The adjustment for temperature-compensation is made by means of a small weight W which is clamped on to one of the magnets. The weight of the moving system is about 3 grams.

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* Watson, Phil. Mag. July 1903, p. 191; Proc. Phys. Soc. of London, xviii. p. 502 (1903).

In order to damp the vibrations of the magnet four copper plates K are placed near the poles. These plates are attached to an arrestor J, which can be raised by a cam X worked by a handle Y. When the arrestor is raised the right-hand magnet rests in two V's, while the left-hand magnet rests on a plane surface.

The instrument is closed in with a wooden case, and a thermometer passes through the upper surface of this case, the bulb being in the neighbourhood of the magnets. A small drawer is fitted at the lower part of the base which serves to contain some desiccating agent when the instrument is used in a damp situation.

An instrument of the above pattern was, through the kindness of the Director (Dr. R. T. Glazebrook), tested at the National Physical Laboratory, and was found to work satisfactorily. Owing to disturbances produced by electrical railways it was impossible to make the temperature-compensation at South Kensington; and Mr. F. E. Smith, of the National Physical Laboratory, kindly undertook to perform the adjustment after the instrument was removed to Bushey House. He performed this adjustment with such skill, that with a change of temperature of 7° C. he was unable to detect any variation in the scale-reading (1 mm. $=1{\cdot}6\gamma$) due to temperature change. When making these temperature adjustments, the trace obtained from a vertical-force magnetograph which was kept at a constant temperature was used to eliminate changes in V. I should like here to express my thanks to Mr. Smith for the trouble he took, and my appreciation of the skill with which he handled what is necessarily a very fragile and delicate instrument. In fig. 7 is given a reproduction of a portion of a trace obtained with the instrument when at Bushey. In the original 1 cm. corresponded to 3 minutes of time and 8γ* respectively. The recording-drum was at a distance of 170 cms. from the instrument.

The following hints as to making and adjusting the moveable system may be of assistance to any one attempting to make one of these instruments. A slab of fused silica is cut by means of a lapidaries' wheel, armed with diamond-dust, and roughly ground to size with carborundum on a metal plate †. Two small tags of silica are then attached to the sides of this slab by fusion with an oxyhydrogen flame. These tags serve to support the stirrups which carry the magnets, and to their ends the fibres will be fused afterwards. The surfaces of the slab and of the tags are ground plane and approximately parallel, and then one surface is ground and polished optically plane. The platinum stirrups for the magnets having been fitted, the small silica upright for the inertia-bob is fused to one of the tags. The surface of the mirror having been platinized, the magnets are fitted in place, a small quantity of fused shellac being used to prevent the stirrups slipping.

To prepare the fibres, a rod of fused silica having a diameter of about 2 mms. is taken and heated in a small oxyhydrogen flame, and then rapidly drawn out. Two stops must be arranged so as to limit the distance the hands can be separated, and hence the length of the fibre produced. With a little practice it is possible to obtain a fibre of a suitable length and diameter (0·008 to 0·010 cm.). The two portions of the original rod must be cut off about a centimetre from the end of the fibre. One of these ends is fused to one of the tags attached to the mirror. The other end is in one case fused to the quartz spring, and in the other is soldered by the process described by Threlfall ‡ to a small brass rod which fits into the clamp attached to the screw H (fig. 1).

To adjust the centre of gravity, the movable system is mounted between two uprights on a wooden board, while a small U-shaped piece of brass, with slots to take the tags attached to the mirror, supports the mirror and magnets. The rods of quartz between the fibre and the mirror are then softened by heating in a very small oxyhydrogen flame, and bent so that the centre of gravity lies a little below the line of support. During this operation the uprights of the U protect the magnets and mirror from the heat of the flame. The operation is one requiring some delicacy of touch, but is really not as difficult as one would suppose.

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* $1\gamma = {\cdot}00001$ C.G.S. unit.

† The methods of working such quartz mirrors and producing the platinum reflecting surface are described in the paper referred to previously.

‡ 'Laboratory Arts,' p. 226 (Macmillan & Co.).

With reference to the adjustment for temperature compensation, the side on which the weight will have to be placed depends on which side of the suspension-line the centre of gravity is situated, since it is almost impossible to get it exactly vertically below this line when adjusting the system. By, however, running the instrument in a situation where the changes of temperature are considerable, it will soon be seen whether it is over- or under-compensated. By making two sets of observations with the weight in two widely different positions, and noting the amount of twist put into the fibre, the position for compensation can be calculated. In this connexion the easiest way is to calculate the correct amount of twist for the fibre, put in this amount of twist, and then adjust the weight till the magnet is horizontal.

The advantages claimed for this form of vertical-force magnetograph are:—

1. The elimination of the knife-edge, and hence the absence of irregularities introduced by the presence of dust &c. on the supporting plane. Also the needle does not move in azimuth owing to tremors of the supports.

2. The number of materials and separate parts entering into the composition of the movable system is small, and, with the exception of the steel of the magnets, they are unoxidizable and unaffected by impurities in the air.

3. The method of compensating for temperature is simple, and does not involve a complicated counterpoise likely to get displaced.

4. The moment of inertia of the movable system is small although the magnetic moment is considerable.

5. The mirror cannot be distorted by its mounting, nor is there any likelihood of the position of the mirror with reference to the magnets being variable with the temperature.

A second instrument has been constructed according to this design, the cost being defrayed by the Government Grant Committee of the Royal Society; and it is proposed to set up this instrument in the new Magnetic Observatory belonging to the National Physical Laboratory.

DISCUSSION.

Mr. C. V. **BOYS** congratulated the author, and said he had had the opportunity and privilege of watching the growth of the instrument. It showed that the properties of quartz-fibres were even more perfect than anything he himself had ever claimed for them. The fact that it was possible to design and construct such an instrument in a physical laboratory showed an advance in scientific education. The theory of the instrument had been carefully worked out, and everything that had to be done practically had been done in the best possible way.

Prof. J. D. **EVERETT** expressed his interest in the paper, and said there was no novelty in using a horizontal fibre for a torsional suspension. Horizontal wires were so used in Kelvin's portable electrometer and in the gauge of an ordinary quadrant electrometer. He thought the instrument would be improved by increasing the number of needles and making them shorter.

Dr. J. A. **HARKER** asked the author for the ordinary working sensitiveness of the instrument. When working with a vertical-force magnetograph of the Eschenhagen type, he had found it had a tendency to become unstable when the sensitiveness was increased to about 3 times the Kew standard sensitiveness.

Mr. T. H. **BLAKESLEY** asked for the period of vibration of the suspended system.

Mr. J. R. **ASHWORTH** sent a letter which was read by the Chairman, stating that the temperature compensation might be effected by using drawn steel wire magnets with an incremental temperature coefficient. Magnets can now be made to have, within limits, any assigned temperature coefficient (from negative, through zero, to positive).

Dr. R. T. **Glazebrook** congratulated the author upon his success, and said that he was glad that Dr. Watson had been able to obtain assistance from the National Physical Laboratory.

Dr. **Watson** said he did not think that the magnets suggested by Mr. Ashworth would be of much advantage for vertical-force magnetographs, as there would have to be a gravity-bob in all cases in order to adjust the centre of gravity of the moving system. They might, however, be of use in a horizontal-force instrument. Replying to Dr. Harker, he said that 1 cm. on the records corresponded to 8 γ, whereas 1 cm. on the Kew records represented 50 γ. Replying to Mr. Blakesley, he said the period of vibration was large, but it was not necessarily important to have as long a period as possible. With regard to Prof. Everett's remarks, he pointed out that all metallic wires were ruled out of the question on account of elastic fatigue. He had used two magnets only because of the experimental difficulties in attaching a larger number to their support.

XIV. *On Stresses in a Magneto-static Field.* *By* GEORGE W. WALKER, *M.A., A.R.C.Sc., Fellow of Trinity College, Cambridge, Lecturer on Physics in the University of Glasgow*[*].

THE usual methods of analysis lead to a system of stress in a magneto-static field which is exactly analogous to the Maxwellian system of stress in an electrostatic field. The stresses may thus be spoken of as stresses of the electrical type.

In his work entitled 'Aberration and the Electromagnetic Field,' p. 76, Mr. G. T. Walker obtains a system of stress in a magneto-static field which differs from the stresses of electrical type in the superposition of a hydrostatic pressure

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[*] Read February 26, 1904.

$\frac{1}{8\pi}\left(\frac{\mu}{\mu_0}-1\right)H^2$. I am not concerned here with the analysis by which this result is obtained; but on page 79 *et seq.* Mr. Walker discusses an experiment due to Quincke, which he considers crucial between the two views.

The experiment was as follows:—A thermometer-shaped vessel with a capillary tube attached was filled with a solution of ferric chloride, so that the liquid stood at a certain height in the capillary tube. The bulb was placed between the poles of a strong electromagnet, and when the magnet was excited the level of the liquid in the capillary tube fell. The bulb was in the form of a disk, the direction of thickness being parallel to the lines of force.

Mr. Walker argues that his system of stress would produce a hydrostatic pressure throughout the liquid "under which it must contract, and the bulb slightly expand; whereas the stresses of electric type would only give a very small force at the surface of the liquid in the capillary tube which would make the column rise."

It appears to me that Mr. Walker's discussion takes no account of the surface forces due to stresses of electrical type which must exist at the surface separating the liquid in the bulb from the glass, and, moreover, involves the assumption that the expansion of the glass under an internal pressure is of less importance than the contraction of the liquid under the same pressure.

It is easy to show that the stresses of electrical type give a force per unit area directed outwards on the inner surface exactly equal to the hydrostatic pressure in the liquid on Mr. Walker's theory.

For assuming that the glass is non-magnetic and that the field is uniform in the liquid and outside, let H_1 be the magnetic force in the glass, H_2 the magnetic force in the liquid, then

$$H_1=\frac{\mu}{\mu_0}H_2,$$

where $\frac{\mu}{\mu_0}$ is the permeability of the liquid. Then the force per unit area directed outwards is

$$\frac{1}{8\pi}\left(H_1^2-\frac{\mu}{\mu_0}H_2^2\right),$$

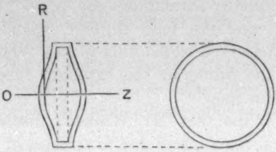
that is $\frac{1}{8\pi}\frac{\mu_0}{\mu}\left(\frac{\mu}{\mu_0}-1\right)H_1^2$, or $\frac{1}{8\pi}\frac{\mu}{\mu_0}\left(\frac{\mu}{\mu_0}-1\right)H_2^2$;

and since $\frac{\mu}{\mu_0}$ is very nearly unity this is practically

$$\frac{1}{8\pi}\left(\frac{\mu}{\mu_0}-1\right)H_2^2.$$

My contention is that the expansion of the glass vessel under this force (of electrical type) is more important than the contraction of the liquid under the equal hydrostatic pressure on Mr. Walker's system.

Under this internal force per unit area which we may call p_0, the walls will bulge outwards, and we may consider them as thin elastic plates clamped at the edge.



Taking O as origin, the equation for the displacement ω parallel to OZ is

$$\nabla^4\omega=\frac{3}{2}\cdot\frac{p_0(1-\mu^2)}{qh^3}\,*,$$

where μ is Poisson's ratio,
q is Young's modulus,
and h is the thickness of the plate;
assuming symmetry about OZ.

The solution is

$$\omega=\frac{3}{2}\frac{p_0(1-\mu^2)}{qh^3}\left\{\frac{1}{32}a^2r^2-\frac{1}{64}r^4\right\},$$

where r is the distance from O and a is the radius of the disk.

Thus the increase of volume due to bulging is

$$4\pi\int_0^a\omega r\,dr$$
$$=\frac{\pi}{32}a^6p_0\,\frac{(1-\mu^2)}{q\times h^3}.$$

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* Rayleigh, 'Theory of Sound,' vol. i. p. 356.

The radius of the bulb was 1·75 cm. and the length ·4 cm. I do not suppose the walls were more than ½ mm. thick, so that we have

$$a = 1{\cdot}75 \text{ cm.}, \qquad l = {\cdot}4 \text{ cm.}, \qquad 2h = {\cdot}05 \text{ cm.};$$

$$\therefore\ a = 4{\cdot}4l, \qquad h = \frac{a}{70}.$$

For glass we may take $q = 6 \times 10^{11}$ and $\mu = \frac{1}{4}$.
Thus the increase in volume of the bulb is

$$\delta v = \pi a^2 l p_0 \times \frac{4{\cdot}4 \times (70)^3}{32 \times 6 \times 10^{11}}$$

$$= \pi a^2 l p_0 \times 7800 \times 10^{-11}.$$

The compressibility of water is about 4×10^{-11}, so that the diminution of volume of the fluid under a hydrostatic pressure is

$$\delta v_0 = \pi a^2 l p_0 \times 4 \times 10^{-11}.$$

The expansion of the glass vessel under the stresses of electrical type is far more important than the contraction of the fluid under the hydrostatic pressure of Mr. Walker's theory. I therefore think that, so far from being crucial, Quincke's experiment gives no support to the latter theory, and can be readily explained by the stresses of electrical type.

Physical Laboratory, Glasgow.
December 12, 1903.

XV. *The Whirling and Transverse Vibrations of Rotating Shafts.* *By* C. CHREE, *Sc.D., LL.D., F.R.S.*[*]

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26–27. CASE 4. Shaft "supported" at one end, and with its direction fixed at the other end.
28–32. CASE 5. Shaft "supported" at both ends and at intermediate point.
33–35. CASE 6. Shaft fixed in direction at both ends.
36–39. General Conclusions.
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Preliminary Discussion.

§ 1. LATERAL vibrations are those executed by a bar when bent to one side and then released. They are connected with the "whirling" of rotating shafts, but in a way which has not, I think, hitherto been clearly recognized. The subject of "whirling" has been treated by Professor Greenhill[†] in a well-known paper dealing with an unloaded shaft rotating under various terminal conditions. More recently the subject has been treated in an elaborate and important paper by Prof. Dunkerley[‡]. He employed two ways of calculating the critical speeds of rotation. The first makes use of the ordinary elastic solid equations applicable to thin rods acted on by "centrifugal force"; this is the method followed by Greenhill. On attempting to apply this method to loaded shafts, Dunkerley reached results which he considered hopelessly complicated. In his second method, which is due apparently to Prof. Osborne Reynolds, Dunkerley calculated a critical speed for the loaded shaft, in which the mass of the shaft itself was neglected. Calling the frequency thus obtained N_2, and that found for the unloaded shaft by the first method N_1, he deduced a final value N for the frequency from the equation

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[*] Read March 11, 1904.

[†] Proc. Institution of Mechanical Engineers, 1883, p. 182.

[‡] Phil. Trans. A, 1894, p. 279.

$$1/N^2=1/N_1^2+1/N_2^2. \quad . \quad . \quad . \quad . \quad (1)$$

By speed or frequency Dunkerley means the number of revolutions per *minute*, i. e.

$$N=30\omega/\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ω is the angle through which the shaft rotates in one second.

When more than one load exists, Dunkerley calculated critical speeds N_2, N_3, ... for each separately, and deduced a critical speed for the whole system from the formula

$$1/N^2=1/N_1^2+1/N_2^2+1/N_3^2+\ldots \quad . \quad . \quad . \quad . \quad (3)$$

He made a number of experiments with a miniature shaft, loaded with one or both of two pulleys, and in many cases the speeds at which whirling commenced agreed remarkably with those calculated. In some cases better agreement was obtained with a formula of the type

$$1/N^2=1/N_2^2+a/N_1^2,$$

in which a is an experimental constant.

Dunkerley (*l. c.* p. 281) noticed that there was a connexion between the speed of whirling and the frequency of the lateral vibrations of the shaft when not rotating; but I see no indication in his paper that he had grasped the real nature of the connexion.

Dunkerley's method (*l. c.* p. 358) of arriving at his working equation (3)—which he characterizes himself as "empirical"—is not convincing mathematically. In fact, the result does not appear to be in general strictly true; and there seems nothing in Dunkerley's work which serves to bring out the limitations. As we shall see later in particular cases, the agreement between theory and observation is not by itself sufficient evidence of the general applicability of formula (1). Again, the applications by Dunkerley of the Euler-Bernoulli elastic theory are cumbrous mathematically, and the formulæ to which they lead, and which were employed by Dunkerley, often admit of great simplification, without appreciable diminution of accuracy, under his experimental conditions. Also, as already explained, the true nature of the relationship to lateral vibrations is not brought out.

For these several reasons I have thought it worth while to examine the whole question afresh from a variety of points of view, making liberal use, however, of Dunkerley's experimental results, and referring to his formulæ in the several cases, so that the present work is in many respects supplementary to his. To go fully into the mathematical investigations in each case would occupy an undue amount of space, and as the same methods are employed in the different cases, I have deemed it sufficient to give one or two illustrations in the Appendix at the end of the paper.

§ 2. It is unnecessary to describe the method employed by Greenhill and Dunkerley for the unloaded shaft, as it is simply the approximate Bernoulli-Eulerian method described in mathematical textbooks treating of "thin" rods. Dunkerley's second method, in so far as it relates to finding the critical speed for a loaded but massless shaft, is really analogous to a method illustrated by Lord Rayleigh * in obtaining approximate frequencies of vibration. Both assume the displacement of the bar to be of a simple algebraic type; but Dunkerley applies ordinary statical equations, whilst Rayleigh applies dynamical equations deduced by means of the principle of energy. Rayleigh, however, advances in justification of his method a result based on very general reasoning, viz. that a considerable departure from the true type of vibration leads to only a small error in the estimate of the frequency.

I am not prepared to say that Rayleigh's general theory is impervious to criticism. A general theorem may pass muster even with acute critics, simply from failing to suggest points of view which decline to be left out of account in actual practice. Again, a theorem may be practically satisfactory within certain limits, and yet those limits may be so difficult to recognize that applications may be fraught with peril to any but one of the very few men who combine profound physical insight with first-rate mathematical ability. Still, taking all these things into account, I think it will be generally recognized that, in view of the empirical nature of Dunkerley's

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* 'Theory of Sound,' vol. i. Arts. 182, 183, &c.

second method, the application of Rayleigh's method to the problem of whirling is, if practicable, highly desirable. Numerous applications of it will be made here, and there is an illustration of the mathematical details in the Appendix.

§ 3. Before treating individual cases, I shall describe in unmathematical language the true nature of the connexion between lateral vibrations and the phenomenon of whirling. Ordinarily, when a shaft held at one or both ends is acted on by forces tending to bend it, on the removal of these forces it tends to return to its original straight position; in doing so it overshoots the mark and vibrates to and fro laterally. The velocity of its approach to the equilibrium position, and the frequency of the vibrations subsequently executed, are greater the larger the elastic stresses produced in the bar by a given lateral displacement. When the bar is rotating round its longitudinal axis, and is displaced laterally, the elastic stresses tend as before to bring it back to the undisturbed position; but the "centrifugal forces" have exactly the opposite tendency; they thus reduce the righting forces, and so diminish the frequency of vibration. If we take the simplest case where there are no complications from the mass of the shaft itself, and where only the mass (not the moment of inertia) of the load requires to be taken into account, it may be shown that if $k/2\pi$ be the frequency of the vibrations which the shaft executes when displaced laterally at a time when it is rotating with uniform angular velocity ω, and $K/2\pi$ be the corresponding frequency in the absence of rotation, then

$$k^2 = K^2 - \omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As ω is increased, the frequency of vibration and so the stability of the bar diminish, until eventually when

$$\omega = K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the frequency becomes *nil*, i. e. the period becomes infinite, or the righting power vanishes. In fact, the position is similar to that of a ship whose C.G. has come to coincide with the metacentre. The case is not one in which forced vibrations are set up with a frequency equal to one of the natural periods. What leads to whirling is the indirect action of the rotation in reducing to zero the righting forces which naturally act on the shaft when displaced laterally. The case of a loaded but massless shaft is of course an extreme one ; but all the other cases which I have examined present similar features. The case selected for mathematical treatment in the Appendix is that of a shaft supported at both ends ; this admits of a variety of sub-cases, illustrative of various points.

[*March* 15.—To prevent misconception, it seems desirable to state explicitly and prove—as was done when the paper was read—that the formula of § 3,

$$k^2 + \omega^2 = K^2,$$

applies exactly to all *unloaded* shafts, to the degree of accuracy possessed by ordinary equations for thin rods. The elastic bodily equation has the following forms :

$d^4y/dx^4 = \Omega^2(\sigma\rho y/EI)$ for rotation with whirling velocity Ω,

$= K^2(\sigma\rho y/EI)$ for vibration without rotation,

$= (k^2+\omega^2)(\sigma\rho y/EI)$ for vibration when velocity of rotation is ω.

The value of μ in the typical equation

$$d^4y/dx^4 = \mu^4 y$$

depends only on the terminal conditions. Thus for any, the same, system of supports we have

$$k^2 + \omega^2 = K^2 = \Omega^2.]$$

§ 4. As stated above, whirling is not really a case of coincidence of period between a vibrating system and disturbing forces. A rotating shaft may, however,—like any other shaft—be acted on by periodic forces which tend directly to set up lateral vibrations. In considering the effect of any such forces, it must be borne in view that what one has to look to is the frequency of the lateral vibrations of the shaft *as reduced by the rotation.* The possibility of forced vibrations of this kind is an additional reason for considering the effects of rotation on the period.

§ 5. In the main I shall follow Dunkerley's classification of the principal cases of whirling shafts, but shall not number separately cases where the shaft is with, and without, a load.

The cases are determined by the number and nature of the supports.

If x be taken parallel, and y perpendicular, to the undisturbed position of the axis of the shaft, the bending being supposed to occur in the plane xy, clearly at any support

$$y=0.$$

If the end of a shaft simply rests on a support, then on the Euler-Bernoulli theory, as the terminal section must be free from a couple,

$$d^2y/dx^2=0.$$

At such an end the shaft is said to be "supported." If, on the other hand, the shaft be constrained to retain a fixed direction at an end, the second terminal condition is

$$dy/dx=0.$$

If a shaft is "supported" at any intermediate point, then clearly y must vanish there, while dy/dx and d^2y/dx^2 must be continuous. A sudden change of dy/dx would imply fracture, while a sudden change of d^2y/dx^2 would imply the action of a couple at the supported section.

When an end is quite free, resting on no support, both stress and couple vanish, and so

$$d^2y/dx^2=d^3y/dx^3=0.$$

Notation used.

$\mathrm{E}=$ Young's modulus for shaft, assumed homogeneous and isotropic.

$\rho=$ density of material of shaft, supposed uniform.

$\sigma=$ cross section (and so $\sigma\rho=$ mass per unit length).

$\mathrm{M}=$ mass of load, when there is one.

$\mathrm{I}=$ moment of inertia of σ about diameter perpendicular to plane of bending.

$\mathrm{I}'=$ moment of inertia of M about an axis through its C.G. perpendicular to plane of bending.

$\omega=$ angular velocity of rotation.

$k/2\pi=$ frequency of lateral vibration, taking rotation into account.

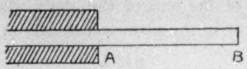
$\mathrm{T}=$ *variable* part of kinetic energy of system.

$\mathrm{V}=$,, ,, potential ,, ,,

$\mu=\omega^{\frac{1}{2}}(\rho\sigma/\mathrm{EI})^{\frac{1}{4}}.$

CASE 1 (Dunkerley's Cases I., VII., and IX.).

§ 6. Overhanging shaft fixed in direction at end A. Length $AB = l$.



The terminal conditions are :

At A (or $x=0$), $y = dy/dx = 0$;
,, B (or $x=l$), $d^2y/dx^2 = d^3y/dx^3 = 0$.

(*a*) Unloaded bar : Euler-Bernoulli solution (*cf.* Dunkerley, *l. c.* p. 288).

The displacement is of the type

$$y = \alpha\{(\cosh \mu x - \cos \mu x)(\sinh \mu l + \sin \mu l) - (\sinh \mu x - \sin \mu x)(\cosh \mu l + \cos \mu l)\}, \quad . \; . \quad (1)$$

where α is a constant.

The equation for μ—and so for ω—(see Dunkerley's equation (A) p. 289) is

$$1 + \cosh \mu l \cos \mu l = 0. \quad . \; . \; . \; . \; . \; . \quad (2)$$

The smallest root (see Rayleigh's 'Sound,' art. 174) is

$$\mu l = 1{\cdot}8751.$$

But by definition of μ,

$$\omega^2 = (EI/\sigma\rho l^4)(\mu l)^4,$$

i. e.
$$\omega^2 = 12{\cdot}36(EI/\sigma\rho l^4), \quad . \; . \; . \; . \; . \; . \quad (3)$$

$$\text{or } \omega = 3{\cdot}516\ (EI/\sigma\rho l^4)^{\frac{1}{2}}. \quad . \; . \; . \; . \; . \; . \quad (4)$$

Usually I shall record only the value of ω^2, as often more convenient than that of ω. In general ω^2 will equal $(EI/\sigma\rho l^4)$ multiplied by some numerical quantity.

The above method leads to a somewhat complicated expression for the displacement, and throws no direct light on the relationship of whirling to lateral vibrations.

§ 7. (*b*) Supposing the bar still unloaded, replace the Euler-Bernoulli expression (1) by the much simpler one

$$y = \eta(x^4 - 4\,lx^3 + 6\,l^2x^2). \quad . \; . \; . \; . \; . \; . \quad (5)$$

If for η we substitute $g\rho\sigma/24EI$, we have the displacement which the shaft would experience if bending under its own weight. For the present purpose we assume

$$\eta \propto \cos kt,$$

where $k/2\pi$ is the frequency of the lateral vibrations.

Neglecting the inertia of the motion of cross-sections relative to their centres of gravity, we find from (5)

$$\left. \begin{aligned} \mathrm{T} &= \tfrac{1}{2}(\dot{\eta}^2 + \omega^2\eta^2)\,\frac{104}{45}\,\sigma\rho l^9, \\ \mathrm{V} &= \tfrac{1}{2}\eta^2\,\frac{144}{5}\,\mathrm{E}\mathrm{I}l^5\,; \end{aligned} \right\} \quad . \;\; . \;\; . \;\; . \;\; (6)$$

whence, by Lagrange's equation,

$$k^2 + \omega^2 = 12{\cdot}46\,(\mathrm{EI}/\sigma\rho l^4). \quad . \;\; . \;\; . \;\; . \;\; (7)$$

This gives the frequency of vibration for any assigned value of ω.

The critical angular velocity answering to whirling is that for which k vanishes, or the motion becomes unstable; it is thus given by

$$\omega^2 = 12{\cdot}46(\mathrm{EI}/\sigma\rho l^4). \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; (8)$$

The values given by (3) and (7) for the critical value of ω differ by less than $\frac{1}{2}$ per cent.

§ 8. (*c*) Mass M, inertia I′, at end of massless shaft.

Assume (*cf.* Rayleigh's 'Sound,' art. 183)

$$y = (3z - l\theta)(x/l)^2 + (l\theta - 2z)(x/l)^3, \quad . \;\; . \;\; . \;\; (9)$$

where z and θ are the values of y and dy/dx (which may be regarded as the inclination of the shaft to its undisturbed direction) at the point of attachment of the load.

By Lagrange's equations, or otherwise, we obtain for the frequency

$$\{(k^2+\omega^2)\mathrm{M} - 12\mathrm{EI}/l^3\}\{(k^2-\omega^2)\mathrm{I}' - 4\mathrm{EI}/l\} = 36(\mathrm{EI}/l^2)^2. \quad (10)$$

For any assigned value of ω, (10) gives two values of k^2, answering to two different types of vibration. Only one of these—which answers normally to the smaller value of k^2—is properly speaking of the lateral type.

For the critical angular velocity answering to whirling, we put $k=0$ and obtain a quadratic equation for ω^2, identical with that obtained otherwise by Dunkerley (*l. c.* p. 304).

One of these values of ω^2 is negative, and has no application to the present problem.

If, as in Dunkerley's experiments, I′ has but little effect, a first approximation to the desired value of ω^2—obtained by omitting I′ altogether—is

$$\omega^2 = 3\mathrm{EI}/\mathrm{M}l^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As a second approximation, neglecting $(\mathrm{I}')^2$ we find

$$\omega^2 = (3\mathrm{EI}/\mathrm{M}l^3)(1 + 9\mathrm{I}'/4\mathrm{M}l^2). \quad . \quad . \quad . \quad (12)$$

§ 9. (*d*) If we know, to start with, that the effect of I′ is small, we can simplify the work by taking in place of (9)

$$y = \eta(3lx^2 - x^3)/2l^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If we substituted $\mathrm{M}gl^3/3\mathrm{EI}$ for η we should have the displacement produced in the shaft by a weight M*g* at the end.

Assuming η in (13) proportional to cos *kt*, and still neglecting the mass of the shaft, we have

$$\left.\begin{array}{l} \mathrm{T} = \frac{1}{2}\mathrm{M}(\dot{\eta}^2 + \omega^2\eta^2) + \frac{1}{2}\mathrm{I}'(\dot{\eta}^2 - \omega^2\eta^2)(9/4l^2), \\ \mathrm{V} = \frac{1}{2}\mathrm{EI} \,.\, 3\eta^2/l^3 \end{array}\right\} \quad . \quad . \quad (14)$$

Thence we have for the frequency equation

$$(k^2 + \omega^2)\mathrm{M} + (k^2 - \omega^2)(9\mathrm{I}'/4l^2) = 3\mathrm{EI}/l^3, \quad . \quad . \quad (15)$$

and for the critical angular velocity

$$\omega^2 = (3\mathrm{EI}/\mathrm{M}l^3)(1 - 9\mathrm{I}'/4\mathrm{M}l^2)^{-1}. \quad . \quad . \quad . \quad (16)$$

Omitting I′ altogether we deduce (11); while retaining I′, but omitting $(\mathrm{I}')^2$, we have (12).

§ 10. (*e*) Loaded shaft of appreciable mass.

Dunkerley's hypothesis (see § 1) supplies as the equation for the critical angular velocity

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where ω_1^2 and ω_2^2 are given by (3) and (10) respectively. Supposing I′ so small that (12) is applicable, we deduce

$$\frac{1}{\omega^2} = \frac{\sigma\rho l^4}{12{\cdot}36\mathrm{EI}} + \frac{\mathrm{M}l^3}{3\mathrm{EI}} - \frac{3}{4}\frac{\mathrm{I}'l}{\mathrm{EI}}. \quad . \quad . \quad . \quad (18)$$

§ 11. (*f*) Instead of assuming the truth of (17) we may, following Rayleigh, assume (13) as the type of displacement, no longer neglecting the mass of the shaft. This adds to the value of T in (14) the term

$$\tfrac{1}{2}(\dot{\eta}^2 + \omega^2\eta^2)\,\frac{33}{140}\sigma\rho l,$$

and so leads to the frequency equation

$$(k^2+\omega^2)\left(M+\frac{33}{140}\sigma\rho l\right)+(k^2-\omega^2)9I'/4l^2=3EI/l^3. \quad (19)$$

For the critical angular velocity, noticing that $140/11=12{\cdot}73$, we have

$$\frac{1}{\omega^2}=\frac{\sigma\rho l^4}{12{\cdot}73EI}+\frac{Ml^3}{3EI}-\frac{3}{4}\frac{I'l}{EI}, \quad . \quad . \quad . \quad (20)$$

which is certainly in close agreement with the result (18) obtained by Dunkerley's hypothesis.

(*g*) If instead of (13) we assume the type (5), but regard I' as small, we obtain in place of (20)

$$\frac{1}{\omega^2}=\frac{\sigma\rho}{12{\cdot}46}\frac{l^4}{EI}+\frac{Ml^3}{3{\cdot}2EI}-\frac{5}{9}\frac{I'l}{EI}. \quad . \quad . \quad . \quad (21)$$

CASE 2 (Dunkerley's Cases II., VIII., and X.).
Shaft *supported* at both ends.

§ 12. At each end we have

$$y=d^2y/dx^2=0.$$

(*a*) Unloaded shaft: Euler-Bernoulli solution.

The displacement is of the type

$$y=\alpha\sin\mu x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α is a constant, and the equation for μ is

$$\mu l=\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where l is the total length of the shaft.

From this we have

$$\omega^2=\pi^4(EI/\sigma\rho l^4)=97{\cdot}41(EI/\sigma\rho l^4). \quad . \quad . \quad . \quad (3)$$

(*b*) Instead of the Euler-Bernoulli method for the unloaded shaft, we may assume

$$y=\eta x(l^3-2lx^2+x^3). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If η were replaced by $g\sigma\rho/24EI$ this would represent the bending of the shaft under its own weight.

Assuming $\eta\propto\cos kt$, we have

$$\left.\begin{aligned} T&=\tfrac{1}{2}(\dot\eta^2+\omega^2\eta^2)\frac{31}{630}\sigma\rho l^7, \\ V&=\tfrac{1}{2}\eta^2(24/5)EIl^5 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (5)$$

whence we have for the frequency equation

$$k^2+\omega^2=(3024/31)(\mathrm{EI}/\sigma\rho l^4), \quad . \quad . \quad . \quad . \quad (6)$$

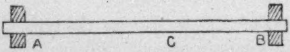
and for the critical velocity

$$\omega^2=97{\cdot}55\mathrm{EI}/\sigma\rho l^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which presents an exceedingly close agreement with (3).

§ 13. (*c*) Mass (M, I′) on massless shaft at a point C

(AC$=a$, BC$=b$, $a+b=l$).



Assuming Rayleigh type formulæ, and employing Lagrange's equations (see Appendix, § 41) we find for the frequency equation

$$\{\mathrm{M}(k^2+\omega^2)-3\mathrm{EI}(a^{-3}+b^{-3})\}\{\mathrm{I}'(k^2-\omega^2)-3\mathrm{EI}(a^{-1}+b^{-1})\}$$
$$=9(\mathrm{EI})^2(b^{-2}-a^{-2})^2. \quad (8)$$

From this, putting $k=0$, we obtain a quadratic equation for the critical angular velocity, identical with that found otherwise by Dunkerley (*l. c.* p. 307).

If in (8) we absolutely neglect I′, we have the simple result

$$k^2+\omega^2=3\mathrm{EI}l/\mathrm{M}a^2b^2\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whilst, retaining I′ but neglecting I′², we have for the critical angular velocity

$$\omega^2=(3\mathrm{EI}l/\mathrm{M}a^2b^2)\left\{1+\frac{\mathrm{I}'}{\mathrm{M}}\left(\frac{a-b}{ab}\right)^2\right\}. \quad . \quad (10)$$

(*d*) If in the last sub-case we start with the assumption that I′ is small, we may take

$$\left.\begin{array}{l}\text{for AC, } y\ =\eta bx\,(l^2-b^2-x^2), \\ \text{for BC, } y'=\eta ax'\,(l^2-a^2-x'^2)\end{array}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

If η were replaced by $g\mathrm{M}/6\mathrm{EI}l$ this would represent the bending of the bar under the weight of M.

Assuming $\eta \propto \cos kt$ we find by Lagrange's equations

$$(k^2+\omega^2)\mathrm{M}a^2b^2+(k^2-\omega^2)\mathrm{I}'(a-b)^2=3\mathrm{EI}l. \quad . \quad (12)$$

For the critical angular velocity, putting $k=0$, we have

$$\omega^2=(3\mathrm{EI}l/\mathrm{M}a^2b^2)\{1-(\mathrm{I}'/\mathrm{M})(a-b)^2/a^2b^2\}^{-1}. \quad . \quad (13)$$

When $(\mathrm{I}'/\mathrm{M})^2$ is neglected this is identical with (10).

§ 14. (*e*) **Loaded shaft of appreciable mass.**

On Dunkerley's hypothesis we have for the critical velocity

$$1/\omega^2 = 1/\omega_1^2 + 1/\omega_2^2,$$

where ω_1^2 is given by (3), and ω_2^2 by (8) with k put $=0$. Thus when I' is so small that (10) is applicable, we have

$$\frac{1}{\omega^2} = \frac{\sigma\rho l^4}{97{\cdot}41\mathrm{EI}} + \frac{\mathrm{M}a^2b^2}{3\mathrm{EI}l} - \frac{\mathrm{I}'(a-b)^2}{3\mathrm{EI}l} \quad . \quad . \quad . \qquad (14)$$

(*f*) If we employ the same Rayleigh formulæ as in (*c*), but do not neglect the mass of the shaft, we replace (8) by

$$\left\{(k^2+\omega^2)\left(\mathrm{M}+\frac{17}{35}\sigma\rho l\right)-3\mathrm{EI}(a^{-3}+b^{-3})\right\} \times$$
$$\left\{(k^2-\omega^2)\mathrm{I}' + (k^2+\omega^2)\frac{2}{105}\sigma\rho(a^3+b^3)-3\mathrm{EI}(a^{-1}+b^{-1})\right\}$$
$$= (a^2-b^2)^2\left\{3\mathrm{EI}a^{-2}b^{-2}+\frac{3}{35}(k^2+\omega^2)\sigma\rho\right\}^2 \quad . \quad . \quad . \quad . \qquad (15)$$

This may be expected to give best results when the load is at or near the centre of the span, as the assumed type of displacement, which answers to the load only, is then nearest to that natural to a massive but unloaded shaft.

If the load is at the exact centre (15) gives

$$k^2+\omega^2 = 48\mathrm{EI} \div \left\{\mathrm{M}l^3 + \frac{17}{35}\sigma\rho l^4\right\} . \quad . \quad . \quad . \qquad (16)$$

When M is neglected this gives

$$k^2+\omega^2 = 98{\cdot}8\mathrm{EI}/\sigma\rho l^4 . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Putting $k=0$ in (16) and (17) we have of course the corresponding critical angular velocities. The value obtained from (17) is a very fair approximation for the case of an unloaded shaft (*cf.* (3)).

§ 15. (*g*) When I' is small, but the load M is not near the centre of the span, and is of the same order as the mass m of the shaft, better results are obtained from the following displacement types :—

$$\left.\begin{aligned} &\text{for AC, } y = \eta x\{m(l-x)(l^2+lx-x^2)+4\mathrm{M}b(l^2-b^2-x^2)\}, \\ &\text{for BC, } y' = \eta x'\{m(l-x')(l^2+lx'-x'^2)+4\mathrm{M}a(l^2-a^2-x'^2)\}. \end{aligned}\right\} \; (18)$$

If η were replaced by $g/24\mathrm{EI}l$ this would give the bending of the shaft under its own weight and that of the load combined. Assuming $\eta \propto \cos kt$ we obtain the frequency equation

$$(k^2+\omega^2)\mathrm{A}+(k^2-\omega^2)\mathrm{I}'\mathrm{B}=\mathrm{C}, \quad . \quad . \quad . \quad (19)$$

giving for the critical angular velocity

$$\omega^2=\mathrm{C}\div(\mathrm{A}-\mathrm{I}'\mathrm{B})\,; \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where for brevity

$$\left.\begin{aligned}
\mathrm{A}=1&+\frac{18}{31}\frac{\mathrm{M}}{m}\left(\frac{ab}{l^2}\right)\left\{17+52\frac{ab}{l^2}+76\left(\frac{ab}{l^2}\right)^2+36\left(\frac{ab}{l^2}\right)^3\right\}\\
&+\frac{96}{31}\left(\frac{\mathrm{M}}{m}\right)^2\left(\frac{ab}{l^2}\right)^2\left\{8+121\frac{ab}{l^2}+117\left(\frac{ab}{l^2}\right)^2\right\}+\frac{40320}{31}\left(\frac{\mathrm{M}}{m}\right)^3\left(\frac{ab}{l^2}\right)^4,\\
\mathrm{B}=&\,\frac{630}{31}\frac{(a-b)^2}{ml^4}\left(1+2\frac{ab}{l^2}+\frac{8\mathrm{M}}{m}\frac{ab}{l^2}\right),\\
\mathrm{C}=&\,\frac{3024}{31}\frac{\mathrm{EI}}{ml^3}\left\{1+10\frac{\mathrm{M}ab}{ml^2}\left(1+\frac{ab}{l^2}\right)+40\left(\frac{\mathrm{M}}{m}\right)^2\left(\frac{ab}{l^2}\right)^2\right\}.
\end{aligned}\right\}\ (21)$$

§ 16. Prof. Dunkerley carried out a number of experiments under the conditions of Case 2. He employed a shaft unloaded, or loaded with one or other of two pulleys of different sizes. As a preliminary to comparing his theory with observation, he had to calculate critical speeds for the pulleys alone, neglecting the mass of the shaft. Table I. compares the number of revolutions per minute which he calculated from a formula equivalent to (8), with k omitted, with the corresponding numbers which are given by the much simpler formulæ (9) and (13). The results apply of course only to the particular shaft and pulleys employed by Dunkerley.

TABLE I.—N_2 (Number of revolutions per minute for whirling).

Pulley.	Formula. $b/l=$	1/2.	1/3.	1/6.	1/11.	1/32.
I.	(8) [Dunkerley's]...	1495	1683	2705	4621	13537
	(9)	1495	1682	2690	4532	12343
	(13)	1495	...	2710	4636	(16364)
II.	(8) [Dunkerley's] ...	997	1122	1808	3116	10355
	(9)	997	1121	1794	3015	8231
	(13)	997	....	1808	3119	(13716)

§ 17. The results from the formula (13) for the smallest value of b/l are put in brackets because the assumption on which the formula is based—viz., that the contribution from

I' is small—is then far from being satisfied, so that the result is *a priori* unsatisfactory. The closeness with which the simplest formula (9) approaches to the results from the complicated formula (8) is rather surprising.

In Dunkerley's case the value given by (3) for N_1—the critical number of revolutions per minute for the shaft when unloaded—is 1122. Thus for the value 1/32 of b/l we have

$$\begin{aligned} \omega_2/\omega_1 = N_2/N_1 &= 13537/1122 \text{ for pulley I. (or 12 roughly),} \\ &= 10355/1122 \quad \text{,,} \quad \text{II. (or 9 roughly).} \end{aligned}$$

Thus the contribution from ω_2, or N_2, to the critical speed for the loaded shaft on Dunkerley's hypothesis (1) § 1 is only about one-eightieth of that from ω_1, even for the case of the heavier pulley II. Under such circumstances an error of even 100 per cent. in the value of ω_2, or N_2, would exert but little influence. It is thus clear that under the conditions of Dunkerley's experiments the simple formula (9) would for all practical purposes be as satisfactory as (8).

§ 18. Table II. compares the speeds at which Dunkerley observed whirling to commence with those which he calculated from (3) and (8) on his hypothesis (1) § 1, and with those which are given by the formulæ (15) and (20) singly.

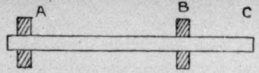
TABLE II.—Critical revolutions per minute.

	Pulley I.				Pulley II.			
bl.	Observed.	(3) & (8) Dunkerley.	(15)	(20)	Observed.	(3) & (8) Dunkerley.	(15)	(20)
1/2	921	897	901	899	769	745	747	769
1/3	952	933	951	940	803	793	...	799
1/6	1044	1036	1113	1050	942	953	...	974
1/8	1101	1069	1179	1079	1007	1013	...	1031
1/11	1123	1089	1241	1100	1046	1055	...	1074
1/32	1150	1117	1362	1121	1130	1115	...	1121

Formula (15) is derived from a type of displacement which treats the shaft as massless, so its failure to agree well with experiment when the load is near one of the ends was to be anticipated. The simple formula (14) would give results in close agreement with those calculated by Dunkerley.

CASE 3 (Dunkerley's Cases III. and XI.).

§ 19. Shaft supported at end A and at a second point B,



with BC overhanging ($AB=l$, $BC=c$). Taking B as origin of coordinates x, y for BC, and x', y' for BA, we have:

At $x=0$, $y=0$ with continuity in dy/dx and d^2y/dx^2,

„ $x=c$, $d^2y/dx^2=d^3y/dx^3=0$ (when there is no load),

„ $x'=l$, $y'=d^2y'/dx'^2=0$.

(a) Unloaded shaft: Euler-Bernoulli solution. The displacements are cumbrous to record. The equation determining μ (*cf.* Dunkerley, *l.c.* equation (A) p. 291) is

$$(\cosh \mu l \sin \mu l - \sinh \mu l \cos \mu l)(\cosh \mu c \sin \mu c - \sinh \mu c \cos \mu c) \\ -2 \sinh \mu l \sin \mu l (1+\cosh \mu c \cos \mu c)=0. \quad . \quad . \quad (1)$$

Except for special values of c/l this is somewhat intractable. On his p. 292 Dunkerley specifies 3·08 as the limiting value to which μl approaches when c/l is indefinitely reduced. This is not quite correct; the true limiting value is π, exactly as in (a) Case 2.

Treating c/l as small though not negligible, I find for the next approximation

$$\mu l=\pi(1-\tfrac{1}{6}\pi^2 c^3/l^3). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Whence, substituting its numerical value for π, we have for the critical angular velocity

$$\omega^2=97{\cdot}41(\mathrm{EI}/\sigma\rho l^4)(1-6{\cdot}6c^3/l^3). \quad . \quad . \quad . \quad (3)$$

This is satisfactory so long as $(c/l)^3(\pi^3/6)\coth\pi$ is small compared with unity.

A way of treating this problem by an assumed type of vibration will be found below in (f).

§ 20 (b) Mass (M, I′) on overhanging part of massless shaft at distance c from the nearest support.

Formulæ of the Rayleigh type are

$$\left.\begin{aligned} &\text{for BC,}\ y=\frac{3z-c\theta}{c(3c+4l)}(2lx+3x^2)+\frac{\theta c(2l+3c)-2z(l+3c)}{3c+4l}\left(\frac{x}{c}\right)^3, \\ &\text{„ BA,}\ y'=\frac{c\theta-3z}{cl(3c+4l)}(2l^2x'-3lx'^2+x'^3), \end{aligned}\right\} (4)$$

By Lagrange's equations we find for the frequency equation

$$\left\{\mathrm{M}(k^2+\omega^2)-12\mathrm{EI}\,\frac{3c+l}{c^3(3c+4l)}\right\}\left\{\mathrm{I}'(k^2-\omega^2)-12\mathrm{EI}\,\frac{c+l}{c(3c+4l)}\right\}$$
$$=\frac{36(\mathrm{EI})^2(3c+2l)^2}{c^4(3c+4l)^2}. \quad . \quad . \quad (5)$$

Putting $k=0$ we obtain an equation for the critical angular velocity which is identical with Dunkerley's (*l. c.* p. 314).

If in (5) we neglect I′ altogether, we find

$$k^2+\omega^2=3\mathrm{EI}\div \mathrm{M}c^2(c+l). \quad . \quad . \quad . \quad . \quad (6)$$

Proceeding to a second approximation, retaining only the lowest power of I′, we find for the critical angular velocity, putting $k^2=0$,

$$\omega^2=\frac{3\mathrm{EI}}{\mathrm{M}c^2(c+l)}\left\{1+\frac{1}{4}\,\frac{\mathrm{I}'}{\mathrm{M}}\,\frac{(3c+2l)^2}{c^2(c+l)^2}\right\}. \quad . \quad . \quad (7)$$

This last result will not be satisfactory when c/l is very small, or the load very close to B; but under these conditions $1/\omega^2$ is very small, so that on Dunkerley's hypothesis the load has but little real influence on the critical velocity.

(*c*) If in the problem treated under (*b*) we assume the effect of I′ small to start with, we may employ the simpler type

$$\left.\begin{array}{l}\text{for BC, } y=\quad\eta(2clx+3cx^2-x^3),\\ \text{,, BA, } y'=-\eta(c/l)(2l^2x'-3lx'^2+x'^3).\end{array}\right\} \quad . \quad . \quad (8)$$

If η were replaced by $g\mathrm{M}/6\mathrm{EI}$ this would represent the bending of the shaft under the weight of M.

Treating I′ as small, we find by assuming $\eta \propto \cos kt$

$$(k^2+\omega^2)4\mathrm{M}c^2(c+l)^2+(k^2-\omega^2)\mathrm{I}'(3c+2l)^2=12\mathrm{EI}(c+l). \qquad (9)$$

When I′ is wholly neglected, this agrees with (6).

For the critical angular velocity, we have from (9)

$$\omega^2=\frac{3\mathrm{EI}}{\mathrm{M}c^2(c+l)}\left\{1-\frac{\mathrm{I}'}{\mathrm{M}}\frac{(3c+2l)^2}{4c^2(c+l)^2}\right\}^{-1}, \quad . \quad . \quad (10)$$

which agrees with (7) when $(\mathrm{I}')^2$ is neglected.

§ 21. (*d*) Loaded shaft of appreciable mass.

On Dunkerley's hypothesis the critical angular velocity is given by

$$1/\omega^2=1/\omega_1^{\,2}+1/\omega_2^{\,2},$$

where ω_1 is given by (1) and ω_2 by (5) with k omitted. So long as the effect of I' is small, and c/l does not exceed 1/4, the following approximation is deducible from (3) and (7)

$$\frac{1}{\omega^2}=\frac{\sigma\rho l^4}{97\cdot 41\mathrm{EI}}\left(1-6\cdot 6\frac{c^3}{l^3}\right)^{-1}+\frac{\mathrm{M}c^2(c+l)}{3\mathrm{EI}}-\frac{\mathrm{I}'(3c+2l)^2}{12\mathrm{EI}(c+l)}. \quad (11)$$

(*e*) If, while assuming the displacements (8), we allow for the mass of the rod, we replace (9) by

$$(k^2+\omega^2)\left[\mathrm{M}\,.\,4c^2(c+l)^2+\sigma\rho\left\{\frac{4}{3}l^2(l^3+c^3)+\frac{33}{35}(l^5+c^5)\right.\right.$$
$$\left.\left.-\frac{11}{5}l(l^4-c^4)\right\}\right]+(k^2-\omega^2)\mathrm{I}'(3c+2l)^2=12\mathrm{EI}(c+l). \quad (12)$$

Putting $k=0$ we have a form of the critical velocity equation in which allowance is made for the inertia of both shaft and load.

(*f*) As an alternative to (*c*) we may assume a type of displacement answering to the form taken by the shaft when bending under its own weight, viz.

$$\left.\begin{array}{l}\text{for BC,}\\ y=\eta\{lx(4c^2-l^2)+6c^2x^2-4cx^3+x^4\}\ ;\\ \text{for BA,}\\ y'=\eta\{-3l^3x'+6l^2x'^2-4lx'^3+x'^4+(2/l)(l^2-c^2)(2l^2x'-3lx'^2+x'^3)\}\end{array}\right\}. \quad (13)$$

In the statical problem $\eta\equiv g\sigma\rho/24\mathrm{EI}$. In the kinetic problem, assuming $\eta\propto\cos kt$, we find

$$(k^2+\omega^2)\left[9\mathrm{M}c^2(c+l)^2(3c^2+cl-l^2)^2+\sigma\rho\left\{l^9-\frac{33}{10}l^7(l^2-c^2)\right.\right.$$
$$\left.\left.+\frac{96}{35}l^5(l^2-c^2)^2+3c^3l^2(4c^2-l^2)^2+\frac{78}{5}c^6l(4c^2-l^2)+\frac{104}{5}c^9\right\}\right]$$
$$+(k^2-\omega^2)\mathrm{I}'\times 9\{4c^2(l+c)-l^3\}^2=\frac{216}{5}\mathrm{EI}(l^5-5l^3c^2+10lc^4+6c^5). \quad (14)$$

Putting $k=0$, we have a second formula for the critical velocity, in which allowance is made for both shaft and load.

If in (14) we omit M and I' we have a result appropriate to the unloaded bar in (*a*). It is not, however, very satisfactory unless c/l is small. When powers of (c/l) above the fourth are neglected it gives for the critical angular velocity

$$\omega^2=97\cdot 55(\mathrm{EI}/\sigma\rho l^4)\{1-0\cdot 06(c/l)^2-6\cdot 8(c/l)^3+3\cdot 5(c/l)^4\}, \quad (15)$$

a result very similar to (3).

(*g*) An assumption which appears more natural at first sight than that made in either (*e*) or (*f*) is that the type of displacement answers to the bending of the shaft under its own weight and that of the load combined. This gives

$$\left.\begin{aligned}&\text{for CB, } y=\eta\left\{\frac{1}{6}\mathrm{M}(\xi^3-c^2\xi)+\frac{1}{24}\sigma\rho(\xi^4-c^3\xi)+(c-\xi)\mathrm{H}\right\},\\ &\text{,, AB, } y'=\eta\left\{\frac{1}{6}\mathrm{P}(\xi'^3-l^2\xi')+\frac{1}{24}\sigma\rho(\xi'^4-l^3\xi')\right\},\end{aligned}\right\}\quad(16)$$

$$\left.\begin{aligned}\text{where }\ \xi&\equiv c-x, & \mathrm{P}l&\equiv\mathrm{M}c-\tfrac{1}{2}\sigma\rho(l^2-c^2),\\ \xi'&\equiv l-x', & \mathrm{H}&\equiv\tfrac{1}{3}(\mathrm{P}l^2+\mathrm{M}c^2)+\tfrac{1}{8}\sigma\rho(l^3+c^3).\end{aligned}\right\}\quad(17)$$

Writing for shortness

$$\left.\begin{aligned}\mathrm{R}&\equiv\mathrm{M}c^2\mathrm{H}^2+\sigma\rho\left\{\frac{2}{945}(\mathrm{P}^2l^7+\mathrm{M}^2c^7)+\frac{11}{8640}\sigma\rho(\mathrm{P}l^8+\mathrm{M}c^8)+\frac{(\sigma\rho)^2}{72^2}(l^9+c^9)\right.\\ &\qquad\left.+\tfrac{1}{3}c^3\mathrm{H}^2-\frac{7}{180}c^5\mathrm{MH}-\frac{1}{90}c^6\sigma\rho\mathrm{H}\right\},\\ \mathrm{Q}&\equiv(\mathrm{H}+\frac{1}{6}c^2\mathrm{M}+\frac{1}{24}c^3\sigma\rho)^2,\\ \mathrm{S}&\equiv\mathrm{EI}\{\tfrac{1}{3}(\mathrm{P}^2l^3+\mathrm{M}^2c^3)+\tfrac{1}{4}\sigma\rho(\mathrm{P}l^4+\mathrm{M}c^4)+\frac{1}{20}(\sigma\rho)^2(l^5+c^5)\},\end{aligned}\right\}\quad(18)$$

we find for the frequency equation

$$(k^2+\omega^2)\mathrm{R}+(k^2-\omega^2)\mathrm{I}'\mathrm{Q}=\mathrm{S},\quad .\ \ .\ \ .\ \ .\quad(19)$$

and so for the critical angular velocity

$$\omega^2=\mathrm{S}\div(\mathrm{R}-\mathrm{QI}').\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\quad(20)$$

Obviously in general the evaluation of R and S is laborious.

§ 22. (*h*) It will be found that none of the types (*e*), (*f*), or (*g*) gives results which invariably accord well with experiment. It is clear that the most natural way of bending may be such that portions of the bar on opposite sides of the support at B have "centrifugal forces" acting on them in opposite directions. This suggests the use of a type of displacement answering to an imaginary gravitational force oppositely directed on opposite sides of B. Such a type is

$$\left.\begin{aligned}&\text{for BC, } y=\eta\{x^4-4cx^3+6c^2x^2+2\mathrm{A}(3cx^2-x^3)-\mathrm{B}x\},\\ &\text{,, BA, } y'=-\eta\{x'^4-4lx'^3+6l^2x'^2+2\mathrm{A}'(3lx'^2-x^3)-\mathrm{B}x'\},\end{aligned}\right\}\quad(21)$$

where

$$\left.\begin{aligned} \mathrm{A}&=2\mathrm{M}/\sigma\rho, \quad \mathrm{A}'=2\mathrm{P}/\sigma\rho, \\ \mathrm{B}&=(8\mathrm{P}l^2+3\sigma\rho l^3)/\sigma\rho, \\ \mathrm{P}l&=-\mathrm{M}c-\tfrac{1}{2}\sigma\rho(l^2+c^2). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

Putting for shortness

$$\left.\begin{aligned} \mathrm{R}'=\;&\mathrm{M}c^2\{3(l^3-c^3)+8(\mathrm{P}l^2-\mathrm{M}c^2)/\sigma\rho\}^2+\frac{104}{45}\sigma\rho(l^9+c^9) \\ &+\frac{59}{5}(\mathrm{P}l^8+\mathrm{M}c^8)-\frac{26}{15}(8\mathrm{P}l^2+3\sigma\rho l^3)(l^6+c^6)+\frac{528}{35\sigma\rho}(\mathrm{P}^2l^7+\mathrm{M}^2c^7) \\ &+\frac{1}{3\sigma\rho}(l^3+c^3)(8\mathrm{P}l^2+3\sigma\rho l^3)^2-\frac{22}{5\sigma\rho}(8\mathrm{P}l^2+3\sigma\rho l^3)(\mathrm{P}l^5+\mathrm{M}c^5), \\ \mathrm{S}'=\;&144\mathrm{EI}\left\{\tfrac{1}{5}(l^5+c^5)+\{\mathrm{P}l^4+\mathrm{M}c^4+\frac{4}{3\sigma\rho}(\mathrm{P}^2l^3+\mathrm{M}^2c^3)\}/\sigma\rho\right\}, \end{aligned}\right\} \quad (23)$$

we find for the frequency equation

$$k^2+\omega^2=\mathrm{S}'/\mathrm{R}', \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and for the critical angular velocity

$$\omega^2=\mathrm{S}'/\mathrm{R}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

§ 23. Table III. compares the results obtainable from Dunkerley's formula, corresponding to (5) with $k=0$, with those given by the simple formulæ (6) and (10) for the case when the mass of the shaft is neglected.

TABLE III.

c/l ……	$\frac{1\cdot00}{30\cdot66}$		$\frac{2\cdot57}{29\cdot10}$		$\frac{3\cdot66}{28\cdot0}$		$\frac{4\cdot99}{26\cdot66}$		$\frac{7\cdot66}{24\cdot00}$		$\frac{10\cdot32}{21\cdot33}$	
Pulley Formula	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
(5) ………	16390	13816	4808	3353	3318	2277	2428	1643	1572	1056	1162	782
(6) ………	12014	8020	4603	3157	3256	2209	2393	1617	1562	1051	1161	779
(10) ………	15214	14733	4747	3350	3307	2275						

The results from (5) are taken from Dunkerley's paper. The values of c/l really differed slightly for the two pulleys; the values given in the headings to the Table are the means for the two cases.

It will be seen that except in the first instance, where the pulley was only an inch from a support, the simple formula (6) differs but little from (5). The difference for the larger values of c/l is so small that the improvement obtained when (10) is substituted for (6) is hardly worth considering.

§ 24. Table IV. gives particulars of the critical number of revolutions per minute of an unloaded shaft as observed and calculated by Dunkerley, and as calculated from several of the other equations advanced above.

TABLE IV.—Unloaded overhanging Shaft, critical speeds.

c/l.	Observed value.	Calculated values from				
		(1) by Dunkerley.	(3)	(14)(M=0).	(15)	(25)(M=0).
1/10 ...	1309	1301	1351			
1/7 ...	1435	1397	1450			
1/5 ...	1472	1516	1571		1574	
1/3 ...	1606	1704		2013		1747
1/2 ...	1558	1606		3825		1627
1 ...	1002	1031		1058		1046

The values calculated by Dunkerley for small values of c/l are apparently affected by the error referred to in (a); if this were corrected his values should practically coincide with those obtained from (3). The large difference between the observed values answering to the values 1/3 and 1/2 of c/l and those calculated from (14) is to be ascribed to the fact that the type of displacement assumed in (f) answers more nearly to a higher harmonic than to the fundamental vibration. This serves to illustrate a contingency which must never be lost sight of when applying Rayleigh's method. We know from his general theory * that the value so calculated for k can never be too low—excluding of course errors of calculation—so that when results have been obtained from more than one assumed type of vibration we need never be at loss which to prefer.

§ 25. Table V. compares the results observed by Dunkerley in a shaft carrying a pulley with those variously calculated.

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* 'Theory of Sound,' vol. i. Art. 89.

TABLE V.—Loaded overhanging Shaft, critical speeds.

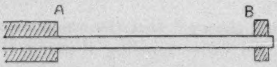
Pulley.	l inches.	c inches.	Observed speed.	Calculated speeds from				
				(1) & (5) by Dunkerley.	(e) (12).	(f) (14).	(g) (20).	(h) (25).
I.	30·70	1·00	1223	1170			1215	
	29·10	2·61	1329	1256			1342	
	28·00	3·69	1384	(1288)	[1678]	1385		
	26·66	5·02	1407	(1286)	(1578)	1454	[1839]	1394
	24·00	7·69	1224	1156	1267			1229
	21·33	10·35	968	941	971		1073	
II.	30·63	1·00	1227	1173			1220	
	29·10	2·54	1276	1213			(1416)	
	28·00	3·63	1281	(1191)	[1473]	1311	[1927]	
	26·66	4·96	1215	(1114)	1280	(1326)		1200
	24·00	7·63	928	898	947			944
	21·33	10·29	712	703	711			

Calculated results differing widely from the observed are put in [ ] brackets ; those whose divergence is less but still conspicuous are put in ( ) brackets. When c/l is small, all the formulæ necessarily supply results which approach closely to those for an unloaded shaft.

When there is a conspicuous difference between observed and calculated values, the latter, in accordance with Rayleigh's principle, are invariably the larger, except in the case of Dunkerley's own calculations.

CASE 4 (Dunkerley's Cases IV. and XII.).

§ 26. Shaft fixed in direction at one end (A, $x=0$), and supported at the other (B, $x=l$).



(a) Unloaded shaft : Euler-Bernoulli solution.

The displacement is given by

$$y=\alpha\{(\cosh\mu x-\cos\mu x)(\sinh\mu l+\sin\mu l)$$
$$-(\sinh\mu x-\sin\mu x)(\cosh\mu l+\cos\mu l)\}, \quad . \quad (1)$$

where α is a constant.

The equation determining μ (*cf.* Dunkerley, *l. c.* p. 294) is

$$\coth \mu l = \cot \mu l. \qquad (2)$$

The least root of this (Rayleigh's 'Sound,' arts. 180 and 174) is

$$\mu l = 3{\cdot}9266. \qquad (3)$$

Answering to which we have for the critical angular velocity

$$\omega^2 = 237{\cdot}7\,(\mathrm{EI}/\sigma\rho l^4). \qquad (4)$$

(*b*) Instead of the Euler-Bernoulli method, we may assume for the unloaded shaft

$$y = \eta x^2 (l - x)(3l - 2x). \qquad (5)$$

If η were replaced by $g\sigma\rho/48\mathrm{EI}$, this would give the bending of the shaft under its own weight.

For the dynamical problem we find

$$\left.\begin{aligned} \mathrm{T} &= \tfrac{1}{2}\sigma\rho(\dot{\eta}^2 + \omega^2\eta^2)(19/630)l^9, \\ \mathrm{V} &= \tfrac{1}{2}\eta^2(36/5)\mathrm{EI}\,l^5\,; \end{aligned}\right\} \qquad (6)$$

whence we have for the frequency equation

$$k^2 + \omega^2 = (4536/19)\,(\mathrm{EI}/\sigma\rho\, l^4), \qquad (7)$$

and for the critical angular velocity

$$\omega^2 = 238{\cdot}7\,\mathrm{EI}/\sigma\rho\, l^4. \qquad (8)$$

The value of ω given by (8) is only 0·2 per cent. in excess of that given by the exact equation (4).

(*c*) Mass (M, I′) on massless shaft.

Supposing the mass at C ($\mathrm{AC} = a$, $\mathrm{BC} = b$), and measuring x from A to C, and x' from B to C, we have for the Rayleigh type of displacement

$$\left.\begin{aligned} &\text{for AC,} \quad y = (3z - a\theta)(x^2/a^2) + (a\theta - 2z)(x^3/a^3), \\ &\text{for BC,} \quad y' = \tfrac{1}{2}(3z + b\theta)(x'/b) - \tfrac{1}{2}(b\theta + z)(x'^3/b^3). \end{aligned}\right\} \qquad (9)$$

By Lagrange's equations we find

$$\left\{ \mathrm{M}(k^2 + \omega^2) - 3\mathrm{EI}\frac{a^3 + 4b^3}{a^3 b^3} \right\} \left\{ \mathrm{I}'(k^2 - \omega^2) - \mathrm{EI}\frac{3a + 4b}{ab} \right\} = (3\mathrm{EI})^2 \left(\frac{a^2 - 2b^2}{a^2 b^2} \right)^2. \qquad (10)$$

When k is omitted this agrees with Dunkerley's equation (*l. c.* p. 321). Equation (10) splits into two factors when

$$a/b = \sqrt{2} = 1{\cdot}414.$$

In this position of the load the frequencies of what may be called the transverse and the oscillational vibrations are respectively given by

$$\left.\begin{aligned} k^2 &= -\omega^2 + (51 + 36\sqrt{2})(\mathrm{EI}/\mathrm{M}l^3) \equiv -\omega^2 + 101{\cdot}9(\mathrm{EI}/\mathrm{M}l^3), \\ k^2 &= \omega^2 + (7 + 5\sqrt{2})(\mathrm{EI}/\mathrm{I}'l) \equiv \omega^2 + 14{\cdot}14(\mathrm{EI}/\mathrm{I}'l). \end{aligned}\right\} \quad (11)$$

If we wholly neglect I′ in (10), we have for the critical velocity

$$\omega^2 = 12\mathrm{EI}\,l^3 \div \{\mathrm{M}a^3b^2(3a+4b)\}\ ; \quad . \quad . \quad (12)$$

while retaining I′, but neglecting (I′)², we have

$$\omega^2 = \frac{12\mathrm{EI}\,l^3}{\mathrm{M}a^3b^2(3a+4b)}\left\{1 + \frac{9\mathrm{I}'(a^2-2b^2)^2}{\mathrm{M}a^2b^2(3a+4b)^2}\right\}. \quad . \quad (13)$$

(*d*) If we assume I′ small to commence with, still neglecting the mass of the shaft, we may take

$$\left.\begin{aligned} &\text{in AC, } y = \eta\{2l^3(3ax^2 - x^3) - a^2(2a+3b)(3lx^2 - x^3)\}, \\ &\text{in BC, } y' = \eta\{3a^2bl^2x' - a^2(2a+3b)x'^3\}. \end{aligned}\right\} \quad (14)$$

When η is replaced by $\mathrm{M}g/12\mathrm{EI}\,l^3$ we have the bending of the shaft under the weight of M.

For the kinetic problem, we find for the frequency equation

$$\begin{aligned} (k^2+\omega^2)\mathrm{M}a^3b^2(3a+4b)^2 &+ (k^2-\omega^2)9\mathrm{I}'a(a^2-2b^2)^2 \\ &= 12\mathrm{EI}\,l^3(3a+4b), \quad . \quad (15) \end{aligned}$$

and for the critical angular velocity

$$\omega^2 = \frac{12\mathrm{EI}\,l^3}{\mathrm{M}a^3b^2(3a+4b)}\left\{1 - \frac{9\mathrm{I}'(a^2-2b^2)^2}{\mathrm{M}a^2b^2(3a+4b)^2}\right\}^{-1}. \quad . \quad (16)$$

This agrees with (12) when I′ is neglected, with (13) when (I′)² only is neglected.

§ 27. (*e*) Load (M, I′) on massive shaft.

On Dunkerley's hypothesis we have

$$1/\omega^2 = 1/\omega_1{}^2 + 1/\omega_2{}^2,$$

where ω_1 is given by (4), and ω_2 by (10), with k omitted.

If the effect of (I′) is small we should get from (4) and (13)

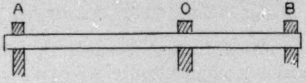
$$\frac{1}{\omega^2} = \frac{\sigma\rho l^4}{237{\cdot}7\,\mathrm{EI}} + \frac{\mathrm{M}a^3b^2(3a+4b)}{12\mathrm{EI}\,l^3} - \frac{3}{4}\frac{\mathrm{I}'a(a^2-2b^2)^2}{\mathrm{EI}\,l^3(3a+4b)}. \quad . \quad (17)$$

In this case I have not worked out an independent dynamical method taking both shaft and load into account; but the difficulties would be less than in case (3).

CASE 5 (Dunkerley's Cases V. and XIII.).

§ 28. Shaft supported at ends A and B, and at intermediate point O

$$(OA=a, \quad OB=b, \quad a+b=l).$$



(*a*) Unloaded shaft: Euler-Bernoulli solution.

As pointed out by Dunkerley, the mathematical conditions are all satisfied if the two relations (*cf.* § 12)

$$\left.\begin{array}{l}\mu a = i\pi, \\ \mu b = j\pi\end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where i and j are integers, can exist simultaneously.

To have a real application to the practical problem, i and j must be small integers, so that (1) is of very limited scope. If, however, the spans are equal, or if the longer, say a, is a multiple of b, we have obviously

$$\mu b = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Answering to this, we obtain for the critical velocity

$$\omega^2 = \pi^4 \,\mathrm{EI}/\sigma p b^4 = 97{\cdot}41 \,\mathrm{EI}/\sigma p b^4. \quad . \quad . \quad . \quad (3)$$

This is the same result (*cf.* (*a*) Case 2) as for a shaft of length b supported at both ends.

Excluding the above special cases, the general equation obtained by Dunkerley (*l. c.* p. 296) is

$$\coth \mu a - \cot \mu a + \coth \mu b - \cot \mu b = 0. \quad . \quad . \quad . \quad (4)$$

If b/a be very small, a first approximation is

$$\coth \mu a - \cot \mu a = 0,$$

the least root of which (*cf.* (*a*) case 4) is

$$\mu a = 3{\cdot}9266.$$

This reduction to (*a*) case 4 only implies, what is physically obvious, that when the support O is close to an end it serves to fix the terminal direction of the shaft. If b/a in (4) is treated as small, and $(b/a)^2$ as negligible, we deduce

$$\mu a = 3{\cdot}9266\,(1-b/3a), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence

$$\omega^2 = 237{\cdot}7\,(1-4b/3a)\,(\mathrm{EI}/\sigma p a^4). \quad . \quad . \quad (6)$$

(*b*) As (4) is somewhat unmanageable except for special cases, I have tried several algebraic types of displacement, amongst them

$$\left.\begin{array}{l}\text{for AO, } y=\eta\{x^4-4a^3x-\mathrm{C}(x^3-3a^2x)-\mathrm{D}x\},\\ \text{for BO, } y'=\eta\{x'^4-4b^3x'-\mathrm{C}'(x'^3-3b^2x')+\mathrm{D}x'\}.\end{array}\right\}. \quad (7)$$

Here x is measured from A towards O, and x' from B towards O, and

$$\mathrm{C}\equiv(3a^2+ab-b^2)/2a, \qquad \mathrm{C}'\equiv(3b^2+ab-a^2)/2b,$$
$$\mathrm{D}\equiv ab(a-b).$$

Replacing η in (7) by $g\sigma\rho/24\mathrm{EI}$ we should have the bending of the shaft under its own weight.

From the kinetic method I find

$$k^2+\omega^2=(3/5)(\mathrm{EI}/\sigma\rho)\{3(a^5+b^5)+5ab(a^3+b^3)-5a^2b^2(a+b)\}\div\Big\{\frac{37}{9}(a^9+b^9)$$
$$+\frac{68}{35}(\mathrm{C}^2a^7+\mathrm{C}'^2b^7)+\frac{1}{3}\mathrm{D}^2(a^3+b^3)-\frac{113}{20}(\mathrm{C}a^8+\mathrm{C}'b^8)+\frac{7}{3}\mathrm{D}(a^6-b^6)-\frac{8}{5}\mathrm{D}(\mathrm{C}a^5-\mathrm{C}'b^5)\Big\}. \quad (8)$$

Putting $k^2=0$ we have the critical angular velocity. The evaluation, though perfectly straightforward, is in general tedious.

For b/a small, however, we easily find

$$k^2+\omega^2=(4536/19)(1-4b/3a)(\mathrm{EI}/\sigma\rho a^4), \quad . \quad . \quad . \quad (9)$$

and thence for the critical velocity

$$\omega^2=238{\cdot}7(1-4b/3a)(\mathrm{EI}/\sigma\rho a^4), \quad . \quad . \quad . \quad (10)$$

a result of course in close agreement with (6) (*cf.* also (8) and (4) of case 4, § 26).

Again, if $b=a$ we obtain from (8) for the period and critical velocity

$$k^2+\omega^2=(4536/19)(\mathrm{EI}/\sigma\rho a^4), \quad . \quad . \quad . \quad (11)$$
$$\omega^2=238{\cdot}7\,(\mathrm{EI}/\sigma\rho a^4). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(*c*) As an alternative type to (7) let us take

$$\left.\begin{array}{l}\text{for AO, } y=-\eta\{x^4-4a^3x-\mathrm{C}(x^3-3a^2x)-\mathrm{D}x\},\\ \text{for BO, } y'=\ \ \eta\{x'^4-4b^3x'-\mathrm{C}'(x'^3-3b^2x')-\mathrm{D}x'\},\end{array}\right\}. \quad (13)$$

where now

$$\mathrm{C}\equiv(3a^3+4a^2b+b^3)\div2a(a+b), \qquad \mathrm{C}'\equiv(a^3+4ab^2+3b^3)\div2b(a+b),$$
$$\mathrm{D}\equiv ab(a^2+b^2)/(a+b).$$

The above displacements answer to an imaginary gravitational force oppositely directed on the two sides of O.

By Lagrange's equations we deduce

$$k^2+\omega^2=(3/5)(\mathrm{EI}/\sigma\rho l)(3l^6-10abl^4-5a^2b^2l^2+20a^3b^3)\div\Big\{\frac{37}{9}(a^9+b^9)+\frac{68}{35}(\mathrm{C}^2a^7+\mathrm{C}'^2b^7)$$
$$+\frac{1}{3}\mathrm{D}^2(a^3+b^3)-\frac{113}{20}(\mathrm{C}a^8+\mathrm{C}'b^8)+\frac{7}{3}\mathrm{D}(a^6+b^6)-\frac{8}{5}\mathrm{D}(\mathrm{C}a^5+\mathrm{C}'b^5)\Big\}. \quad . \quad . \quad . \quad (14)$$

Omitting k^2, we have the critical angular velocity.

When b/a is small (14) agrees with (8) in giving (9) and (10). For $b=a$, however, it gives the widely different result

$$k^2+\omega^2=(48\times 63/31)(\mathrm{EI}/\sigma\rho a^4), \quad . \quad . \quad . \quad (15)$$

whence for the critical velocity

$$\omega^2=97{\cdot}55(\mathrm{EI}/\sigma\rho a^4). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This is identical with (7) of case (2), which applies to a single span of length a, and is in close agreement—as it should be—with (3), when b is replaced by a. The divergence of (12) really means (*cf.* § 24 and Table IV.) that its assumed type of vibration answers not to the fundamental note but to an harmonic.

§ 29. Mass (M, I′) on massless shaft.

(*d*) Supposing the load at C, between O and A, at a distance c from O, Rayleigh type displacements are :

$$\left.\begin{aligned}
&\text{for OC, } y=\frac{2b(3z-c\theta)}{c(4b+3c)}\left(x+\frac{3x^2}{2b}\right)+\frac{\theta c(2b+3c)-2z(b+3c)}{4b+3c}\frac{x^3}{c^3},\\
&\text{for AC, } y'=\frac{3z+(a-c)\theta}{2(a-c)}x'-\frac{(a-c)\theta+z}{2(a-c)^3}x'^3,\\
&\text{for OB, } y''=\frac{2b(c\theta-3z)}{c(4b+3c)}\left(x''-\frac{3}{2}\frac{x''^2}{b}+\frac{1}{2}\frac{x''^3}{b^2}\right),
\end{aligned}\right\}(17)$$

where x and x'' are measured from O, and x' from A.

Lagrange's equations lead to

$$\left[\mathrm{M}(k^2+\omega^2)-3\mathrm{EI}\left\{\frac{1}{(a-c)^3}+\frac{4(b+3c)}{c^3(4b+3c)}\right\}\right]$$
$$\times\left[\mathrm{I}'(k^2-\omega^2)-3\mathrm{EI}\left\{\frac{1}{a-c}+\frac{4(b+c)}{c(4b+3c)}\right\}\right]$$
$$-9(\mathrm{EI})^2\left\{\frac{1}{(a-c)^2}-\frac{2(2b+3c)}{c^2(4b+3c)}\right\}^2=0. \quad . \quad (18)$$

Omitting k^2, we obtain an equation for the critical angular velocity which—allowing for a misprint—agrees with Dunkerley's equation A (*l. c.* p. 326). It is worth noticing that the terms independent of k or ω in (18) reduce to

$$36(\mathrm{EI})^2a^2(a+b) \div \{c^3(a-c)^3(4b+3c)\}.$$

The quadratic (18) splits into two factors, representing pure transverse and oscillational vibrations, when

$$\begin{aligned} c^2/(a-c)^2 &= (4b+6c)/(4b+3c), \\ &= 1 \text{ when } b/c \text{ is very big } (cf.\ (c) \text{ case } 2), \\ &= 2 \text{ when } b/c \text{ is very small } (cf.\ (c) \text{ case } 4). \end{aligned}$$

If in (18) we altogether neglect I′ we find for the critical velocity

$$\omega^2 = 12\mathrm{EI}a^2(a+b) \div [\mathrm{M}c^2(a-c)^2\{4a(b+c)-c^2\}]. \qquad (19)$$

(*e*) If we assume I′ small to begin with, we may replace the displacements in (*d*) by the simpler type

$$\left.\begin{aligned} &\text{in OC, } y = \eta[-2abc(3ac-2a^2-c^2)x + 2a^2(a+b)(3cx^2-x^3) \\ &\qquad\qquad -c(3ac+2ab-c^2)(3ax^2-x^3)], \\ &\text{in CA, } y = \eta[-2c^3a^2(a+b) + 2ac(c^2b+3a^2c+2a^2b)x \\ &\qquad\qquad -c(3ac+2ab-c^2)(3ax^2-x^3)], \\ &\text{in OB, } y' = \eta[2abc(3ac-2a^2-c^2)x' \\ &\qquad\qquad -(ac/b)(3ac-2a^2-c^2)(3bx'^2-x'^3)]. \end{aligned}\right\} . \qquad (20)$$

In the above x is measured from O to A, and x' from O to B.

Replacing η by $(g\mathrm{M}/\mathrm{EI})/12a^2(a+b)$, we should have the bending of the shaft due to the weight of M at C.

From Lagrange's equations I find for the frequency equation

$$\begin{aligned} &(k^2+\omega^2)\mathrm{M}c^2(a-c)^2(4ab+4ac-c^2)^2 \\ &\quad + (k^2-\omega^2)\mathrm{I}'\{4ab(a-2c)+3c(2a^2-4ac+c^2)\}^2 \\ &\quad = 12\mathrm{EI}a^2(a+b)(4ab+4ac-c^2), \quad . \quad . \quad . \quad . \quad . \qquad (21) \end{aligned}$$

and for the critical velocity

$$\omega^2 = \frac{12\mathrm{EI}a^2(a+b)}{\mathrm{M}c^2(a-c)^2(4ab+4ac-c^2)} \times \left[1 - \frac{\mathrm{I}'\{4ab(a-2c)+3c(2a^2-4ac+c^2)\}^2}{\mathrm{M}c^2(a-c)^2(4ab+4ac-c^2)^2}\right]^{-1}. \qquad (22)$$

Neglecting I′ altogether, we obtain (19).

In the case of two equal spans, or $b=a$, omitting higher powers of I' in (22), we find

$$\omega^2=\frac{24EIa^3}{Mc^2(a-c)^2(4a^2+4ac-c^2)}\times\left[1+\frac{I'(4a^3-2a^2c-12ac^2+3c^3)^2}{Mc^2(a-c)^2(4a^2+4ac-c^2)^2}\right]. \quad . \quad . \quad (23)$$

An identical result is deducible—but not so easily—from (18).

§ 30. Load (M, I′) on massive shaft.

(*f*) On Dunkerley's hypothesis the critical velocity is given by

$$1/\omega^2=1/\omega_1{}^2+1/\omega_2{}^2,$$

where ω_1 is given by (3) or (4), and ω_2 by (18) with k omitted.

In the case of equal spans, supposing the effect of I' small, we thus find

$$\frac{1}{\omega^2}=\frac{\sigma\rho a^4}{97{\cdot}41EI}+\frac{Mc^2(a-c)^2(4a^2+4ac-c^2)}{24EIa^3}-\frac{I'(4a^3-2a^2c-12ac^2+3c^3)^2}{24EIa^3(4a^2+4ac-c^2)} \quad . \quad . \quad (24)$$

(*g*) The best algebraic type of displacement would probably answer to the bending of the shaft under a gravitational force supposed to act on both shaft and load, but oppositely directed on the two spans. I have only worked out results from the simpler type (13), which neglects the influence of the load on the displacement. This leads to the frequency equation

$$(k^2+\omega^2)R+(k^2-\omega^2)QI'=S, \quad . \quad . \quad . \quad (25)$$

where

$$\left.\begin{aligned}R=M[&(a-c)^4-4a^3(a-c)\\&-C\{(a-c)^3-3a^2(a-c)\}-D(a-c)]^2\\&+\sigma\rho\Big[\frac{37}{9}(a^9+b^9)+\frac{68}{35}(C^2a^7+C'^2b^7)+\frac{1}{3}D^2(a^3+b^3)\\&-\frac{113}{20}(Ca^8+C'b^8)+\frac{7}{3}D(a^6+b^6)-\frac{8}{5}D(Ca^5+C'b^5)\Big],\\Q=[&4\{a^3-(a-c)^3\}-3\{a^2-(a-c)^2\}C+D]^2,\\S=&\frac{3EI}{5l}\{3l^6-10abl^4-5a^2b^2l^2+20a^3b^3\},\end{aligned}\right\}\;(26)$$

the notation being the same as in (*c*).

Omitting k in (25) we obtain the critical angular velocity. If, for instance, the spans are equal, and I' is negligible, we obtain

$$\omega^2 = \frac{97\cdot55\mathrm{EI}}{\sigma\rho a^4} \div \left[1 + \frac{315}{31}\frac{\mathrm{M}}{\sigma\rho a}\left\{\frac{c(a-c)}{a^2}\right\}^2 \left\{1 + \frac{c(a-c)}{a^2}\right\}^2\right] \quad (27)$$

Numerical calculations are here simplified by noticing that the coefficient of M involves c and $a-c$ symmetrically; this implies that the critical velocity is unaltered when the distances of the load from the central and terminal piers are interchanged.

As (27) is based on a displacement which omits the effect of the load, we may expect it to prove less exact the larger the load.

§ 31. Table VI. compares the results calculated by Dunkerley from (18), with k omitted, for the critical number of revolutions per minute in his shaft, supposed massless but carrying one of his pulleys, with corresponding results from the simpler formulæ (19) and (23). The spans are supposed equal.

TABLE VI.

c/a......	1/16.		1/4.		1/2.		3/4.		15/16.	
Pulley ...	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
Dunkerley from (18)...	31664	24173	8114	4842	4987	3325	6318	4288	24550	18816
„ (19)...	24760	16510	7175	4784	4987	3325	6284	4190	19464	12978
„ (23)...	32676	27219	...	...	...	...	...	...	24976	20096

When the pulley is close to a support it exerts—*cf.* the analogous case in § 17—an exceedingly small effect on the critical velocity in the practical case of shaft and pulley combined. Except when the pulley is close to a support, we see in Table VI. a close agreement between even the simplest formula (19) and (18). The more complicated formula (23) agrees pretty closely with (18) even for the values 1/16 and 15/16 of c/a; for the other values the two would be in practical agreement.

§ 32. Table VII. compares the critical number of revolutions actually observed by Dunkerley for the case of equal spans, with those which he calculated from (2) and (18), and the corresponding results from (27), which altogether neglects I′.

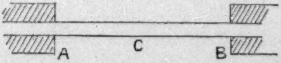
TABLE VII.

c/a	1/16.		1/4.		1/2.		3/4.		15/16.	
Pulley ...	I.	II.	I.	II.	I.	II.	I.	II.	I.	II.
Observed	4430	4524	3930	3213	3420	2600	3846	3056	4402	4220
Dunkerley from (2) & (18)	4440	4411	3925	3286	3334	2671	3657	3100	4411	4362
„ (27)...	4438	4381	3967	3516	3604	3000	3967	3516	4438	4381

As theory led us to expect, the agreement of (27) with observation is not so good for the heavier pulley II. as for the lighter.

CASE 6 (Dunkerley's Cases VI. and XIV.).

§ 33. Shaft fixed in direction at both ends.



(*a*) Unloaded shaft: Euler-Bernoulli method (Dunkerley, *l. c.* p. 298).

Measuring x from an end A, and denoting AB by l, we have

$$y=\alpha\{(\cosh \mu x-\cos \mu x)(\sinh \mu l-\sin \mu l) - (\sinh \mu x-\sin \mu x)(\cosh \mu l-\cos \mu l)\}, \quad . \quad . \quad (1)$$

where α is a constant, and the value of μ is given by

$$\cosh \mu l \cos \mu l-1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The least root of (2) (see Rayleigh's Sound, art. 174) is

$$\mu l=4{\cdot}73004, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives for the critical velocity

$$\omega^2=500{\cdot}6(\mathrm{EI}/\sigma\rho l^4). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(*b*) In the above case of no carried load, an approximate frequency equation is deducible from the simple type

$$y=\eta x^2(l-x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Replacing η by $g\sigma\rho/24\mathrm{EI}$ we should have the bending of the shaft under its own weight.

From the kinetic treatment we easily find

$$k^2+\omega^2=504(\mathrm{EI}/\sigma\rho l^4), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

giving for the critical velocity

$$\omega^2=504(\mathrm{EI}/\sigma\rho l^4). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The values given by (4) and (7) for ω differ by only about 0·3 per cent.

§ 34. Load (M, I′) on massless shaft.

(*c*) If the load be at C ($\mathrm{AC}=a$, $\mathrm{BC}=b$) we have for Rayleigh type displacements, measuring x from A and x' from B,

$$\left.\begin{array}{l}\text{in AC, } y=(3z-a\theta)(x/a)^2+(a\theta-2z)(x/a)^3, \\ \text{in BC, } y'=(3z+b\theta)(x'/b)^2-(b\theta+2z)(x'/b)^3\end{array}\right\} \quad . \quad . \quad . \qquad (8)$$

Applying Lagrange's equations we have

$$\{\mathrm{M}(k^2+\omega^2)-12\mathrm{EI}(a^{-3}+b^{-3})\}\{\mathrm{I}'(k^2-\omega^2)-4\mathrm{EI}(a^{-1}+b^{-1})\}$$
$$=36(\mathrm{EI})^2(b^{-2}-a^{-2})^2 \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Omitting k^2, we obtain for the critical velocity a result agreeing with Dunkerley's (*l. c.* p. 337).

(9) splits into factors, representing pure transverse and oscillational vibrations, when $b=a$. In this case we have for the respective frequencies (*cf.* (11) of §26, and (13) and (14) of §42)

$$\left.\begin{array}{l}k^2=-\omega^2+192\mathrm{EI}/\mathrm{M}l^3, \\ k^2=\ \ \ \omega^2+16\mathrm{EI}/\mathrm{I}'l.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

If we omit I′ altogether in (9) we obtain for the critical velocity

$$\omega^2=3\mathrm{EI}l^3/\mathrm{M}a^3b^3. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

As a second approximation, when I′ is small, we have

$$\omega^2=(3\mathrm{EI}l^3/\mathrm{M}a^3b^3)\{1+9\mathrm{I}'(a-b)^2/4\mathrm{M}a^2b^2\}. \quad . \qquad (12)$$

(d) If we can assume I' small to begin with, we may replace (8) by

$$\left.\begin{aligned}&\text{in AC, } y=\eta\{ab^2(a+b)x^2-\tfrac{1}{3}b^2(3a+b)x^3\},\\&\text{in BC, } y'=\eta\{a^2b(a+b)x'^2-\tfrac{1}{3}a^2(a+3b)x'^3\}\end{aligned}\right\} \quad . \quad (13)$$

Applying Lagrange's equations we find

$$(k^2+\omega^2)\tfrac{4}{9}\mathrm{M}a^3b^3+(k^2-\omega^2)\mathrm{I}'ab(a-b)^2=\tfrac{4}{3}\mathrm{EI}(a+b)^3. \quad (14)$$

For the critical angular velocity, writing l for $a+b$, we have

$$\omega^2=(3\mathrm{EI}l^3/\mathrm{M}a^3b^3)\{1-9\mathrm{I}'(a-b)^2/4\mathrm{M}a^2b^2\}^{-1} \quad . \quad (15)$$

Neglecting I' this agrees with (11), neglecting $(\mathrm{I}')^2$ with (12).

§ 35. (c) Load (M, I′) on massive shaft.

On Dunkerley's hypothesis we have

$$1/\omega^2=1/\omega_1^2+1/\omega_2^2,$$

where ω_1 is given by (4) and ω_2 by (9) with k omitted.

When I' is small we thus obtain

$$\frac{1}{\omega^2}=\frac{\sigma\rho l^4}{500{\cdot}6\mathrm{EI}}+\frac{\mathrm{M}a^3b^3}{3\mathrm{EI}l^3}-\frac{3}{4}\frac{\mathrm{I}'ab(a-b)^2}{\mathrm{EI}l^3} \quad . \quad . \quad (16)$$

I have not worked out a frequency equation based on an assumed type of displacement, but it would present no difficulty. The displacements would combine terms of the types (5) and (13) according to the relative masses of the shaft and load.

General Conclusions.

§ 36. In every case here treated when the effect of the moment of inertia of the load has been small—as was true invariably in Dunkerley's experiments, and probably often is in practice—the frequency equation has proved to be of the type

$$k^2=\mathrm{K}^2-\nu\omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\mathrm{K}/2\pi$ is the frequency of the fundamental transverse vibration of the system when not rotating. There would thus seem grounds for supposing that a formula of type (1) will often prove a close approximation to the truth. When this is the case, we can arrive at a close approximation to the velocity answering to whirling without endangering the shaft by actually pushing the velocity to this point. All that is necessary is to determine the frequency of the lowest natural transverse vibration in the shaft when not rotating, and when rotating with any convenient velocity ω_1.

If these frequencies be $K/2\pi$ and $k_1/2\pi$ respectively, then it is easily found from (1) that the frequency answering to any arbitrary value of ω is given by

$$k^2 = K^2 - (K^2 - k_1^2)\omega^2/\omega_1^2 . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the critical angular velocity, Ω say, being that for which k vanishes, is given by

$$\Omega^2 = \omega_1^2 K^2/(K^2 - k_1^2) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As a check on the applicability of (2), it would in general be advisable to determine the frequency $k_2/2\pi$ answering to a second angular velocity ω_2. If (2) is strictly true, we should obviously have

$$(K^2 - k_2^2)/\omega_2^2 = (K^2 - k_1^2)/\omega_1^2, \quad . \quad . \quad . \quad (4)$$

the quantity on either side of the equation being a value for K^2/Ω^2.

In all the cases solved, the quantity ν in (1) has approached unity as a limiting value when the moment of inertia of the load has been indefinitely diminished; *i. e.* the angular velocity answering to whirling has approached the limiting value $2\pi n$, where n is the number of transverse vibrations of the fundamental type executed by the system when not rotating in unit of time.

The feasibility of determining frequencies of vibration in actual shaft systems, or models, is a question which I must leave to those experienced in Acoustics and practical Engineering.

§ 37. When a shaft is carried on more than two supports, it is not easy to lay down a suitable basis for the comparison of the critical velocities answering to different terminal conditions. A comparison is, however, easily instituted in Cases 1, 2, 4, and 6, when the shaft is unloaded. In all four cases, suppose the total length l, the mass $m (\equiv \sigma\rho l)$, the stiffness EI, and let ω be the critical angular velocity, $N (= 30\omega/\pi)$ the corresponding number of revolutions per minute. Then we have the results given in Table VIII.

TABLE VIII.

Case.	State of ends.	Value of $\omega^2 \div (EI/ml^3)$.	Value of $N \div (EI/ml^3)^{\frac{1}{2}}$.
1.	One fixed in direction, other free.	12·36	33·58
2.	Both supported	97·41	94·25
4.	One fixed in direction, other supported...	237·7	147·2
6.	Both fixed in direction	500·6	213·7

The smaller ω, or N, the more easily is the shaft caused to whirl. Table VIII. thus serves to bring out the great reduction in the tendency to whirl caused by fixing the direction at the ends of the shaft.

§ 38. A comparison is also readily made for a loaded but massless shaft under the conditions of cases 1, 2, 4, and 6, when the moment I′ of the load is negligible. This is done in Table IX. The load M is supposed at the end in case 1; in the other cases its distances from the ends A and B are a and b. The so-called "minimum values" of ω^2 and N answer to that position of M which gives the smallest critical velocity. The rest of the notation has the same significance as in Table VIII.

TABLE IX,

Case.	End A.	End B.	General value of $\omega \div (EI/Ml^3)$.	Value of b/a giving minimum value of ω.	Minimum Values of	
					$\omega^2 \div (EI/Ml^3)$.	$N \div (EI/Ml^3)^{\frac{1}{2}}$.
1. ...	Direction fixed.	Free.	3	....	3	16·54
2. ...	Supported.	Supported.	$3l^4/a^2b^2$	1	48	66·16
4. ...	Direction fixed.	Supported.	$12\dfrac{l^6}{a^3b^2(3a+4b)}$	$(1/2)^{\frac{1}{2}}$	101·9	96·4
6. ...	Direction fixed.	Direction fixed.	$3l^6/a^3b^3$	1	192·0	132·3

In cases 2, 4, and 6, the position of the load which supplies the minimum value is precisely that for which Dunkerley's equations and mine agree in making the effect of I′ vanish. Again, in every case where I have treated I′ as small, the corrective term in I′ has increased ω. There are thus reasons for regarding the above minimum values of N as pretty safe measures of the lowest speed at which whirling is likely to arise when a load is attached anywhere to a shaft of much smaller mass.

If we compare the results for ω^2 or N in Table VIII. with the corresponding minimum values in Table IX., we obtain the following as the ratios of m to M for which the critical velocities in the two cases are equal :—

Case	1.	2.	4.	6.
m/M	4·12	2·03	2·33	2·61

It follows that when a load is near its most effective position, it must be taken into account as well as the shaft if we aim at a close approximation to the critical velocity.

The general values of ω^2, however, in Table IX. show how very rapidly the tendency in M to produce whirling falls off as its position approaches a support, especially when the support is such as to fix the direction of the shaft.

§ 39. In considering liability to whirl, we must not lose sight of the possibility of the elastic strains and stresses exceeding the limit of safety before whirling is reached. As an example, let us suppose the shafts circular (solid, or hollow), of external radius a and perimeter p ($p = 2\pi a$), of a material for which

$E = 20 \times 10^8$ grammes weight per sq. cm. (1270 tons per sq. inch),
$\eta = 1/4$,
$\rho = 7{\cdot}8$ times the density of water.

Suppose the shafts unloaded, and denote the maximum stress-difference answering to the critical angular velocity of whirling by $\bar{S}$, then the following results may be proved :—

TABLE X.

	Solid Shaft, both ends		Very thin-walled hollow Shaft, both ends	
	Supported.	Fixed.	Supported.	Fixed.
$\bar{S}$ (in tons per sq. inch) ...	$6{\cdot}61(p/l)^4$	$34{\cdot}0(p/l)^4$	$39{\cdot}7(p/l)^4$	$204{\cdot}0(p/l)^4$
$\bar{S}=2$ tons .... $l/p=$	1·35	2·03	2·11	3·18
$l/a=$	8·5	12·8	13·3	20·0

The value of $\overline{S}$ for the very thin-walled shaft is really 6 times that for the solid shaft, for the assumed value 1/4 of Poisson's ratio. Since the maximum stress-difference answering to the critical velocity varies as the fourth power of the ratio borne by the perimeter (or radius) of the shaft to its length, it increases with great rapidity as the length is reduced, the section remaining unaltered. The physical properties assumed answer fairly to steel; but the results, it may be remarked, depend only on the elasticity, not on the density of the material. Under statical conditions, a stress-difference of 2 tons on the square inch is but a trifle compared to what good steel will stand; but in a rotating shaft, where there are ordinarily rapid alternations of stresses from various sources, it is probably at least as large a contribution from "centrifugal forces" as a cautious engineer will care to see.

When the ratio of the length to the circumference, or radius, is less than the values recorded in the two last lines of Table X., the stress-difference will exceed 2 tons on the square inch before the shaft whirls.

MATHEMATICAL APPENDIX.

§ 40. The kinetic energy of a body rotating about an axis through its C.G. is given by

$$T_r = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2),$$

where I_1, I_2, I_3 are the principal moments of inertia, and ω_1, ω_2, ω_3 the component angular velocities about the three principal axes. Supposing the body one of revolution, and that it rotates with angular velocity ω about a fixed direction with which its axis of figure makes a small angle θ, then

$$\omega_1 = \omega\cos\theta, \qquad \omega_2 = \omega\sin\theta, \qquad \omega_3 = \dot{\theta},$$

and

$$T_r = \tfrac{1}{2}\{\omega^2 I_1 - (I_1 - I_2)\omega^2\sin^2\theta + I_2\dot{\theta}^2\}. \quad . \quad . \quad . \quad (1)$$

If the body be a flat disk $I_1 = 2I_2$; and if θ be very small $\sin\theta$ may be replaced by θ, thus leading to

$$T_r = \tfrac{1}{2}\omega^2 I_1 + \tfrac{1}{2}I_2(\dot{\theta}^2 - \omega^2\theta^2). \quad . \quad . \quad . \quad . \quad (2)$$

This result is applicable to a plate-shaped pulley carried on a rotating shaft, at a place where the tangent to the axis of the shaft is inclined at an angle θ to its undisturbed position. It is of course additional to the energy of the mass supposed collected at its centre of gravity. Unless the thickness of the pulley is small, variations of θ throughout it may not be negligible; and unless it closely resembles a cylinder of revolution, it may be necessary to allow for I_1 not being double I_2.

In the text, and subsequently in the Appendix, I' is used for I_2.

§ 41. For illustrative purposes I shall take the case of a massless shaft of length l, supported at its two ends A and B ($AB=l$), carrying a load of mass M, and inertia I', at an intermediate point C ($AC=a$, $BC=b$).

For AC we measure x from A, and for BC we measure x' from B. At any time t suppose C to be at a distance z from AB, in the xy plane, and let the tangent at C to the axis of the shaft make an angle θ with AB. Then we must have

$$\left.\begin{array}{lll} \text{at } x=0, & y=d^2y/dx^2=0, & \\ x=a, & y=z, & dy/dx=\theta, \\ x'=0, & y'=d^2y'/dx'^2=0, & \\ x'=b, & y'=z, & dy'/dx'=-\theta. \end{array}\right\} \quad . \quad . \quad (3)$$

These conditions are easily seen to be satisfied by

$$\left.\begin{array}{l} y=(3z-a\theta)(x/2a)+(a\theta-z)(x^3/2a^3), \\ y'=(3z+b\theta)(x'/2b)-(b\theta+z)(x'^3/2b^3), \end{array}\right\} \quad . \quad . \quad (4)$$

when we treat z and θ as constants. Under the same assumption these expressions satisfy

$$d^4y/dx^4=0, \qquad d^4y'/dx'^4=0,$$

equations answering to the absence of external forces on the shaft itself.

The components of velocity at the C.G. C of the load are $\dot{z}$ perpendicular to AB in the plane of bending, and ωz perpendicular to the plane of bending. Thus, treating θ as small, we have for the kinetic energy of the system

$$T=\tfrac{1}{2}M(\dot{z}^2+\omega^2z^2)+\tfrac{1}{2}I_1\omega^2+\tfrac{1}{2}I'(\dot{\theta}^2-\omega^2\theta^2). \quad . \quad . \quad (5)$$

The shaft being supposed massless, contributes nothing to T. It is, however, the seat of the potential energy, V, which is given on the ordinary Euler-Bernoulli theory by

$$V=\tfrac{1}{2}\mathrm{EI}\left\{\int_0^a (d^2y/dx^2)^2dx+\int_0^b (d^2y'/dx'^2)^2dx'\right\}. \ . \quad (6)$$

Substituting the values of d^2y/dx^2 and d^2y'/dx'^2 from (4), and carrying out the integrations, we easily find

$$V=\tfrac{1}{2}\mathrm{EI}\{3(a\theta-z)^2/a^3+3(b\theta+z)^2/b^3\}. \ . \ . \ . \quad (7)$$

Employing these values of T and V in the two Lagrangian equations

$$\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\dot{z}}\right)-\frac{d\mathrm{T}}{dz}+\frac{d\mathrm{V}}{dz}=0, \ . \ . \ . \ . \ . \quad (8)$$

$$\frac{d}{dt}\left(\frac{d\mathrm{T}}{d\dot{\theta}}\right)-\frac{d\mathrm{T}}{d\theta}+\frac{d\mathrm{V}}{d\theta}=0, \ . \ . \ . \ . \ . \quad (9)$$

we have

$$\mathrm{M}(\ddot{z}-\omega^2z)+3\mathrm{EI}z(a^{-3}+b^{-3})+3\mathrm{EI}\theta(b^{-2}-a^{-2})=0, \ . \quad (10)$$

$$\mathrm{I}'(\ddot{\theta}+\omega^2\theta)+3\mathrm{EI}z(b^{-2}-a^{-2})+3\mathrm{EI}\theta(b^{-1}+a^{-1})=0. \ . \quad (11)$$

For a vibration of frequency $k/2\pi$ we have

$$\ddot{z}=-k^2z, \qquad \ddot{\theta}=-k^2\theta.$$

Substituting for $\ddot{z}$ and $\ddot{\theta}$ in (10) and (11), and eliminating z and θ between these two equations, we have, as in (8) of § 13,

$$\{\mathrm{M}(k^2+\omega^2)-3\mathrm{EI}(a^{-3}+b^{-3})\}\{\mathrm{I}'(k^2-\omega^2)-3\mathrm{EI}(a^{-1}+b^{-1})\}$$
$$=9(\mathrm{EI})^2(b^{-2}-a^{-2})^2 \ . \ . \quad (12)$$

The term $\tfrac{1}{2}\mathrm{I}_1\omega^2$ contributes nothing to Lagrange's equations and is for this reason omitted in the text.

Assigning any arbitrary value to ω, we obtain two values of k^2—one of which may be imaginary—answering to two different types of vibration.

§ 42. In general each of the equations (10) and (11) contains both z and θ, and each root of k^2 given by (12) depends on both M and I′. There is obviously, however, a complete separation of the transverse and oscillatory movements when the load is at the centre of the span. For putting $b=a$, we have z only in (10) and θ only in (11); while the right-hand side of (12) vanishes, and we have for the transverse vibration

$$k_1^2 = 48(\mathrm{EI}/\mathrm{M}l^3) - \omega^2, \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

for the oscillatory vibration

$$k_2^2 = 12(\mathrm{EI}/\mathrm{I}'l) + \omega^2. \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Answering to $\omega = 0$ we have

$$\begin{aligned} k_2^2/k_1^2 &= \mathrm{M}l^2/4\mathrm{I}', \\ &= l^2/r^2, \end{aligned}$$

if the load be a thin circular disk of radius r.

Thus even when $\omega = 0$, k_2 will exceed k_1 unless the radius of the disk be equal to the span.

As ω increases, k_2 increases while k_1 diminishes; thus under ordinary circumstances the frequency of the transverse vibration is much the less of the two.

In the above special case it is obvious that k_2 cannot vanish, and that it is only the transverse vibration in connexion with which instability can arise. The critical angular velocity, answering to k_1 becoming nil, is given by

$$\omega^2 = 48(\mathrm{EI}/\mathrm{M}l^3). \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Even in the general case it is easily shown that one only of the two values of k^2 supplied by (12) can possibly vanish. For assuming k zero, we find the equation to reduce to

$$\omega^4 \mathrm{MI}' + \omega^2 . 3\mathrm{EI}\{\mathrm{M}(a^{-1} + b^{-1}) - \mathrm{I}'(a^{-3} + b^{-3})\} \\ - 9(\mathrm{EI})^2 (a+b)^2 / a^3 b^3 = 0, \qquad (16)$$

a quadratic in ω^2 whose roots are of opposite sign. As a negative value of ω^2 supplies an imaginary value of ω, there is only one real value of ω for which k can vanish. And as (12), regarded as an equation in k^2, cannot have equal roots, unless $b = a$, only one of the two values of k^2 can be made to vanish.

§ 43. When there is no load, and an algebraic type of vibration is assumed, the application of the Lagrangian equations is even simpler. Taking, for example, the case of a shaft supported at both ends, we have for the displacement (*cf.* (*b*) case 2)

$$y = \eta x(l^3 - 2lx^2 + x^3), \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

whence

$$T=\tfrac{1}{2}\sigma\rho\int_0^l(\dot{y}^2+\omega^2y^2)dx=\tfrac{1}{2}\sigma\rho(\dot{\eta}^2+\omega^2\eta^2)\int_0^l x^2(l^3-2lx^2+x^3)^2dx$$

$$=\tfrac{1}{2}(\dot{\eta}^2+\omega^2\eta^2)(31/630)\sigma\rho l^9, \quad . \quad . \quad (18)$$

$$V=\tfrac{1}{2}EI\int_0^l(d^2y/dx^2)^2dx=\tfrac{1}{2}EI\eta^2\int_0^l 144\,x^2(x-l)^2dx$$

$$=\tfrac{1}{2}\eta^2(24/5)EIl^5. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Lagrange's equation

$$\frac{d}{dt}\frac{dT}{d\dot{\eta}}-\frac{dT}{d\eta}+\frac{dV}{d\eta}=0$$

gives

$$(\ddot{\eta}-\omega^2\eta)\,(31/630)\sigma\rho l^9+(24/5)\,\eta EI\,l^5=0.$$

Assuming $\eta\propto\cos kt$, and so $\ddot{\eta}/\eta=-k^2$, we have (*cf.* (6), § 12)

$$k^2+\omega^2=\frac{24\times126}{31}\,\frac{EI}{\sigma\rho l^4}=\frac{3024}{31}\,(EI/\sigma\rho l^4). \quad . \quad (20)$$

§ 44. Under certain circumstances an equation of type (1), § 1, may be shown to be true for vibration frequencies. The ordinary differential equation for a frictionless simple harmonic motion is

$$Md^2x/dt^2+Fx=0, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where M is a quantity of the nature of a mass, and F a force of restitution, such as is exerted by a spring. The frequency $k/2\pi$ of the corresponding vibration is given by

$$1/k^2=M/F. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Suppose, now, that the force of restitution remains the same whether we apply one or a series of loads, M_1, M_2, &c. When the loads are put on one at a time, the corresponding frequency equations are

$$1/k_1^2=M_1/F, \qquad 1/k_2^2=M_2/F; \quad . \quad . \quad . \quad (23)$$

when put on all together we have for the frequency equation

$$1/k^2=(M_1+M_2+\ldots)/F$$

$$=1/k_1^2+1/k_2^2+\ldots \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This is analogous of course to Dunkerley's hypothesis, but it is far from amounting to a proof. Even if we assumed that what is true of transverse vibration frequencies is true of whirling velocities, we should have to prove that the addition of pulleys at different parts of a shaft is equivalent to varying the load without affecting the forces of restitution.

§ 45. The following investigation would seem to show that the result is not in general strictly true, though it may be, and not improbably often is, a close approximation to the truth.

Suppose that a massless shaft of length l, supported at its ends A and B, carries a mass M_1 at C ($AC = a$), and a second mass M_2 at D ($CD = c$, $BD = b$), the effect of the moment of inertia being negligible in either case.

Measuring x from A, and x' from B, we may assume the following types of displacement—derived by considering the bending of the shaft under the weight of the two loads :—

$$\left.\begin{array}{ll} \text{from A to C,} & y=\eta[M_1(b+c)x\{l^2-(b+c)^2-x^2\} \\ & \qquad + M_2bx(l^2-b^2-x^2)], \\ \text{,, C to D,} & y=\eta[M_1ax'(l^2-a^2-x'^2) \\ & \qquad + M_2bx(l^2-b^2-x^2)], \\ \text{,, D to B,} & y=\eta[M_1ax'(l^2-a^2-x'^2) \\ & \qquad + M_2(a+c)x'\{l^2-(a+c)^2-x'^2\}]. \end{array}\right\} \quad . \quad . \quad (25)$$

Taking ω as usual for the angular velocity, and applying Lagrange's equations, we find after algebraic manipulation

$$1/(k^2+\omega^2) = M_1\{a^2(b+c)^2/3EIl\} + M_2\{b^2(a+c)^2/3EIl\} - R, \quad (26)$$

where

$$\begin{aligned} R = {} & M_1M_2(M_1+M_2)a^2b^2c^2(4ab+4ac+4bc+3c^2) \\ & \div 12EIl\{M_1^2a^2(b+c)^2 + M_2^2b^2(a+c)^2 \\ & + M_1M_2ab(2ab+2ac+2bc+c^2)\} \quad . \quad . \quad (27) \end{aligned}$$

For the critical angular velocity answering to whirling we put $k=0$, and find

$$1/\omega^2 = 1/\omega_1{}^2 + 1/\omega_2{}^2 - R, \quad . \quad . \quad . \quad (28)$$

where

$$\omega_1{}^2 = 3EIl \div M_1a^2(b+c)^2, \quad \omega_2{}^2 = 3EIl \div M_2b^2(a+c)^2.$$

Referring to (9) or (10) §13, we see that ω_1 and ω_2 are the critical velocities for the shaft when loaded with the mass M_1 and when loaded with the mass M_2.

In order that (28) should agree with Dunkerley's hypothesis R should vanish. It is, however, obvious that R is positive for all values of M_2/M_1, and for all values of a, b, and c. It vanishes, it is true, when c vanishes, but the two loads then coincide in position. As R is positive, the application of Dunkerley's hypothesis gives a larger value for $1/\omega^2$, and so a smaller value for ω^2, than does (28).

This does not entirely disprove Dunkerley's hypothesis, because we are not entitled to assume that (25) accords absolutely with the true type of displacement, and we know from Rayleigh's general theorem that, unless this is the case, the value given by (26) for k^2 must be somewhat in excess, and consequently the value (28) for $1/\omega^2$ somewhat too low. We may however expect, in accordance with Rayleigh's general reasoning, that (26) is a very close approach to the truth; and whilst R is usually much smaller than $1/\omega_1^2+1/\omega_2^2$, it is by no means negligible, unless one of the loads be much less than the other, or one of the three lengths, a, b, and c be small.

Considering, however, the various sources of uncertainty, it must be allowed that in the present instance Dunkerley's hypothesis gives at least a fair first approximation. Taking, for example, the fairly representative case presented when the two loads are equal, and a, b, c all equal, we find

$$1/\omega^2=(15/16)(1/\omega_1^2+1/\omega_2^2).$$

§ 46. In carrying out investigations in cases where there are two, three, or more loads, the physical significance of the processes is more easily seen by adopting a generalized notation. In the above case, for instance, it will be found that the displacements at the points where the loads occur are really of the types

$$\eta'(M_1y_{11}+M_2y_{12}) \text{ and } \eta'(M_1y_{12}+M_2y_{22}),$$

where y_{11} and y_{12} are the displacements at the point where M_1 occurs, due respectively to unit loads at this point and at the point where M_2 occurs. (By a well-known general theorem y_{12} and y_{21} are equal.) The kinetic and the potential energies vary respectively as

$$M_1(M_1y_{11}+M_2y_{12})^2+M_2(M_1y_{12}+M_2y_{22})^2 \text{ and } (M_1^2y_{11}+2M_1M_2y_{12}+M_2^2y_{22}),$$

and the function which appears in the expression for $1/\omega^2$ really varies as

$$M_1y_{11}+M_2y_{22}- (y_{11}y_{22}-y_{12}^2)M_1M_2(M_1+M_2)\div\{M_1^2y_{11}+2M_1M_2y_{12}+M_2^2y_{22}\}.$$

The sign of R in (26) and (28) really turns on the sign of $(y_{11}y_{22}-y_{12}^2)$.

DISCUSSION.

Dr. GLAZEBROOK said there was one point he would like to bring out in connexion with Dr. Chree's paper. The author had referred to a theorem due to Lord Rayleigh: a theorem which he knew well in its application to sound, although he had never seen it employed in connexion with rotating shafts. The theorem is that in any vibrating system the period of natural vibrations is a stationary period, so that the application of any small external force produces only a second-order change in the period. By use of this theorem it is possible to assume some definite law and calculate periods with sufficient accuracy by working with expressions much less complicated than would otherwise be the case.

XVI. *Note on the Diffraction Theory of the Microscope as applied to the case when the Object is in Motion.* By R. T. Glazebrook, *D.Sc.*, *F.R.S.*[*]

ACCORDING to the Abbe theory of Microscopic Vision, when a grating is placed on the stage of a microscope and illuminated by plane waves, diffraction images are formed in the focal plane of the object-glass and the images in the view-plane result from these, and this is undoubtedly true.

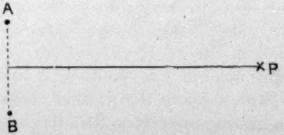
The following difficulty has, however, been raised: if the grating be moved in its own plane in a direction perpendicular to the ruling the positions of the diffraction images do not change, those seen in the view-plane move. How, then, can the latter images be due to the former?

The answer lies in the fact that in the above argument, the effect of the differences of phase among the diffracted images has been neglected.

The diffracted images are not all in the same phase, their relative phases are altered by shifting the grating and the image pattern in the view-field is altered in consequence.

Thus, if A, B be two diffraction images in the same phase,

Fig. 1.



and P any point in a line bisecting A, B at right angles, then P is a point of maximum illumination. If, however, the phases at A and B are opposite, the illumination at P is zero, the point of maximum illumination has shifted. Now it is easy to prove that as the grating is shifted the relative phases at A and B do alter.

For let L_1, L_2, L_3, L_4, be a series of consecutive bright spaces of the grating, A and B the central and first diffracted image, E, F the lens, and C, D a line at right angles to the direction of the parallel rays which go to form the first image B. And let L_1M_1, L_2M_2 . . . etc. these rays meet CD in M_1, M_2 . . . Then the time to B from any point in CD is the same, and the series of differences $L_1M_1 - L_2M_2$, $L_2M_2 - L_3M_3$ etc., are each equal to one wave-length.

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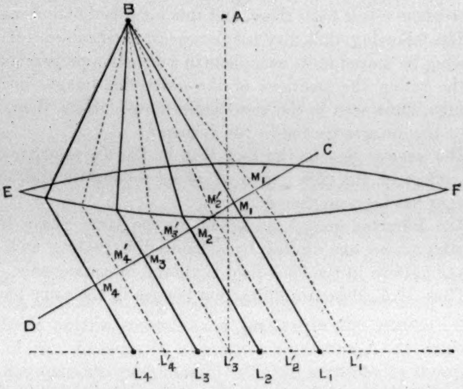
[*] Read April 22, 1904.

Fig. 2.



Let us assume that in this position A and B have the same phase. Now, suppose the grating shifted to the right through half a grating interval so that the bright spaces are now $L'_1L'_2$... midway between their former positions. This produces no change in the light coming to A but in that to B the phase is altered.

The length of path for the light from the first space has been increased by the amount $L'_1M'_1 - L_1M_1$, that from the next space by $L'_2M'_2 - L_2M_2$, and these increments are all equal and each is half a wave-length. For as we have seen $L_1M_1 - L_2M_2 = \lambda$, and since L'_2 is midway between L_1 and L_2, $L'_2M'_2 - L_2M_2 = \frac{1}{2}(L_1M_1 - L_2M_2) = \frac{1}{2}\lambda$.

Thus the length of path for each component of the light coming to B has been increased by half a wave, and the resultant phase at B is therefore altered by half a period.

Hence A and B are now in opposite phase.

But if the image in the view-plane be due to A and B only, it is clear that the dark lines in the image are by this change converted into bright lines and *vice versâ*, the image shifts in a corresponding manner to the object. If we take into account the second diffraction image, it can be shown similarly that its phase is not altered by the assumed shift, while the third is changed, and so on ; and the result is that in all cases the change in the image corresponds with that in the object.

The fact that there will be an alteration in phase of half a period at B when the grating is shifted can be seen readily thus.

With the grating in the first position light from $L_1L_2\ldots L_4$ reaches B, and the waves from these points are all in the same phase; the respective paths differ by a wave-length. When the grating is moved to its second position, $L'_1L'_2\ldots$ become the sources of the light ; the waves from these points being again in the same phase, the respective path differences being again a wave-length. If the grating is moved entirely away so that both series of points $L_1L_2\ldots$ $L'_1L'_2\ldots$ are effective, no light reaches B because the resultant phase of the disturbance from $L_1L_2\ldots$ is exactly opposite to that from $L'_1L'_2\ldots$ *i. e.*, the effect of shifting the grating is to alter the phase at B by half a period.

Thus it is seen in this simple case how it comes about that the image in the view-plane may change without an alteration in the *position* of the diffraction images, the change in phase of these images suffices to explain it.

XVII. *Note on the Measurement of Small Inductances and Capacities, and on a Standard of Small Inductance. By* J. A. Fleming, *D.Sc., F.R.S., Professor of Electrical Engineering in University College, London**.

Last year a paper was read before the Physical Society by the present writer and Mr. W. C. Clinton, on the "Measurement of Small Capacities and Inductances" †.

In that paper we described two forms of motor-driven commutator for the measurement of small capacities and inductances. Since that date, these appliances have been extensively used for this purpose in the Pender Electrical Laboratory at the University College.

In the measurement of small inductances lying in value between 100,000 and 10,000 centimetres, it is essential to use in connexion with the modification of the Anderson method ‡, described in our paper, a very sensitive galvanometer; and when small inductances of this order are being measured we have since found that the stray field from the motor employed to drive the commutator produces, by a dynamo action, a small electromotive force in the commutator which makes itself evident in the galvanometer circuit, and so gives rise to an irregularity, vitiating the results. The remedy for this, of course, is to employ an enclosed iron-clad motor, or else to place the commutator at a greater distance from the motor, connecting the two by a long shaft. This has already been done and has been found to be effective.

Meanwhile, in the course of the experiments to overcome these difficulties, the attempt was made to use a telephone in place of the galvanometer and a simple interrupted current in the battery-circuit. In the bridge arrangement described by Prof. Anderson (*loc. cit.*) we substituted an ordinary buzzer in the battery-circuit to interrupt the current at the rate of about 100 per second, and in the bridge-circuit an ordinary Bell telephone for the galvanometer, the commutator being abolished. Under these circumstances, it was found that an observer with sharp hearing could obtain a very good balance when a coil having small inductance was placed in one arm of the bridge, and a condenser of suitable capacity placed as described by Prof. Anderson (see fig. 1).

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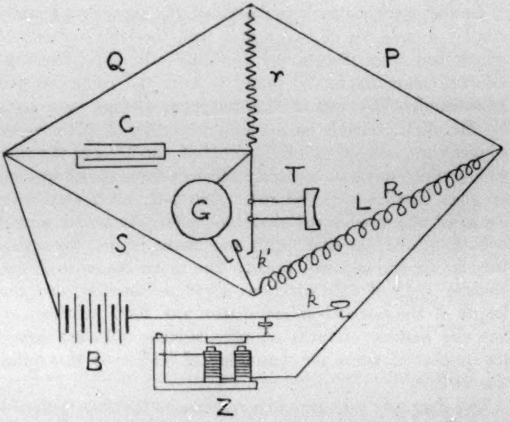
* Read March 25, 1904.

† See Proc. Phys. Soc. Lond. vol. xviii. p. 386; also Phil. Mag. May 1903, p. 493.

‡ See Phil. Mag. vol. xxxi. p. 329 (1891).

Fig. 1.



Mr. J. C. Shields, who has been engaged in experiments on this matter in the Pender Laboratory, found that with this arrangement he could make very quick and fairly accurate determinations of small inductances, the accuracy of the reading being determined by the limits within which a value could be assigned to r in the equation given by Prof. Anderson, viz.:

$$L = C\{r(R+S)+RQ\},$$

the value of r being that of a resistance inserted in the bridge-circuit, which is varied until no sound is heard in the telephone.

In the above equation L is the inductance and R the resistance of the coil being measured, C the capacity of the condenser, and S and Q the resistances of the adjacent and opposite bridge-arms, and r the resistance inserted in series with the telephone in the bridge-circuit.

When r was adjusted to produce silence in the telephone, it was found that variations to the extent of about 1 per cent. either way, and sometimes much less, caused the sound to reappear in the telephone, and hence gave the limits within which the inductance L could be determined.

In the experiments here described, the capacity generally employed consisted of two leyden-jars, the joint capacity of which had been determined carefully with the Fleming-Clinton commutator, and found to have the value 0·00272 microfarad. The tests of this telephone method were made by Mr. J. C. Shields on a number of coils of silk-covered copper wire, each of which consisted of one layer of the wire wound uniformly and in closely adjacent turns upon a wooden or glass circular-sectioned rod. One coil, much employed, consisted of a wooden rod about two metres in length wound over as above described with one layer of no. 32 S.W.G. wire in closely adjacent turns. The mean diameter of one circular turn of this wire was 4·096 centimetres, and the length of the solenoid or spiral wire was 200·3 centimetres, and the number of turns of wire 5000 in all, and hence the number of turns per centimetre of length of the spiral was 24·96.

This long coil belonged to a resonance apparatus designed by Seibt, and is hence alluded to as the long Seibt coil. The resistance of the wire on this coil was about 152 ohms, and it was connected to a Wheatstone's bridge (as shown in fig. 1), the other arms of which are denoted by P, Q, and S.

The arrangement of apparatus used, consisted therefore of an ordinary Post-Office plug Wheatstone's bridge having the spiral of which the inductance was to be determined connected to it. The battery-circuit contained the buzzer, and the bridge-circuit a telephone in series with a plug resistance-box, affording values for r. The condenser consisted of one or more leyden-jars or a mica condenser. The steady balance was obtained first in the usual way with a galvanometer and steady current.

The following Table gives the values of the bridge-arms, the bridge resistance r, the capacity used, and the inductance L calculated from the formula given by Anderson.

The Table contains two sets of measurements, one set marked A, made by Mr. Shields with the Fleming-Clinton commutator, and the other marked B, made with the telephone and buzzer as above described.

Table I.—Results of Inductance Measurements of a Long Coil, having a Dimension ratio of 50 : 1.

P.	Q.	R.	S.	r.	C. in mfds.	L. observed in cms.	
100	1000	152·26	1522·6	4260	0·00272	19,900,000	A
100	1000	152·31	1523·1	7675	0·00149	19,400,000	
100	1000	151·1	1511	4400 ± 50	0·00272	20,300,000	B
1000	10,000	151·5	1515	3350 ± 50	0·00272	19,200,000	
100	10,000	151·5	15150	365 ± 5	0·00272	19,300,000	
1000	1000	152	152	24200 ± 100	0·00272	20,100,000	
100	1000	151·4	1514	4400 ± 50	0·00272	20,300,000	
1000	10,000	151·4	1514	3330 ± 20	0·00272	19,200,000	
10	1000	151·7	15170	485 ± 5	0·00272	20,600,000	
100	10,000	151·7	15170	365 ± 5	0·00272	19,300,000	
100	100	152	152	217 ± 1	0·256	20,800,000	

Mean of A readings $= 19{\cdot}7 \times 10^6$ cms.
Mean of B readings $= 19{\cdot}9 \times 10^6$ cms.
Value calculated from the formula $L = (\pi D t)(\pi D N) = 20{\cdot}6 \times 10^6$ cms.

By numerous observations on coils of this kind, sometimes 50 diameters long or even less, the wire being wound in a single layer and in closely adjacent turns, the writer has found that a very simple formula enables the inductance to be calculated very approximately.

If D is the mean diameter of the axis of one circular turn of the wire forming the solenoid, and if l is the length of the solenoid, and N the total number of turns on the solenoid, then it is clear that the magnetic force per unit of current in the central portions of the interior of the coil is equal to $4\pi N/l$, and the total self-linked flux is $4\pi N^2 \pi D^2/4l$. Hence, if we neglect the variation of flux at the ends and consider that all up the coil it has the same value as at the centre, the inductance L of the coil is given by the formula

$$L = \pi^2 D^2 t^2 l ;$$

where t is written for the turns per unit of length of the solenoid $=N/l$.

Hence the above equation may be written

$$L=(\pi Dt)(\pi DN).$$

The first factor πDt is the length of wire wound on one unit of length of the cylindrical rod used as a core. This factor is of no dimensions and is a mere numeric.

The second factor πDN is the total length of wire used. Hence, we have for such a solenoid:—

$$\text{Inductance}=\left(\begin{matrix}\text{length of wire per}\\\text{unit length of solenoid}\end{matrix}\right)\times\left(\begin{matrix}\text{total length of wire}\\\text{used to form solenoid}\end{matrix}\right).$$

Applying this rule to the above mentioned long Seibt coil, we have for the total length of wire used

$$3{\cdot}1416\times 4{\cdot}096\times 5000\text{ cms.}=64340\text{ cms.},$$

and the length wound on per centimetre of the rod is

$$3{\cdot}1416\times 4{\cdot}096\times 5000\div 200{\cdot}3=321\text{ cms.},$$

and hence

$$L=20{\cdot}6\times 10^6\text{ cms nearly, or }20{\cdot}6\text{ millihenrys.}$$

It will be seen from Table I. that the average observed value of L for this coil, as calculated from nine bridge-readings made with the telephone method, is $19{\cdot}9\times 10^6$ cms. nearly; and hence the inductance calculated by the above rule agrees within 3·5 per cent. with that obtained by actual measurement.

This rule affords a very simple and convenient guide for constructing small known inductances. All that it is necessary to do is to wind silk-covered copper wire in one layer and in closely adjacent turns on a glass rod of measured diameter, and make the length of the solenoid at least 50 times the diameter. The inductance can then be predetermined to within say 2 per cent. and adjusted to be of required value by varying the length and diameter of the rod. A series of tuning inductances can in this manner be easily made, which, when associated with known capacities, give circuits having known oscillation frequencies.

When the coils have a smaller dimension-ratio, being only

6 or 7 diameters long, then the above rule always gives, as it should do, an inductance value which is too large, but even in the case of such short coils not by very many per cent. This may be seen from the inductance measurements made by the telephone method on four short coils called for distinction A, B, C, and D, which had the following dimensions and windings. The coils were made of silk-covered no. 36 or no. 38 wire wound on glass tubes.

TABLE II.

Coil.	Length of Solenoid $=l$.	Diameter of Solenoid $=D$.	Number of Windings $=N$.	Calculated Inductance $=L$ from the formula $L=(\pi Dl)(\pi DN)$.
A	20·5 cms.	3·3 cms.	513	1,379,800 cms.
B	20·4 cms.	3·5 cms.	813	3,918,200 cms.
C	20·8 cms.	3·4 cms.	847	3,935,200 cms.
D	20·9 cms.	3·7 cms.	850	4,670,800 cms.

The inductance of these coils was measured with the bridge and telephone, using a capacity of 0·00272 microfarad and the bridge values were as in Table III. (p. 166).

The mean values of the observed inductances of each coil differ from the extreme values in some cases by less than 1 per cent., and in no case by more than 2 per cent. The mean observed values are less (as they should be) than the values calculated by the formula $L=(\pi Dl)(\pi DN)$ by about 6 or 7 per cent.

It has always been found that when the dimension ratio is as much as 50 : 1 or more, then there is a close agreement between the observed and calculated value of the inductance. The above described telephone and buzzer modification of the Anderson method can be therefore used to calculate the value of the capacity used in the bridge, assuming the calculated value of the inductance of the inductive arm. Thus, if in the measurements recorded in Table I. we take the inductance

Table III.—Results of Inductance Measurements on Short Coils, having a Dimension ratio of 7 : 1.

(Telephone and buzzer in combination with Anderson's method.)

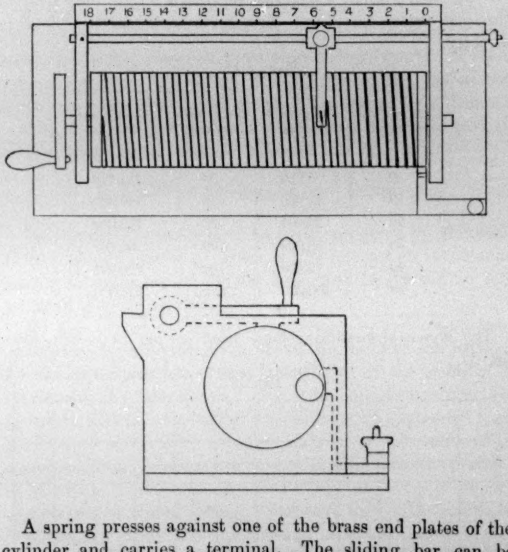
Coil.	Bridge-arms in ohms.				r. ohms.	C. mfds.	L. Inductance observed. cms.	Mean value and greatest deviation from mean.	L. Calculated by formula.
	P.	Q.	R.	S.					
A	10	1000	12·23	1223	384+2	·00272	1,321,000	1,295,000 +26,000 −13,000	1,379,800
	100	110	12·73	14	164½	·2222	1,288,000		
	100	100	13·00	13	173	·2222	1,288,000		
	10	1000	12·22	1222	370±2	·00272	1,282,000		
B	100	1000	76·9	769	1450±4	·00272	3,550,000	3,537,000 +13,000 −13,000	3,918,200
	100	1000	77·1	771	1440±10	·00272	3,530,000		
	100	100	78·0	78	51+<1	·2222	3,524,000		
	10	10	78·0	78	96±<1	·2222	3,543,000		
C	100	1000	80·3	803	1440+5	·00272	3,680,000	3,661,000 +19,000 −11,000	3,935,200
	100	100	83·0	83	49∓1	·2222	3,652,000		
	10	10	83·0	83	94∓1	·2222	3,652,000		
D	100	1000	86·5	865	1555+5	·00272	4,257,000	4,300,000 +41,000 −43,000	4,670,800
	100	100	88·0	88	60+1	·2222	4,302,000		
	10	10	88·0	88	106±1	·2222	4,341,000		

of the long coil to be 20·6 millihenrys, and use the bridge readings to calculate the value of the capacity of the leyden-jars, we find it to be 0·00282 mfd. The values of this capacity measured by the intermittent discharge method was found to be 0·00272 mfd. Hence, the bridge-telephone-buzzer method, which needs no special appliances other than those found in every laboratory, provides a means of determining with fair accuracy quite small capacities very easily, and can be applied to the measurement of the capacities of telegraph wires or aerials or very short lengths of cables.

The method has been used for the calibration of a variable inductance made for certain resonance experiments. This inductance standard was constructed as follows :—

A cylinder of boxwood about 10 centimetres in diameter and 45 cms. in length is provided at the ends with brass plates carrying centre pins, by means of which it is suspended in bearings. This cylinder is cut with a screw of 6 threads to the inch, and in this is wound a no. 14 S.W.G. copper wire, the ends of which are attached to the end plates. Parallel with the cylinder (see fig. 2) is fixed a brass rod about 1 cm. in diameter on which slides a travelling bar, the end of which carries a brush which makes contact with the copper spiral at one place.

Fig. 2.



A spring presses against one of the brass end plates of the cylinder and carries a terminal. The sliding bar can be moved along to any position, its setting being determined by a scale, and the amount of inductance included between the two terminals can therefore be varied. The sliding bar can be lifted and moved quickly from one position to another, or moved slowly by turning the cylinder, in which case a gradual variation of inductance takes place.

The inductance of the whole spiral was measured by the telephone-buzzer-bridge method, and found to be 227,000 cms., and it was also measured for every 5 or 10 turns and found to be as shown in Table IV.

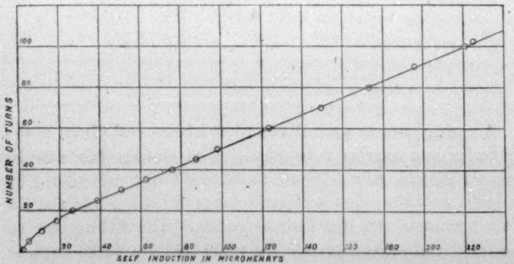
The value of the inductance predetermined by the formula above is 246,000 cms. for 100 turns, thus showing about nine per cent. excess over the real value. The above-mentioned formula cannot of course be applied to the case of a spiral having such a small dimension ratio as 4·5 : 1. The scale of the instrument has been divided to read in microhenrys directly.

TABLE IV.

No. of turns of wire.	Inductance in cms.	No. of turns of wire.	Inductance in cms.
0	0	60	124,000
10	10,000	70	148,000
20	28,000	80	172,000
30	50,000	90	196,000
40	75,000	100	220,000
50	100,000	Total	227,000

The above observations have been set out in a curve (see fig. 3).

Fig. 3.



Such a graduated standard of inductance is useful in tuning wireless telegraph-circuits and in experiments on resonance.

By the above described method the measurement of inductances, even as small as 2 or 3 microhenrys, is reduced to an extremely simple straightforward method, capable of being carried out without any special appliances, other than those found in every testing-room and laboratory. By its aid inductance as small as 10 microhenrys can be measured with an accuracy of about 5 per cent., and inductances of the order of a millihenry with an accuracy of at least 1 per cent.

A good method of constructing a small inductance of known value is to stretch two round wires of diameter *d* cms. parallel to each other at a distance D cms. apart. If these are short-circuited at the far end by a cross bar, first at one place and then at another nearer place closer by a distance *l* cms., then the difference of the inductances measured in the two cases has a value L such that

$$L=2l\left\{4\cdot606\log_{10}\frac{2D}{d}+\frac{1}{2}\right\}\text{ cms.}$$

This formula is easily derived from one given by Maxwell. It is a simple matter to obtain in this manner an inductance having a value say of 30,000 cms., and by its aid to test methods of measurement.

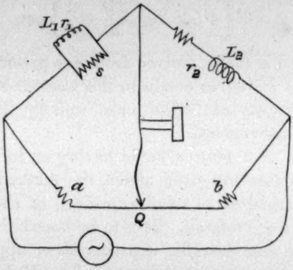
By the use of a long solenoid having an inductance predetermined by the rule given above, the method can be used for the determination of small capacities of the order of a thousandth of a microfarad. It is to be hoped, therefore, that in future those who describe experiments or appliances such as wireless telegraphy arrangements in which such small capacities or inductances are used, will cease from the practice of speaking of jars with so many "square inches or square centimetres of coated surface," and take the slight trouble to measure and record the capacity and inductances, and in this way afford the means of testing theories of the operation of the appliances.

It can hardly be said that the practical problem of measuring with great accuracy very small inductances of the order of 1 microhenry or less has been satisfactorily solved.

Probably in the case of inductances of very low resistance the best method to adopt would be to measure the fall of potential down the conductor first, with a continuous current, and then with a high-frequency sine form alternating current. Professor W. Stroud and Mr. J. H. Oates recently described a bridge method employing alternating currents, which they stated could be employed for the measurement of very small inductances *.

Discussion.

Mr. Campbell said that at the National Physical Laboratory they had lately had to measure some self-inductances of the order of 100 microhenries (100,000 cms.). The method used was practically that of Max Wien as modified by Dolezalek. As Wien's methods are not well known in this country, a short description may be of interest. In the original method the connexions are as shown in the figure.



Two contiguous arms of a Wheatstone's bridge consist of (1) the inductive coil (under test) shunted by a known non-inductive resistance, and (2) an unknown (preferably variable) inductance with an adjustable resistance in series. The other two arms (a and b) are non-inductive resistances with a slide-wire between them. The indicating instrument (either an optical telephone or a vibration galvanometer) is tuned to respond only to a particular frequency, and the

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* See Prof. W. Stroud and Mr. J. H. Oates on the "Application of Alternating Currents to the Calibration of Capacities and Inductances," Phil. Mag. ser. 6, vol. vi. p. 707 (1903).

source of current is worked at this frequency. Thus, although the source may give a quite irregular wave-form, the instrument practically ignores all but one sine-curve component, and the formulæ can therefore be easily obtained. By adjustments of r_2 and the position of the slider Q a balance is arrived at. Using the operator of the form $r+ip\mathrm{L}$ for each inductive coil, we have

$$\frac{(r_1+ip\mathrm{L}_1)s}{r_1+ip\mathrm{L}_1+s}\bigg/(r_2+ip\mathrm{L}_2)=\frac{a}{b}.$$

By simplifying this equation and equating the real and imaginary parts each to zero we obtain two equations, one of which gives $\mathrm{L}_1\mathrm{L}_2$ and the other $\mathrm{L}_1/\mathrm{L}_2$. Thus we obtain L_1 and L_2 in terms of the non-inductive resistances and p, where $p=2\pi n$, n being the frequency (alternations per second) of the current source.

The modification described by Dolezalek* consists in using a current source giving a nearly pure sinoidal wave, and as indicating instrument an ordinary telephone. The current generator is a "microphone-hummer." In this a telephone plate (P) carries a microphone (M). A current from two accumulators is passed through M and the primary of an induction-coil, the secondary being connected with a polarized magnet acting on P. For obtaining frequencies above 1000 ∼ per second I have found it desirable to replace the telephone plate by a steel bar 2·5 cms. in diameter. In order to make the note in the detecting telephone pure, I tune the latter by means of a small screw pressing against the ferrotype plate. This screw is gradually tightened till a good note is attained. Dolezalek states that, by the use of a "hummer" and a variable inductance standard, inductances of about 0·1 microhenry (100 cms.) can be readily measured to within 1 or 2 per cent.

Mr. W. Duddell expressed his interest in the paper, and asked what precautions were taken to ensure that the resistances and connexions used were free from self-induction and capacity. He thought that some of the variations in the figures given in Table I. of the paper might be due to these effects. He had used an alternator giving 100,000 ~ per second to measure small self-inductions.

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* *Zeitschrift für Instrumentenkunde*, August 1903.

Dr. **Fleming**, in reply, said that in connexion with all measurements of small inductances it was essential to continually check the measurements by the aid of a small known inductance made as he described. We were all apt to forget that the ideal "bridges" drawn on paper were not identical with real "bridges," in that the latter had small inductance and capacity in all the arms and connexions, and hence formulæ obtained on the assumption that certain branches were absolutely inductionless and without capacity were not strictly correct. In some experiments he had found that great errors were introduced merely by using ordinary flexible twin wires for connexions, as these had quite a sensible capacity per metre. In all methods employing the telephone as a detector we were testing the observer quite as much as the method, and not only a perfectly quiet room, but a very acute ear, were necessary if good results were to to be obtained. It was becoming important to possess methods of measuring accurately inductances of less than 1 microhenry. The author sketched a plan which he said he had not yet put into practice, but which seemed promising. It involved the use of a Hertz rectangle, and the measurements were made to depend upon the identity in time-period between two conducting paths connecting the same points.

FIG. 1.

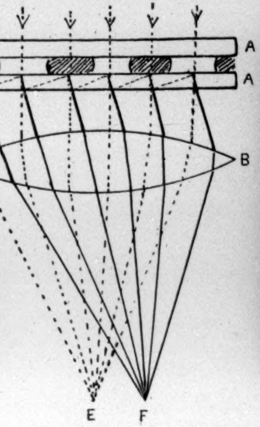


FIG. 2.

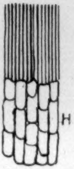
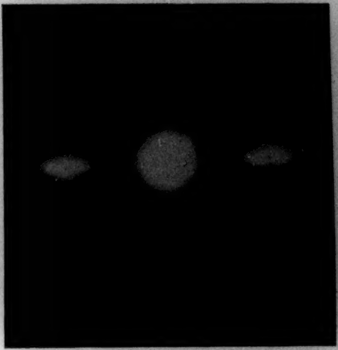
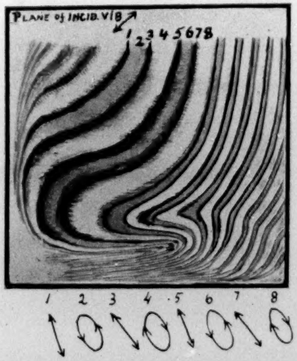


FIG. 3.



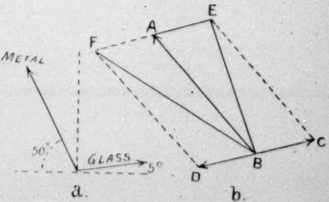
Diffraction “ Parhelia.”

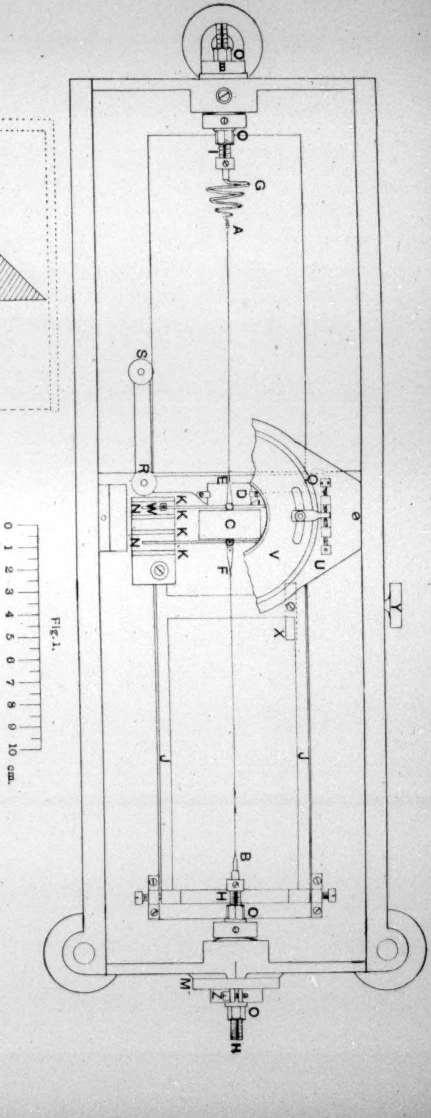
FIG. 4.


PLANE of INCID. VIB
1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8

State of Polarization along the fringes.

FIG. 5.


METAL
50°
GLASS
5°
a.
F
A
E
C
B
D
b.


O
T
G
A
S
R
E
D
C
K K K K
W
N
N
F
V
U
Q
e
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O
M
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0 1 2 3 4 5 6 7 8 9 10 cm.

Fig. 1.

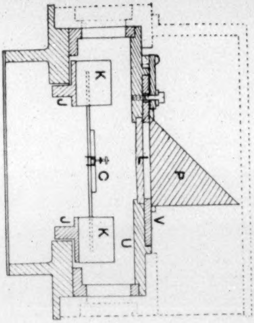


Fig. 2.

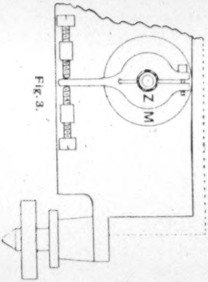


Fig. 3.

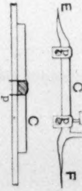


Fig. 4.

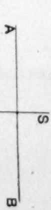


Fig. 5.

Fig. 6.

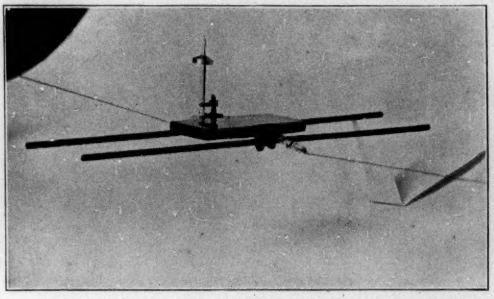
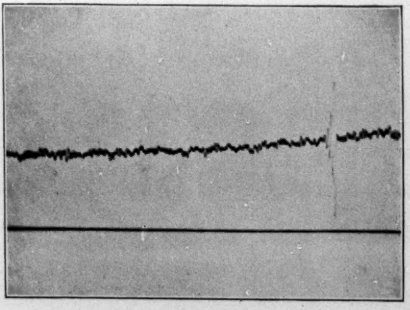


Fig. 7.



XVIII. *On a Hot-Wire Ammeter for the Measurement of very small Alternating Currents.* *By* J. A. Fleming, *D.Sc., F.R.S., Professor of Electrical Engineering in University College, London**.

There are many occasions on which it becomes necessary to measure a small alternating current of the order of one-hundredth of an ampere.

In taking the magnetizing currents of small transformers of one kilowatt size or less for the determination of the power-factor on the high-tension side, say, at 2000 volts, the magnetizing current to be measured may be at most about 0·02 of an ampere, or less in proportion to the size of the transformer.

The capacity of a small condenser or short length of cable or wireless telegraph aerial can be measured by means of a simple harmonic electromotive force, if we have the means of measuring a small alternating current and a high voltage. For if a simple harmonic electromotive force of frequency $n=2\pi/p$, and R.M.S. value V, is applied to a condenser of capacity C microfarads, then the alternating current (R.M.S. value) flowing into it has a value of $CpV/10^6$ amperes, *provided there is no sensible resonance.*

If the frequency is about 80 so that $p=500$ and if $V=2000$ volts, then a capacity as small as 1/500 of a microfarad can be measured in this manner, provided we have the means of measuring the voltage V and an alternating current of the above-named magnitude. This can of course be done by any form of electro-dynamometer adapted for measuring very small currents, but the hot-wire ammeter here described is much simpler to construct.

The following form of hot-wire ammeter can be so made as to measure currents as small as two milliamperes, and is easily calibrated at the time of using it.

The ammeter consists of a wooden box (A B, see fig. 1) 104 cms. in length, 8 cms. in height, and 6 cms. in width.

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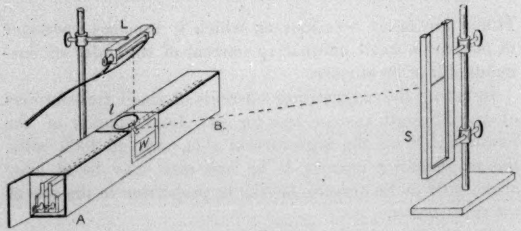
* Read March 25, 1904.

The top of this box opens on hinges, and in the centre is fixed an achromatic convex lens *l* having a focal length of 10 cms. The front of the box is cut down to form a window, W, which is glazed with a sheet of thin transparent mica (see fig. 1). In the box is fixed a square rod of well-seasoned pine, a metre in length and 2·5 cms. in width and breadth. To each end of this rod are fixed two small brass uprights to which terminal screws are attached and also small spring pieces of brass, *p p*, which are pressed in by screws passing through the uprights (see figs. 1 and 2).

Fig. 1.



To these springs at each end of the rod are attached fine wires, either of pure silver or of some high resistance alloy, such as constantan, platinoid, &c., according to the use to which the instrument is to be placed.

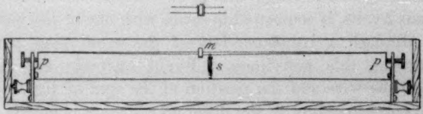
In the instrument I have already constructed, these wires are of platinoid, the length of the wires being one metre and the diameter 0·05 of a millimetre. The distance apart of these wires is about 5 millimetres. The extremities of these wires are soldered to the two spring pieces at the ends of the wooden rod, and the tension of these wires can be adjusted by means of the screws passing through the small uprights and pressing against the spring pieces.

To the centre of the wooden rod carrying the above-mentioned fine wires are fastened two very delicate spiral springs, *s*, which have their other ends looped over the long straight wires. These spiral springs are made of extremely fine platinoid wire, and they serve to keep the ammeter wires tight (see fig. 2).

If one of the wires is heated by passing a current through it, it sags down slightly. This sag is indicated in the following manner :—The two wires are embraced by an exceedingly small loop of paper *m* made from a strip of paper a couple of millimetres in width and about 12 or 15 millimetres in length.

To this loop of paper is attached with a touch of shellac a fragment of silvered microscopic glass about a couple of millimetres in width and 5 millimetres in length.

Fig. 2.



The tension of one of the wires is so adjusted that when no current is passing through either of them one wire sags more than the other, and this little loop of paper and its attached mirror sets itself at an angle of about 45 degrees to the horizontal. This is attained by slightly relaxing the tension on one of the wires. Upon the lid of the containing box is carried an incandescent lamp, having a straight or horseshoe-shaped filament, and in front of the box is placed (see fig. 1) a vertical strip of ground glass S, carried in a brass grooved frame which can be adjusted to any height on a vertical metal rod. The height of the incandescent lamp is so adjusted that the lens forms a clear image of the filament or of one leg of the filament upon the ground glass in the form of a horizontal line of light. With a good lens this image can be made very sharp. The lens actually used was the objective of an old opera-glass. A hood of metal or asbestos placed over the lamp prevents the direct rays of the lamp falling on the ground-glass screen. The screen can be conveniently placed about a metre from the wire box.

If, then, a small current is passed through the slacker of the two measuring wires, its sag will increase and the small mirror attached to the two wires will be tilted, and the image of the filament on the ground glass will move down, but return again to its original zero, as soon as the current is removed.

As a preliminary step, both the wires must be aged by sending intermittently a small current through them for a considerable time, this current being continually interrupted.

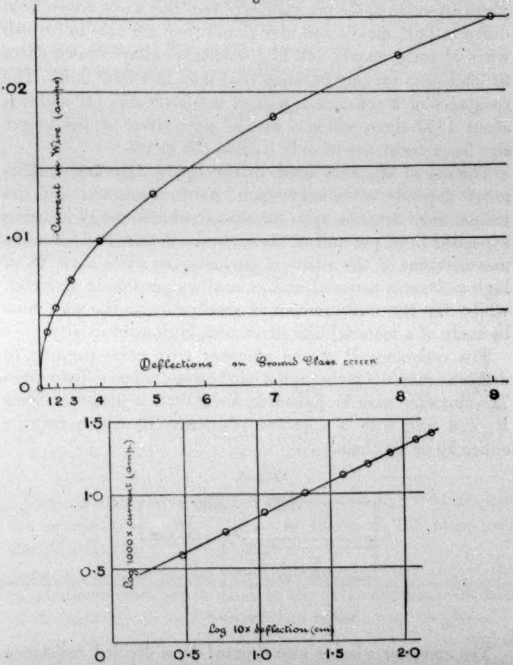
In the instrument actually made, the platinoid wires have a resistance of about 168 ohms each; hence, if an electromotive force of 2 volts is applied to the ends of the wires, a current of about 1/84 of an ampere passes through them.

The instrument is calibrated in the following manner:—A secondary cell having a measured electromotive force, say, of about 2 volts, is connected in series with one of the working wires through a resistance-box of the usual plug pattern. By varying this resistance, different currents are passed through the wire and the position of the spot of light on the screen corresponding to the different currents is noted.

If the wire employed is of platinoid or of constantan, its resistance will not be altered appreciably by different small currents passed through it, and hence the resistance of the wire can be determined once for all, with a sufficient degree of approximation for practical purposes, by means of a potentiometer. When this has once been done, a few observations taken with a cell of known electromotive force and a plug resistance-box used as above, enable the observer to mark off on the ground-glass strip with a pencil the position of the line of light for various known currents lying within a certain range. The strip of ground glass may then be removed and applied to a sheet of squared paper, and a curve plotted down showing the deflexions in terms of the actual currents. This curve proves to be a parabola (see fig. 3) because, if we plot the logarithms of the deflexions and the logarithms of the currents, we have a straight line delineated, making an angle with the horizontal, the tangent of which is equal to 2. If, then, we replace the ground-glass screen in its original position and pass through the ammeter wire any current, continuous or alternating, lying within the range of the graduation, the resulting deflexion of the line of light on the screen can be at once marked off on the ground glass, and from the curve of calibration obtained as above described the ampere value of this current becomes at once known.

In the instrument actually used the deflexion of the line of light on the scale placed at a distance of about 80 cms. from the mirror, produced by an application of two volts to the wire, is about 3 centimetres, and 4 volts produce about

Fig. 3.



12 centimetres deflexion: hence, current of about 1/100 of an ampere or 10 milliamperes produces a deflexion which can be accurately read to within 2 or 3 per cent., and a current of as small as 5 milliamperes thus can be measured.

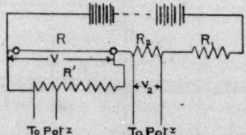
The particular class of wire with which the instrument should be strung depends on the uses to which it is to be put. If the object is to read a current of as small a value as possible, then the wire must be as fine as possible and made of a material of high specific resistance, such as constantan.

Messrs. Hartmann & Braun of Frankfurt have recently given attention to the production of very fine wires drawn from different pure metals and alloys, and they are able to furnish wires of pure metals and high-resistance alloys drawn down to diameters varying between 0·05 mm. and 0·02 mm. The resistance of a constantan wire of the latter size per metre is about 1350 ohms, whilst a wire of pure silver of the larger size has a resistance of only 8 ohms per metre.

The sag of the wire used in the above described instrument depends essentially upon its temperature, and its temperature depends upon the rate at which energy is being expended in it, per unit of its surface. Accordingly, for the measurement of the smallest currents the wires must be of high-resistance material and as small as possible in diameter, whilst for the measurement of small voltages the wire must be made of a material like silver with high conductivity.

The resistance R of the ammeter wire corresponding to different currents A through it can be determined as follows:—The ammeter wire is joined in series with a plug-resistance R_1, and also with a constant resistance R_2 which may be either 20 or 40 ohms.

Fig. 4.



The ammeter wire is also shunted by a divided resistance R′ (see fig. 4), and from a section of this resistance and from the terminals of the resistance R_2 wires are taken to a potentiometer. A battery of 100 volts is connected up, so as to send a small current through the ammeter wire, and this is adjusted until the terminal potential-difference of the ends of the ammeter wire is either 2, 4, or 6 volts as required. Let this last potential-difference be called V, and the P.D. down the resistance R_2 be called V_2. Also let the resistance of the ammeter wire under the working circumstances be R. Then V/R is the current through the ammeter wire and V/R′ the current through the divided resistance, and the sum V/R + V/R′ is equal to V_2/R_2 which is measured. Hence R can be determined corresponding to various values of V/R.

In the case of the actual calibration of the hot-wire ammeter already made, the resistance R′ was 497 ohms, and hence the resistance R was obtained from the equation

$$\frac{V}{R} + \frac{V}{497} = \frac{V_2}{R_2}.$$

The following table gives the observed values of V, V_2, R_2, and R_1, and the calculated values of the wire-resistance R and current A for the two platinoid wires (1) and (2) in the instrument already made.

TABLE I.

	Wire (1).			Wire (2).		
V (in volts)	2	4	6	2	4	6
V_2 "	0·639	1·277	0·9575	0·634	1·2675	0·9500
R_1 (in ohms)	6372	3191	2070	6415	3212	2088
R_2 "	40	40	20	40	40	20
V_2/R_2 (in amperes).	0·0159	0·0319	0·0478	0·0158	0·0317	0·0475
A "	0·0119	0·0238	0·0357	0·0118	0·0236	0·0354
R (calc.) in ohms ...	168·3	167·8	167·8	169·8	169·1	169·3

Between the above limits of current 0·01 and 0·04 ampere the resistance of wire (1) may be taken as 168 ohms and that of wire (2) as 169·4 ohms.

As an instance of the use of this ammeter and also of the precautions necessary in using it, the following measurements of the capacity of leyden-jars taken with it may be given.

Two leyden-jars were employed which will be called respectively no. 1 and no. 2. A small alternating current transformer having a transformation ratio of 1 to 118 was employed to step up the voltage of an alternator having an electromotive force curve of nearly simple sine form.

The jar to be measured was connected across the high-tension terminals of the transformer, and the hot-wire ammeter inserted so as to measure the alternating current taken by the jar. The high and low tension voltages were measured by Kelvin electrostatic voltmeters. The capacity of the jar or jars was calculated from the condenser-current A, as read on the ammeter from the expression $A=CpV/106$, where V is the reading of the electrostatic voltmeter across the high-tension terminals of the transformer. The following Table II. gives the observations :—

TABLE II.

Condenser.	Terminal Voltages of Transformer.		Frequency used $=n$.	$p = 2\pi n$.	Observed Condenser Current $=A$.	Calculated Capacity in Microfarads $=C$.
	L.T.	H.T.				
Jar no. 1 ...	97·5	13,750	73·7	463	0·0112	0·00176
	101·8	14,400	76·7	483	0·0121	0·00174
Jar no. 2 ...	97·0	13,650	73·7	463	0·0107	0·00169
	101·7	14,250	76·7	483	0·0110	0·00159
Jars nos. 1 & 2 in parallel.	97·0	15,300	73·7	462	0·0204	0·00287
	101·8	16,350	76·7	483	0·0226	0·00285

The capacities of the jars as measured by the rotating commutator described by the author and Mr. Clinton (see Proc. Phys. Soc. Lond. vol. xviii. p. 386, also Phil. Mag. May 1903, p. 493) were as follows:—Jar no. 1=0·00154 mfd., Jar no. 2=0·00124 mfd., Jars nos. 1 & 2 in parallel=0·0028 mfd.

Hence it will be seen that the hot-wire ammeter gave a capacity in close approximation to that given by the commutator method for the joint capacity of the jars, but that there was an apparent increase of capacity when the jars were measured separately by the ammeter method. This is clearly due to resonance. The changes in the transformation ratio of the transformer due to the addition of the capacity are apparent from the figures in Table II. Hence, in using this method, the precaution must always be taken of having sufficiently large capacity in parallel with the one to be measured to destroy resonance, but the current into the capacity, which it is desired to measure, is alone measured by the hot-wire ammeter.

In connexion with this subject, I have had occasion to notice the manner in which even the small capacity of a Kelvin high-tension voltmeter will, in conjunction with the inductance of some transformers of high inductance, produce a marked alteration of the change ratio of transformation, by reason of resonance, when the transformer is used at various voltages. The small capacity of a high-tension electrostatic voltmeter across the high-tension terminals of a transformer, will, as shown by the figures in Table III. below, sensibly affect the change-ratio.

The transformer tested was worked at a frequency of about 80, and the voltage of the high-tension and low-tension terminals measured by electrostatic voltmeters over a certain range of voltage, no other load being on the high-tension side.

The change-ratio is far from constant, as shown by the figures in Table III., and this is clearly due to the resonance produced by the capacity of the high-tension voltmeter. Under such circumstances, when brush-discharges are taking place in the interior of the voltmeter it possesses a spurious and increased capacity, and hence, in using electrostatic voltmeters, a single reading should not be accepted too readily as giving the change-ratio of a transformer.

TABLE III.

Transformer Terminal Voltages measured by Kelvin Electrostatic Voltmeters.		Transformation Ratio. Frequency =80.
H.T.	L.T.	
16,350	515	31·7 : 1
16,650	503	33·1 : 1
16,850	460	36·6 : 1
16,900	435	38·8 : 1
16,750	415	40·3 : 1
15,000	360	41·6 ; 1
12,000	303	39·6 : 1

As an instance of the application of this hot-wire ammeter in measuring the magnetizing currents of small transformers, the following figures may be given:—A transformer of 1 K.W. size had the hot-wire ammeter above described placed in series with its primary circuit, and as a check and test of accuracy a known high resistance R of 3450 ohms was also put in series with the hot-wire ammeter, and a Kelvin electrostatic voltmeter was connected to the terminals of this high resistance. The following readings were then taken:—The voltage V at the primary terminals of the transformer, the voltage V′ at the terminals of the high resistance R, and the deflexion of the hot-wire ammeter, were found to be as follows:—

$$R=3450 \text{ ohms}, \quad V'=114{\cdot}5 \text{ volts}. \quad V=2000 \text{ volts}.$$

Hence the magnetizing current of the transformer was $114{\cdot}5/3450=0{\cdot}033$ ampere.

The hot-wire ammeter was calibrated as above described, and the reading of the magnetizing current taken with it was found to be 0·034 of an ampere. Hence the value of the same current as determined by the electrostatic voltmeter, and the high resistance by the ratio V/R and that determined by the hot-wire instrument were in very fair agreement.

The hot-wire ammeter has, however, the advantage over the high-resistance and electrostatic voltmeter method, that it is easier to calibrate and also involves a less drop in voltage, the more so as the current to be measured is smaller.

This form of hot-wire ammeter has many uses. It can be employed as a relay operated by small alternating currents to set in action appliances only workable with continuous currents. In this case the sagging wire is made to drop a steel needle on to a mercury surface and thus close the circuit required.

As a research instrument, it may be useful because it appears that the deflexion of the ray of light created by the sag of the wire within useful limits of working is almost exactly proportional to the square of the current passing through the wire, and hence standardization is effected by passing one current of known value through the instrument, whilst if a suitable wire is employed, it can be calibrated with sufficient accuracy for most technical purposes by applying to the ends of the ammeter wire 2, 4, or 6 volts obtained from 1, 2, or 3 secondary cells, if need be a suitable resistance being interposed. For measuring alternating currents of a few milliamperes for electromedical purposes, it should also be useful.

When currents larger than a few milliamperes have to be measured, it can easily be accomplished by the same fine wire ammeter, by shunting the terminals with a resistance either equal to or one-half, one-third, or one-*n*th of that of the wire resistance. Thus, if the above described ammeter is shunted with 168 ohms, or with 84 ohms, or 42 ohms, then the ammeter readings have to be multiplied by 2, by 3, or by 5, to evaluate the whole current, and if shunted by a shunt of one-*n*th of its own resistance, then the readings have to be multiplied by $n+1$.

The above described ammeter has been skilfully made for me by my assistant, Mr. A. Blok, who has also carried out for me the measurements described.

DISCUSSION.

Mr. W. DUDDELL expressed his interest in the paper and said it was difficult to make a satisfactory instrument. The author had said that with a fine constantin wire it was possible to measure currents as small as 2 milliamperes. He asked if this was calculated or observed. There should be a standard method of comparing these instruments, and he suggested using the current necessary to give a deflexion equal to one-quarter of the scale-distance. Mr. Duddell pointed out that many dynamometers were more sensitive than the instrument shown, and asked the author the order of magnitude of the currents which it was necessary to measure in aerial conductors.

Dr. W. WATSON exhibited and described a form of ammeter for small alternating currents. The current to be measured flows through a piece of fine iron wire bent into the form of a right angle. This is linked with a similar shaped piece of nickel wire forming part of a galvanometer circuit. The thermo-E.M.F. at the junction, produced by the heating effect of the current, sends a current through the galvanometer which can be measured in the usual way. The current to be measured is practically proportional to the deflexion of the galvanometer.

Mr. A. CAMPBELL described a form of ammeter by Rubens; and Mr. W. A. PRICE referred to an instrument similar to that shown by the author, in which the sag of the wire is measured by the movement of an aluminium pointer.

Dr. FLEMING said he was quite aware that electrodynamometers had been designed for measuring very small alternating currents, but they were more difficult to make and calibrate than the simple form of hot-wire instrument he now exhibited. He had no doubt that the use of a constantin wire of 0·02 mm. diameter would greatly increase the sensitiveness of the instrument, though he had not actually tried it for want of time. As regards Mr. Duddell's question about the magnitude of the currents flowing into a Wireless Telegraph aerial, that of course depended upon the size of the aerial. In the case of an ordinary 180 foot single aerial the average value was about 0·5 to 1 ampere, but that depended of course on the frequency of the groups of oscillations. The author said that although in the ammeter he exhibited the sag of the working wire was indicated by a mirror, yet he had also used an index-needle, and he had no doubt a good commercial instrument might be made on this plan.

XIX. *Energy of Secondary Röntgen Radiation.* *By* CHARLES G. BARKLA, *M.Sc.* (*Vict.*), *B.A.* (*Cantab.*), *King's College, Cambridge*; *Oliver Lodge Fellow, University of Liverpool*[*].

IN a paper on "Secondary Radiation from Gases subject to X-rays"[†] experiments were described which led to the following conclusions:—

All gases subject to X-rays are a source of secondary radiation, the nature of which is similar to that of the primary radiation. The absorbability of the secondary radiation is (within the limits of possible error—about 10 per cent. of the absorption coefficient for aluminium) the same as that of the primary radiation producing it.

For a given primary radiation the intensity of secondary radiation from different gases at the same pressure and temperature is proportional to the density of the gas from which it proceeds.

The opinion was expressed that the secondary radiation is due to a kind of scattering of the primary by the corpuscles constituting the molecules of the gas.

Results similar to those which led Sagnac[‡] to conclude that the secondary radiation from air was more absorbable than the primary radiation producing it had been obtained, but the evidence was then considered insufficient to lead to a definite conclusion as to the difference in character of the two radiations. A direct method of comparison did not indicate the slightest difference in the absorption of the primary and secondary radiations by similar plates of aluminium.

As the experiments of Townsend[§] and Sagnac[‖] on secondary radiation from metals showed that this radiation was more absorbable than the primary radiation producing it—the change in penetrating power, however, depending on the metal—and as the results referred to led to the probability of a transformation of the radiation by air, further and more careful experiments were made on the subject.

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[*] Read March 25, 1904.

[†] C. G. Barkla, Phil. Mag. [6] v. p. 685 (1903).

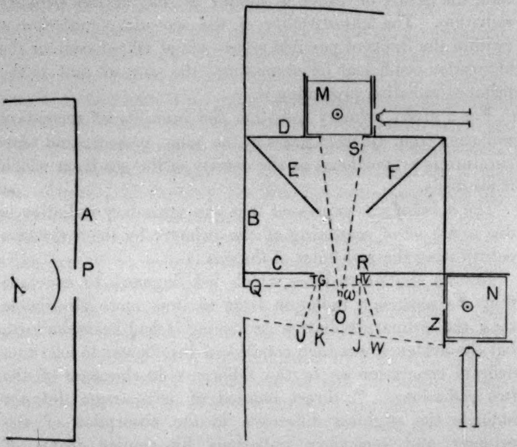
[‡] *Comptes Rendus*, cxxvi. pp. 521-523 (1898).

[§] Proc. Camb. Phil. Soc. x. p. 217 (1899).

[‖] *Comptes Rendus*, cxxv. p. 942 (1897).

The following method was employed :—

A beam of X-rays passed through rectangular apertures in two parallel lead screens A and B (see figure). Two screens, C and D, were placed in planes perpendicular to the others, in positions shown in the figure, the screen C being just outside the primary beam and D parallel to C at a distance of 15 centimetres.



In C and D were two square apertures, R and S, 5 cm. sq., placed so that the lines joining corresponding points of the two were perpendicular to their planes.

Behind the aperture S was placed an electroscope M* which was carefully shielded from radiations proceeding from all directions except through the two apertures R and S. This radiation entered by a thin paper and aluminium window. Lead plates E and F protected it from secondary and tertiary radiation from metals. The rate of fall of the gold-leaf in electroscope M was then only affected by secondary radiation from air in the primary beam opposite aperture R.

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* For description see previous paper.

A second similar electroscope N was placed behind thick leaden screens Z in the primary beam, and received a narrow pencil of primary radiation through a small hole in this screen. Absorbing plates could be placed immediately in front of both electroscopes. The rays then entered the primary electroscope N by passing normally through the absorbing plate. The secondary rays of greatest obliquity entering the electroscope M, in their passage through the plate suffered absorption which would be produced by going normally through a plate of thickness $t(1{\cdot}106)$, t being the actual thickness of the plate.

The average effective thickness of the plate was not more than $1{\cdot}03\,t$. To compare the absorbability of the secondary rays with that of the primary, the ratio of the deflexions of the two electroscopes, consequent upon the ionizations produced in these electroscopes by the primary and secondary radiations, was first obtained. From the observation of the deflexion of one of these, the deflexion of the other could then be calculated to within about 3 per cent.

Plates of aluminium of equal thickness were then placed before the electroscopes, so that both the primary and secondary beams suffered absorption by transmission through aluminium. The ratio between the two deflexions consequent upon the passage of the primary and secondary beams through the electroscopes was again determined. In measuring the deflexions, corrections were always made for the result of the normal ionization taking place in the electroscopes. Many experiments were made with rays differing in penetrative power. In some of these the possible error was very small, and they showed that within 4 or 5 per cent. the absorption coefficients for the primary and secondary radiations were the same. Some of the results are given in Table I. (p. 188).

In the second, third, and fourth sets of experiments given in the table, the order of magnitude of the absorption may be seen from a comparison of the duration of experiment and deflexion of the primary electroscope in each case.

A sheet of aluminium of ·04 cm. thickness reduced the rate of ionization in the electroscope by about 60 per cent.

TABLE I.

Conditions of Exp.	Duration of Exp.	Deflexion of Primary Electroscope.	Deflexion of Secondary Electroscope.	Ratio of Secondary Deflexion to Primary Deflexion.
Without absorbing plates	minutes. 24	31·05	10·6	34·1 to 100
2 sheets of ·01 cm Al before each electroscope	28	31	10·45	33·7 to 100
Without absorbing plates	26	30·4	10·5	34·5 to 100
Without absorbing plates	21	34·75	10·3	29·6 to 100
Sheet of ·04 cm. Al before each electroscope	50	30·4	8·15	26·8 to 100
Without absorbing plates	30	34·6	9·7	28 to 100
Without absorbing plates	25	29·05	11·3	38·9 to 100
Sheet of ·04 cm. Al before each electroscope	40	19·5	7·5	38·4 to 100
Without absorbing plates	30	23·75	9·5	40 to 100
Without absorbing plates	25	26·7	11·6	43·4 to 100
Sheet of ·04 cm. Al before each electroscope	46	20·1	8·45	42·4 to 100
Without absorbing plates	20	21·2	9·2	43·4 to 100

In the first set the intensity of radiation was altered in the different experiments, so that the duration of experiments was varied little. The ionization in an electroscope was diminished by about 40 per cent. by a sheet of aluminium of ·02 cm. thickness.

From the results given in the fifth column, it will be seen that in each case the rate of ionization in the secondary electroscope was diminished by the aluminium plate slightly more than that in the primary electroscope. But the effective thickness of the plate before the secondary electroscope was probably between 2 and 3 per cent. greater than that before the primary, so that the actual difference in the absorptions must have been exceedingly small if any difference existed.

Corrections have been made for the normal ionization in each electroscope.

In order to test this equality in penetrative power for rays differing enormously in character from those previously dealt with, experiments were made with secondary radiation from metals to get a secondary beam of sufficient intensity to use as a primary in experiments similar to those described. The radiation from copper was found to be sufficiently intense to use as a primary beam, as its great ionizing power in some degree compensated for the weakness of the radiation. It was, however, impossible to adopt the method employed with the more intense primary beams, as the ionization produced by what was in this case a tertiary radiation in electroscope M, in the position shown in the figure, was only a small fraction of the normal ionization when the electroscope was not in the path of any known radiation. Instead of placing the electroscope M so far away from the primary beam in order to get the rays normal to the absorbing plate, and so greatly diminishing the energy of secondary radiation entering the electroscope, it was put immediately behind the aperture in screen C. The effective thickness of the absorbing plate was then much greater than before, as most of the radiation passed through it in a direction making a considerable angle with the normal. Before any accurate results could be obtained of the absorption of this secondary radiation, it was necessary to determine the effective thickness of the plate.

First using a primary beam consisting of rays of moderate penetrative power, such as had been experimented upon previously, it was possible, by comparing the rates of ionization in electroscope M when no absorbing plates were used, when an aluminium plate was placed in the primary beam by aperture P, and when the same plate was placed in front of the electroscope, to calculate, on the assumption that the beam was homogeneous in character, the effective thickness of a plate placed before the secondary electroscope. By using thin plates, the error due to the primary beam being a mixture of radiations differing in character was reduced to a minimum.

To obtain the very absorbable primary beam a copper plate was placed in the induction-coil box opposite the aperture, with the X-ray bulb as close as possible without emitting any direct radiation through the aperture Q. The secondary radiation from copper then constituted the new primary beam, and the tertiary radiation from air was the new secondary radiation. The boundaries of this primary beam were approximately the same as before, so that, neglecting the small error due to want of homogeneity, the effective thickness was as before. From the diminution in the rate of ionization in M when an absorbing plate was put at P, the corresponding diminution when the plate was placed at R was calculated on the assumption that the secondary and primary radiations again had the same penetrative power. This was then compared with the actual absorption produced by the plate at R.

As this test would only hold with accuracy provided the primary and secondary beams consisted of rays of one character, the thickness of the absorbing plate at P which produced the same absorption as a plate of given thickness at R was determined experimentally for the moderately penetrating rays, and the same comparisons were made as before.

This method was still open to the objection that the effective thickness of the plate at R determined experimentally for rays coming direct from an X-ray bulb was not the same as that for a primary beam coming from a copper plate, for the radiation from metals is more heterogeneous than the primary radiation producing it. The two methods, however, would give different results if want of homogeneity affected them to an appreciable extent.

Both experiments gave within the limits of possible error the same absorption for the secondary as for the primary, showing that even for these very easily absorbed rays, the character of the secondary radiation differs little from that of the primary producing it.

The possible error was in this case naturally much greater than in the previous experiments, and amounted to fully 15 per cent. of the absorption coefficient.

Comparisons were then made between the rates of ionization produced in the two electroscopes when neither primary nor secondary beam was intercepted by absorbing plates, when similar aluminium plates were placed in the primary and secondary beams just before the electroscopes, and when an aluminium plate was placed in the primary beam at the aperture P. As stated previously, the same ratio was given (1) when no plate intercepted either beam as (2) when plates of equal thickness were placed before each electroscope. (3) When a plate was placed in the primary beam at P, the ratio of the rates of ionization in the two electroscopes secondary to primary was increased.

In these experiments the ratio was increased by about 10 per cent. when an aluminium plate ·04 cm. thick was placed at P. This plate reduced the ionization in the primary electroscope to about 36 per cent. of its initial value, while the ionization produced by the transmitted radiation was reduced to about 54 per cent. by a second similar plate.

Experiments (1) and (3) showed that the rays which got through the aluminium produced a greater proportional secondary effect than the whole direct primary beam. This result must have been due either to a change in the intensity or in the character of the primary beam in its passage through aluminium. There was a possibility that the intensity of secondary radiation was not proportional to that of the primary radiation, even though the radiations were of the same character; and thus that the diminished intensity accounted for the proportional increase in the ratio of secondary to primary ionization. A given bulb was therefore worked in very different ways, currents of various strengths being passed through, so that in some cases the intensity of radiation was four or five times that in others. The differences in penetrative powers in the different experiments were small. It was found that the ratio of the rates of ionization was practically the same in each case, showing it to be independent of the intensity of primary radiation.

The increase in the ratio of secondary to primary ionization by placing an absorbing plate at P was therefore due to the change in the character of the primary radiation by transmission through the plate. But it was shown to be independent of the position of the plate in the primary beam, as the ratio was unaltered when the plate was moved from P to Q. Hence the effect of secondary radiation from aluminium was negligible.

The conclusion was then that the rays of higher penetrating power produced a greater secondary ionization in proportion to the primary ionization produced by them than the more absorbable rays. This, again, could not have been due simply to scattering of these constituents in different proportions, for the results of experiments (2) and (3) would then have been identical. We thus arrive at the conclusion that there must have been a greater transformation* of the penetrating than of the more absorbable rays, the increase in ionizing power accounting for the proportional increase in the secondary ionization.

It should be noticed that experiments (2) and (3) do not in any way give a comparison of the absorption of the primary and secondary beams. If the primary beam were homogeneous in character, *i. e.* consisted of rays of one penetrating power, then the ratio of the secondary ionizations in the two cases would be the ratio of the fractions of primary and secondary radiations transmitted by a plate of the thickness used. Such a test would, however, be no more delicate than that given by experiments (1) and (2). If the different constituents of the primary beam were transformed equally, then again this test would be no more delicate than that given by (1) and (2).

The result depends merely on the difference in the transformation of the penetrating and of the more absorbable rays.

If the absorption by aluminium of different radiations were proportional to the ionization produced by these radiations in a given volume of air, then the absorption by aluminium would have been easily measured.

We thus conclude that for the rays experimented upon a variation in the absorption by aluminium is accompanied by a much greater proportional change in the ionizing power in air.

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* The measure of transformation used is the ratio of the ionization produced in a given volume of air by the transformed beam to that produced by the primary beam if of the same intensity. This is not the ratio of the absorption coefficients for the rays by air, as the total absorption is not proportional to the ionization, but it is the ratio of what may be called the ionizing powers.

As it was evident that there was a greater transformation of the more penetrating rays, further experiments were made to detect if possible a difference in the absorption of the secondary rays produced by a primary beam of intensely penetrating rays, and of the primary rays themselves, by plates of aluminium of equal thickness. As the average penetrating power of the radiations from different bulbs used did not differ sufficiently, one bulb was used, and the more absorbable rays were cut off by plates of aluminium. As the intensity of the primary and secondary beams was considerably reduced and the percentage possible error consequently increased, the accuracy of the former experiments was not attained. These experiments, however, seemed to indicate a slightly greater absorption of the secondary radiation than of the primary.

Having proved the approximate equality in penetrative powers of the primary and secondary radiation from air, it became a simple matter to investigate the relation between the energy lost in secondary radiation by a primary beam in its passage through air and the penetrating power of that radiation. The apparatus was arranged as shown in the figure, but with plates C and D much nearer. The ratio of the rates of deflexion of the secondary and primary electroscopes was determined for a given bulb in a fixed position when the current through it was varied, also when the bulb was soft and when hard. The ratio of secondary to primary ionization was apparently slightly greater for the penetrating than for the more absorbable rays. To obtain a beam of very absorbable rays the secondary radiation from copper was again used as the primary. A copper plate was placed as nearly as possible in the position previously occupied by the anticathode of the bulb, so that the boundaries of the beam were approximately the same as before. This was not obtainable with perfect accuracy, because in the former case the radiation had proceeded from practically a point source, while to obtain a beam of sufficient intensity from the copper, a plate several square centimetres in size was used. The difference in the two cases was, however, not considerable.

The ratio of secondary to primary ionization was again determined, and was found to be of the same order as when the direct beam from the bulb was used as the primary.

The actual readings obtained are given below, but as the ionization produced in the secondary electroscope was only of the same order of magnitude as that occurring under normal conditions, the possible error was great.

TABLE II.

Primary Radiation in Experiment.	Duration of Experiment.	Primary Electroscope Deflexion.	Secondary Electroscope Deflexion.
Rays direct from bulb	minutes. 1·3	164·6	8·9
Secondary rays from copper ...	50	34·4	3·8
None	60	2·4	1·8
Rays direct from bulb	·5	187	7·5
Secondary rays from copper ...	60	71	5·5
None: screened by lead plate...	40	1·4	1·4

After correction for the normal ionization, and for absorption of the secondary rays from copper in their passage from aperture R to electroscope N of the order of 10 per cent., these readings were :—

Primary Radiation.	Ratio of Primary to Secondary Electroscope Deflexion.
Rays direct from bulb	100 to 5·4
Secondary rays from copper . . .	100 to 6·4
Rays direct from bulb	100 to 4
Secondary rays from copper . . .	100 to 4·4

As the boundaries of the beam proceeding from copper were necessarily not so well defined as those of the beam direct from the bulb, on account of the much greater area of the source of radiation, the two were not accurately compared, so that the agreement between these results is quite as close as could have been expected. Quantitative results were thus obtained, showing that the secondary radiation from copper is of the same nature as the primary radiation, for the secondary radiation is productive of a tertiary radiation whose intensity bears the same relation to the intensity of the secondary as that of a secondary from air bears to the intensity of the primary producing it.

It has been shown that the intensely penetrating rays transmitted through sheets of aluminium give a slightly higher ratio of secondary ionization to primary ionization than the more absorbable radiation; and that this result is due principally, if not entirely, to the greater difference in ionizing power between the secondary and primary radiations when penetrating rays constituted the primary beam than when it consisted of more absorbable rays. It is not due to a greater fraction of the energy going as secondary radiation.

We are thus led to the conclusion that—

The energy of secondary radiation from a definite mass of air through which a primary beam of uniform intensity passes is a definite fraction of the energy of the primary beam, whatever be the penetrative power of the primary radiation; or the intensity of secondary radiation depends on the intensity of the primary radiation, but not on the character of that radiation. This verifies the theoretical result given by J. J. Thomson ('Conduction of Electricity through Gases,' p. 271).

Energy of Secondary Radiation.

In the passage of Röntgen radiation through gases, part of the energy of this radiation is spent in the production of ions and subsequently appears as heat in the gas when the ions recombine, part appears more directly as heat, and part is productive of secondary radiation. The approximate proportionality between the absorption and the ionization led to the assumption that practically all the energy was spent in the work of ionization; but the experiments of H. A. Wilson * and of Townsend together with those of Rutherford and McClung †, giving the energy required to ionize a molecule and that absorbed per molecule ionized respectively, lead to the conclusion that only a small fraction of the energy absorbed is spent in the work of ionization. The energy of secondary radiation has been regarded as negligible. The experiments described show that the secondary radiation from gases differs little in absorbability from the primary radiation. It is hence an easy matter to compare the intensities of the secondary and primary radiations, for these are proportional to the ionizations produced by them in equal volumes of the same gas through which they pass, and hence the total energy of secondary radiation may be compared with that of the primary.

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* Phil. Trans. A. 197, p. 415 (1901).

† Phil. Trans. 196, p. 25 (1901).

The electroscope used to measure the intensity of the secondary radiation was placed behind a square aperture 5 cm. by 5 cm. in a lead screen, in such a position as to receive the secondary rays proceeding from the air in a direction approximately at right angles to the direction of propagation of the primary rays (fig. p. 186).

The solid angle subtended by this aperture at the centre of the source of secondary radiation distant about 18·5 cm. was approximately

$$\frac{5^2}{(18{\cdot}5)^2} = \left(\frac{1}{3{\cdot}7}\right)^2 = \omega \text{ say.}$$

Evidence was given in a previous paper leading to the theory that this radiation has its origin in the charged corpuscles constituting the molecules during the passage of X-rays through the gas. The perturbations of these charged corpuscles by the strong electric field in the Röntgen pulses are in directions perpendicular to that of propagation of the primary rays, and the intensity of secondary radiation proceeding from a single corpuscle is different in different directions, being expressed by

$$\frac{1}{4\pi}\frac{e^2 f^2 \sin^2\theta}{\mathrm{V}\,.\,r^2}\text{*},$$

e being the charge on the corpuscle, f the acceleration of that corpuscle, V the velocity of light, r the distance of the point considered from the corpuscle, and θ the angle between the line joining these two with the direction of acceleration of the corpuscle.

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* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 269.

The rate at which energy is radiated from the corpuscle is

$$\frac{2}{3}\frac{e^2f^2}{V}\text{*},$$

Consequently the energy received by a small area subtending a solid angle ω' at the corpuscle =

$$\frac{3\omega'}{8\pi}\sin^2\theta \text{ of the total energy of secondary radiation proceeding from that corpuscle.}$$

Now, as the electric displacements in the Röntgen radiation are in all directions perpendicular to the direction of propagation, and neglecting partial polarization of the primary beam †, if the line joining the small area to the source of secondary radiation be in the plane of electric displacements, the energy received by this area=

$$\frac{3\omega'}{8\pi}\cdot\frac{1}{2} \text{ of the total energy of secondary radiation}$$

proceeding from that corpuscle (the average of $\sin^2\theta$ being $\frac{1}{2}$).

The energy of secondary radiation passing through the secondary electroscope was therefore

$$\Sigma\,\frac{1}{2}\cdot\frac{3}{8\pi}\,\omega'.E_x,$$

the summation being over all elements of the aperture and all the corpuscles which contribute to the radiation through that aperture.

The effect of radiation from the air in the space G H J K is approximately the same as if all the corpuscles in that space radiated from the central point O. The angle subtended by aperture S at any corpuscle in the outer space shown in the figure depends on the position of the corpuscle, varying from ω just outside the space G H J K to zero at the outer boundaries of T U V W.

A calculation giving a result which can only be regarded as approximate, showed the total effect to be equivalent to that of all the corpuscles in a volume of air equal to about 194 c.c. radiating from O.

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* Through an error in integration this is given as $\frac{1}{3}\frac{e^2f^2}{V}$ in 'Conduction of Electricity through Gases.'

† C. G. Barkla, 'Nature,' March 17, 1904.

Therefore the energy of secondary radiation passing through electroscope M

$$= \frac{3}{16\pi}\omega E \triangleq \frac{1}{230}E,$$

E being the total energy of secondary radiation from 194 c.c. of air.

The intensities of the primary and secondary beams were compared by using apertures of various sizes for the primary and secondary electroscopes, and comparing the areas of the two apertures which gave the same rate of leak in similar electroscopes. The intensities of radiation were then inversely as the areas of the apertures. As the electroscopes were not exactly similar they were standardized by finding the rate of leak produced in each by a beam of given cross section and intensity. The proportionality between the ionization produced in a given electroscope and the area of the aperture through which the beam was admitted was verified, so that local effects, as of a small portion of the beam being intercepted by the gold-leaf itself, were negligible.

The result was

$$\frac{\text{Intensity of Sec. Beam at S}}{\text{Intensity of Prim. Beam at R}} = \cdot 00000816$$

$$\therefore E = \cdot 00000816 \times 230 \times \{\text{Energy of Primary Beam of same cross section } (5^2 \text{ cm.})\}$$

Now the length of the primary beam passing through air and of cross section 25 sq. cm. giving rise to this secondary radiation

$$= \frac{194}{25} = 7 \cdot 8 \text{ cm.}$$

Hence the energy of secondary radiation proceeding from 1 cm. length of the primary beam passing through air at atmospheric pressure and about 15° C.

$$= \frac{\cdot 00000816 \times 230}{7 \cdot 8}$$

$$= \cdot 00024 \text{ of energy of primary radiation passing through.}$$

We thus see that the energy lost by secondary radiation is a considerable fraction of the total absorption, which is of the order ·001 of the energy of the primary beam per centimetre.

As this radiation is most intense in the direction of propagation of the primary radiation, the primary beam is reinforced by a portion of the secondary so that the total loss of intensity of a beam whose breadth is not small in comparison with its length is not so much as indicated above.

It was shown in the previous paper that the energy of secondary radiation is proportional to the density of the gas through which the primary beam passes. When we take into consideration the loss of energy due to this secondary radiation, a considerable fraction of the absorption coefficients is accounted for and the ratio of ionization produced to absorption proper (neglecting the absorption due to secondary radiation) is more nearly a constant than the values given by Rutherford show. I think that the variations from this were all within the limits of experimental error *.

The close agreement between the ratio of absorptions found by Rutherford and the ionizations found by J. J. Thomson † for carbonic acid gas and air is accounted for by the fact that for the gases named, the ionizations for a given radiation are proportional to the densities of the gases. But the energy of secondary radiation is also proportional to the density of the gas, hence the sums of the energies lost in ionization and by secondary radiation are in the same ratio.

We thus find that the energy of Röntgen radiation lost by secondary radiation from gases at atmospheric pressure and temperature is very large compared with the energy lost in the work of ionization.

From the results of experiments of Townsend and H. A. Wilson with those of Rutherford and McClung, J. J. Thomson ‡ concluded that about $\frac{1}{80}$ of the energy of the rays is expended in the ionization of the gas, the rest being converted into heat. We see, however, that a large fraction—for moderately absorbable rays of the order $\frac{1}{4}$—goes in secondary radiation. In the case of very penetrating rays the fraction is much larger.

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* Phil. Mag. [5] xliii. p. 241 (1897).

† 'Conduction of Electricity through Gases,' p. 251.

‡ *Ibid.* p. 255.

J. J. Thomson has shown that if Röntgen radiation passes through a medium in which there are N ions per cubic centimetre each of mass m and possessing a charge e, then the radiation from each ion as its motion is accelerated by the intense electric fields in the Röntgen pulses, produces a diminution in the energy of the primary radiation, the rate of change of intensity of the primary beam due to this secondary radiation alone being given by the expression

$$\frac{d\mathrm{E}}{dx} = -\frac{8\pi}{3}\frac{\mathrm{N}e^4}{m^2}\mathrm{E},$$

E being the intensity of primary radiation and $\frac{d\mathrm{E}}{dx}$ the rate of change of intensity.

Now it was experimentally shown that the radiation was independent of the ionization in the gas from which the secondary radiation proceeded, but was proportional to the number of corpuscles or electrons in a given volume of the gas, and hence it was concluded that the corpuscles constituting the molecules were the sources of secondary radiation. Substituting the values experimentally determined for $\frac{d\mathrm{E}}{\mathrm{E}dx}$ in the above equation, together with the accepted values of e and m, we may calculate on this theory the number of corpuscles per cubic centimetre of air at atmospheric pressure and temperature, and compare this with the number assigned by the electronic theory of matter.

Taking $\frac{e}{m} = 7 \times 10^6$ and $e = 10^{-20}$,

$$\frac{d\mathrm{E}}{dx} = -4 \times 10^{-26}\,\mathrm{NE}.$$

But by experiment $\frac{d\mathrm{E}}{dx} = -{\cdot}00024\,\mathrm{E}$,

$$\therefore\ \mathrm{N} = {\cdot}6 \times 10^{22}.$$

Considering the range of possible values of e and m, there is a close agreement between the number thus obtained and that calculated on the electronic theory of matter from the determined values of the number of molecules per cubic centimetre.

The number of ions (here denoting a molecule + or − an electron) necessary to produce the intensity of radiation experimentally determined would be between 10^{30} and 10^{31} per cubic centimetre. This is approximately 10^{11} times the actual number of molecules.

Quantitative measurements thus support the theory that the constituents of the molecules are the radiators.

The laws governing the secondary radiation from gases are thus in perfect agreement with the theory that this radiation proceeds from the negative corpuscles or electrons during the acceleration of their motion by the intense electric fields in the Röntgen pulses passing through the medium containing them.

No quantitative measurements of value have previously been made on the radiation from solids.

An experimental determination of the energy of secondary radiation proceeding from a brass plate which totally absorbed a primary beam was made by H. S. Allen *. He found that the number of ions produced in sulphuretted hydrogen by the secondary rays from brass was about $\frac{1}{2000}$ part of the number which would have been produced by the primary beam if it had been totally absorbed by the gas. This result has no special significance regarding the amount of transformation into secondary radiation, as an unknown fraction —but a very large one—of the total energy of secondary radiation was absorbed by the metal itself and transformed into heat. Only by using very thin sheets can an approximation be made to the fraction of the energy used up in the production of secondary radiation, or even to the character of the radiation proceeding from the metal. Sheets of metal of greater thickness give secondary beams of greater average penetrative power, for the deeper layers are only reached by the penetrating rays, and the most penetrating secondary rays penetrate to the surface in greatest proportion. Consequently the addition of more layers of metal results in the superposition of simply the most penetrating secondary rays, and the composition of the radiation proceeding from a thick plate is thus entirely different from that which is set up in each layer.

To measure the energy of secondary radiation from a solid,

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* Phil. Mag. [6] iii. p. 126 (1902).

the radiation from which differed little in character from that of the primary, the same method was employed as has been described for air.

A sheet of paper weighing ·373 gr. was placed in the primary beam with its plane inclined at an angle of 45° with the direction of propagation of the primary rays and with the line joining corresponding points of the two apertures R and S. The paper was so placed that the primary beam entering electroscope N was not intercepted by it. The absorption of the secondary radiation by plates of aluminium was, within 2 or 3 per cent., the same as that of the primary radiation, showing that the difference in character between these radiations was exceedingly small. The energy of secondary radiation was then compared with that proceeding from air. Correction was made for the absorption of primary and secondary radiations in the paper. The absorption produced by a sheet of paper of the same material and thickness as that used in the experiment amounted to about 7 per cent. of the energy of the radiation normally incident upon it. There was also a small correction made for the absorption of part of the secondary radiation from air which passed through the paper.

The results are given below :—

Secondary Radiator.	Ratio of Secondary Elec. Deflexion to Deflex. of Primary Electroscope.
·373 gr. paper	38·9 to 100
194 c.c. of air at 74 cms. press. and 17° C. (·239 gr.)	25·3 to 100

Correcting for absorption in the paper:—

$$\frac{\text{Energy of sec. radiation from ·373 gr. paper}}{\text{Energy of sec. radiation from ·239 gr. air}} = \frac{42·8}{25·3}.$$

This gives as the ratio of the energies of secondary radiation from equal masses of paper and air 113 to 110. These are as nearly equal as the possible errors in estimating the volume of air productive of secondary radiation and in the electroscope deflexions admit.

We thus arrive at the conclusion that the energy of radiation from gases and light solids, the radiation from which differs little from that of the primary producing it, is for thin layers of equal thickness proportional to the density of the substance; that is, for radiation of given intensity the energy lost in secondary radiation is proportional to the quantity of matter passed through.

Recent experiments on secondary radiation proceeding from thin sheets of metal have also led me to believe—though accurate measurements have not yet been made—that from these also, though the character of the secondary radiation in some cases differs enormously from that of the primary, the energy of secondary radiation is proportional simply to the quantity of matter passed through by a primary beam of given intensity. Using sheets of different metals, equal weights are productive of a greater secondary ionization the greater the absorbability of this radiation by sheets of aluminium; so that on the assumption that the ionizing-power varies for these radiations at a rate of 5 or 6 times the absorbability in aluminium—evidence for which I have previously given—the energy of radiation from equal masses is approximately the same in each case. As an example, the radiation from tin was approximately 5 times as absorbable by aluminium as the primary radiation, while the ionization produced in the secondary electroscope by this radiation was about 29 times the ionization produced by the secondary radiation proceeding from an equal mass of air.

For the few metals experimented upon, the energy of secondary radiation is of the order of magnitude which would bring it into agreement with the law found for gases and light solids. Further experiments are being made to test this more accurately.

It should be observed that in these measurements the radiation from metals which is absorbed by a few millimetres of air under normal conditions has not been taken account of. This may or may not form a part of the radiation accounted for by the acceleration of the negative corpuscles or electrons in the intense electric fields in the primary Röntgen pulses. Further experiments on the subject are necessary.

The results of these experiments may be summarized thus:—

(1) The character of secondary X-radiation from gases differs slightly from that of the primary producing it. (From air the secondary has greater ionizing power in air.)

(2) The penetrating rays are transformed to a greater extent than the more absorbable rays. (See previous note.)

(3) The energy of secondary radiation from a given gas through which a primary beam of given intensity is passing is independent of the character of the primary radiation.

(4) The energy of secondary radiation from gases and those light solids which are the source of a radiation differing little in character from the primary, is proportional to the quantity of matter through whtch the primary beam of given intensity passes.

(5) In the passage of X-radiation through air at 0° C. and 76 cms. of the pressure, the diminution of intensity due to secondary radiation is of the order of magnitude ·02 per cent. per centimetre.

> N.B. This is a large fraction of the total loss of intensity due to all causes for fairly penetrating rays.

(6) Applying experimental results to the expression given by J. J. Thomson for the loss of energy per centimetre (due to radiation) in passing through a medium containing ions, and taking the negative corpuscles with $\frac{e}{m}=7\times10^6$ and $e=10^{-20}$ as the sources of the radiation, the number of these corpuscles or electrons per c.c. for air under normal conditions is of the order 10^{22}.

(7) *Quantitative* results show that the secondary radiation from metals, though of different penetrative power, is of the same nature as the primary X-radiation.

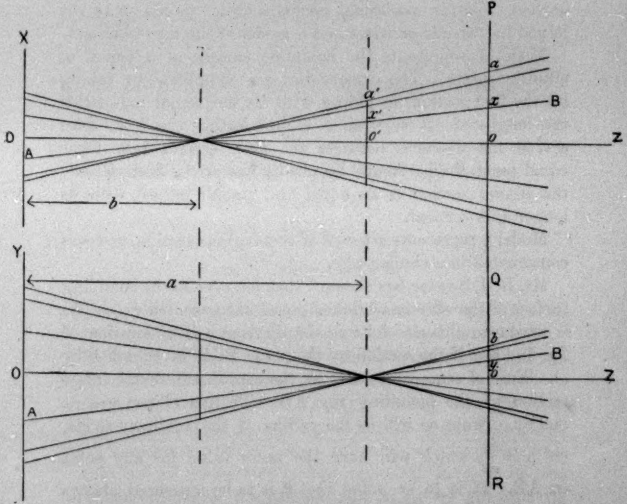
(8) Results of the order of magnitude of those given above have been obtained with all metals experimented upon, though the secondary radiation from them differs considerably in character from the primary.

University of Liverpool.

XX. *Notes on Non-homocentric Pencils, and the Shadows produced by them.*—I. *An Elementary Treatment of the Standard Astigmatic Pencil.* *By* WILLIAM BENNETT*.

A STANDARD astigmatic pencil is one of which all the rays pass through two focal lines, at right angles to one another and to the axis of the pencil. Let the axis of the pencil be taken as the axis of Z, and let the focal lines be parallel to the axes of X and Y respectively, and at distances a and b from the origin. Then the projections of the pencil on the planes of XZ and YZ are as shown in fig. 1.

Fig. 1.



It is evident from this figure that the change in the transverse sections of the pencil consists in a uniform stretching or compression in a direction perpendicular to each of the focal lines, the stretchings or compressions proceeding at unequal rates. Thus, if the section is anywhere a conic with axes parallel to the focal lines, it is everywhere so. The pencil is also of rectangular section with sides parallel to the focal lines at all points if at any; and this is the section obtained if the extent of the pencil is defined by limiting the lengths of the focal lines.

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* Read January 22, 1904.

Any form being assigned to the section by a plane perpendicular to the axis, and the position of the focal lines relative to the plane being defined, it is easy, by drawing plans and elevations of the rays through a number of points on the boundary of the section, to obtain graphically the section by any other plane parallel to the first. The two sections being drawn on card or metal, and erected in the correct relative positions, corresponding points may be joined by threads or wires and a model of the rays obtained.

Model 1 represents the bounding surface of a pencil of elliptic section. The construction was simplified by taking for the first section an ellipse with its horizontal axis twice the length of its vertical axis, and letting the focal lines divide the distance between the two sections into three equal parts, the horizontal line being nearer the first section; the second section is an equal and similar ellipse, with its longer axis vertical.

Model 2 represents a pencil of rectangular section, and was constructed in a similar way.

Mr. R. J. Sowter has shown [*] that for rays on the bounding surface of the elliptic-sectioned pencil the eccentric angle (ϕ) is constant: this also follows simply from a consideration of fig. 1. For if the section by the plane PQR is bounded by an ellipse of semiaxes oa and ob, the coordinates of the intersection of the bounding ray AB with this ellipse are ox and oy. Since oa will be the radius of the auxiliary circle, $\cos\phi = \frac{ox}{oa}$, which will have the same value for any point on AB. It is to be noted that ϕ is to be measured always from the axis parallel to OX, whether this be the major or the minor.

Prof. S. P. Thompson has called attention to the rotation

[*] Phil. Mag. Oct. 1903.

of the shadow of a straight wire placed across the pencil: this also appears from fig. 1. Let $x_0 y_0$ be the coordinates of any point in the plane Z=0. The ray through this point is given by

$$\begin{cases} x = \dfrac{b-z}{b}\,x_0, \\ y = \dfrac{a-z}{a}\,y_0. \end{cases}$$

So that, if

$$y = mx + c$$

be the equation of a line in the plane Z=0, the rays passing through this line rule the surface

$$\frac{ay}{a-z} = m\frac{bx}{b-z} + c.$$

This equation gives also the shadow thrown on a plane perpendicular to the axis—that is, the section of the pencil by the plane. This is a straight line whose inclination is

$$\tan^{-1}\frac{b(a-z)}{a(b-z)}\,m.$$

This inclination varies continually with z. Its tangent

$$\begin{aligned} &= 0 \text{ when } z = a, \\ &= \infty \text{ when } z = b, \\ &= \frac{bm}{a} \text{ when } z = \pm\infty. \end{aligned}$$

The whole amount of rotation between $z=\infty$ and $z=-\infty$ is 180°, 90° of which occurs between the focal lines.

Model 3 represents the shadow-surface, and Model 4 shows its relation to the boundary of the pencil.

As this surface is ruled by rays passing through three straight lines, it is a ruled quadric; further, since these three lines lie in parallel planes, it is a hyperbolic paraboloid.

One set of generators is the rays. If p, q, r be the direction-cosines of the ray through x_0, y_0, we get from its equations

$$\frac{p}{\frac{x_0}{b}} = \frac{q}{\frac{y_0}{a}} = \frac{r}{-1}.$$

Since $y_0=mx_0+c$, we have, eliminating x_0 and y_0,

$$aq-mbp+cr=0\,;$$

that is, the rays are parallel to the plane

$$ay-mbx+cz=0.$$

If $c=0$, that is if the object-line meets the axis, this becomes

$$y=\frac{b}{a}mx.$$

This case is shown in Model 5, which consists of the shadow-surface and the plane $y=\frac{b}{a}mx$.

The other set of generators are the shadow-lines in planes parallel to $z=0$.

The section by any plane not perpendicular to the axis will be an hyperbola whose equation can be obtained from the equations of the plane and the surface.

If the object-line does not lie in a plane perpendicular to the axis, the ray-surface is an hyperboloid of one sheet, and the shadow on a plane at right angles to the axis becomes an hyperbola. But since a generator of each set passes through every point on the surface, the shadow can always be reduced to a straight line by tilting the plane upon which it is received.

The equation of the ray-surface in this case can be obtained as follows :—

The equations to a ray through the point (x_0, y_0, z_0) are

$$\left.\begin{aligned} x&=\frac{z-b}{z_0-b}x_0, \\ y&=\frac{z-a}{z_0-a}y_0, \end{aligned}\right\}.$$

Let the object-line be defined by the equations

$$\left.\begin{aligned} x&=mz+h \\ y&=nz+k \end{aligned}\right\}.$$

Then, since (x_0, y_0, z_0) lies on this line, the ray becomes

$$\left.\begin{aligned} x&=\frac{z-b}{z_0-b}(mz_0+h) \\ y&=\frac{z-a}{z_0-a}(nz_0+k) \end{aligned}\right\}.$$

Equating the values of z_0 obtained from these, we get for the equation of the ruled surface :

$$\frac{xb+h(z-b)}{x-m(z-b)}=\frac{ya+k(z-a)}{y-n(z-a)} \quad (=z_0)\ ;$$

i. e.,

$$(a-b)xy-(am+h)(z-b)y+(bn+k)(z-a)x$$
$$+(nh-mk)(z-a)(z-b)=0.$$

This represents a ruled quadric whose section by a plane $z=$const. is the rectangular hyperbola whose equation is obtained by substituting for z its constant value.

The asymptotes of this section lie in the planes

$$x=\frac{am+h}{a-b}(z-b)$$

and

$$y=\frac{bn+k}{a-b}(z-a)$$

respectively, each of which passes through one of the focal lines.

The two sets of generators are given by the two pairs of equations :

$$\left.\begin{aligned}\frac{xb+h(z-b)}{x-m(z-b)}&=\lambda\\ \frac{ya+k(z-a)}{y-n(z-a)}&=\lambda\end{aligned}\right\}$$

and

$$\left.\begin{aligned}\frac{xb+h(z-b)}{ya+k(z-a)}&=\mu\\ \frac{x-m(z-b)}{y-n(z-a)}&=\mu\end{aligned}\right\}$$

respectively.

The first set is the rays, and $\lambda=z_0$. The object-line and the focal lines are members of the second set. The two focal lines are given by $\mu=0$ and $\mu=\infty$ respectively, and the object-line by $\mu=\dfrac{mb+h}{na+k}$. This case is illustrated in model 6. It will be seen that the object-wire may be moved into the position of any generator of the second set, except the two focal lines, without altering the form of the shadow.

If the plane on which the shadow is taken is not perpendicular to the axis of z, the section is still a conic. If it is turned so as to contain one of the generators of the second set, the section will become two straight lines; the other being the ray which lies wholly in that plane and comes from the point in which the plane is cut by the object-line.

This is an example of a two-part shadow, of which one part is due to a single point of the object.

It is to be noted that although the complete shadow is an hyperbola, it does not follow that both branches will be seen with a pencil limited by any aperture. In fact only one branch will be seen, unless some of the rays which meet it have and some have not passed through one of the focal lines.

The positions of the focal lines produced by reflexion or refraction of a small pencil may be often found by use of the method of sagittæ due to Prof. Thompson. The case of refraction through the centre of a thin lens is taken as an example.

Let r_1 and r_2 be the radii of curvature of a small thin convex lens of diameter $2a$, and refractive index μ.

Then the thickness (t) of the lens is given by

$$t = \frac{a^2}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

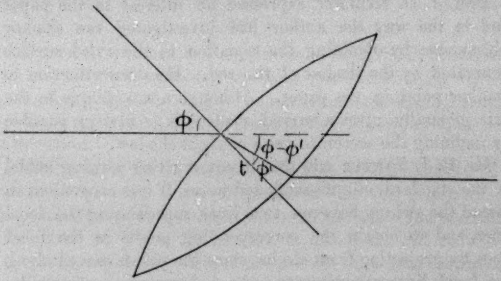
Let a small pencil inclined at an angle ϕ to the axis of the lens fall centrally upon it.

Then if a plane wave-front fall upon the glass, the centre of it travels (see fig. 2) through glass for a distance $\frac{t}{\cos\phi_1}$, where $\sin\phi_1 = \frac{\sin\phi}{\mu}$. But this is in a direction making an angle $\phi - \phi_1$ with the direction of the pencil; so it is only advanced thereby a distance $\frac{t}{\cos\phi_1}\cos(\phi - \phi_1)$. Meanwhile the edges of the wave-front travelling in air have advanced a distance $\frac{\mu t}{\cos\phi_1}$; so that the relative retardation of the centre is $\frac{t}{\cos\phi_1}\{\mu - \cos\overline{(\phi - \phi_1)}\}$, which reduces to

$$\frac{a^2}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)(\mu\cos\phi_1 - \cos\phi) \equiv A \,.\, \frac{a^2}{2}.$$

This is the relative retardation of the centre over the edges of a piece of the incident wave-front whose width is $2a$ in a direction perpendicular to the plane of fig. 2 and $2a \cos \phi$ in that plane.

Fig. 2.



The changes in the curvatures of the principal sections of the wave-front are A and $\frac{A}{\cos^2 \phi}$. That is to say, the lens behaves like a bicylindrical lens whose principal converging powers are A and $A \sec^2 \phi$. This is a standard result derived in Heath's Optics by the use of the characteristic function. It only holds, however, for very narrow pencils and thin lenses, as it assumes that a is a small quantity of the first, and hence t of the second order.

Discussion.

Prof. S. P. Thompson congratulated the author, and said that although many people accustomed to these pencils could easily follow the passage of the rays by the inspection of diagrams, there could be no doubt that there were others to whom the subject could only be made clear by the use of models. Geometrical models had an advantage because they could be so constructed as to show the characteristics of the particular aberration under consideration, free from the presence of other aberrations, a condition of things which could not be realized in the case of actual pencils produced by the passage of light through a lens system. The study of the peculiar shadows produced by these beams was one of the out of the way branches of physics which could be pursued with advantage. The results were so striking that they often engendered a curiosity leading to a further study of optics.

Prof. J. D. Everett expressed his interest in the paper and in the way the author had investigated the shadow phenomena by obtaining the equation to the ruled surface generated by the shadow of the rod. He drew attention to another point in the paper. Although a rod oblique to the axis generally gives a curved shadow, it is always possible by inclining the screen to get a straight shadow.

Mr. R. J. Sowter said that in constructing a string model of the standard elliptic-sectioned beam, it was convenient to thread the strings between two lines representing the focal lines, and to obtain the corresponding points in the focal lines by projection from circles, since the points moved along the focal lines according to a simple harmonic motion rule. Further, it was useful to take stereoscopic photographs of the string models, so that the solid effects could be reproduced in the stereoscope, and the models thus replaced by the photographs.

The Author, in reply to Prof. Thompson, said the geometrical method gave results very simply when the aberration was purely astigmatic, but was less readily applicable when other aberrations were present. The formation of shadows could then be more readily followed by considering the form of the wave front, the way in which it meets the object wire, and the consequent evolutions of the trace left upon it by the object wire. Replying to Mr. Sowter, he said that the general method of constructing the models was simplified in many cases, both end plates being produced from a single easily-constructed template.

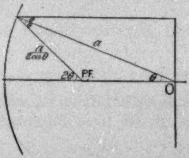
XXI. *Notes on Non-homocentric Pencils, and the Shadows produced by them.—II. Shadows produced by Axially Symmetrical Pencils possessing Spherical Aberration.* By WILLIAM BENNETT *.

[Plate XII.]

ATTENTION has recently been drawn by Prof. S. P. Thompson to some curious shadows produced when a straight wire is introduced into an axially symmetrical converging pencil proceeding from a lens or mirror and possessing spherical aberration.

The case to be investigated is that of the shadows of a straight wire in a pencil produced by the reflexion of a parallel beam at a concave spherical mirror. The wire, and the plane on which the shadow is received, will be taken perpendicular to the axis of the pencil. Let a be the radius of curvature of the mirror (fig. 1), then a ray which meets the mirror at an angular distance θ from its pole will, after reflexion, make an angle 2θ with the axis, and will meet it at a distance $\frac{a}{2\cos\theta}$ from the centre. Values of $\frac{a}{2\cos\theta}$ being tabulated, it is easy to make a drawing of a part of the pencil to any scale. The most important shadows are those produced when the wire is near the principal focus and between it and the mirror; and in the consideration of these it is convenient to think of the form of the wave-front which meets the wire, and of the trace left upon it after it has passed the wire. A method for drawing the wave-front

Fig. 1.



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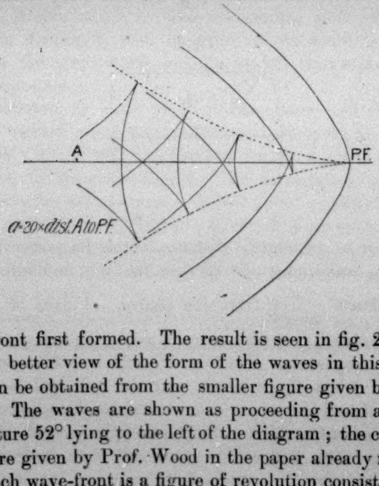
* Read March 11, 1904.

has been given by Prof. R. W. Wood*; but as this method is inconvenient for large-scale drawings, a modification of it was adopted. The optical distance from the incident wave-front, which passes through the centre of curvature, to the axis is $a\left(\cos\theta + \frac{1}{2\cos\theta}\right)$. The distance from this wave-front to that which passes after reflexion through the principal focus is $\frac{3a}{2}$. A reflected ray therefore meets the latter wave-front at a distance $a\left(\frac{3}{2} - \cos\theta - \frac{1}{2\cos\theta}\right)$ beyond its intersection with the axis. The values of this quantity were calculated for different values of 2θ, and measured off on a large scale drawing of the rays near the principal focus. The other wave-fronts were obtained by measuring back a series of equal distances along the rays, starting from the wave-front first formed. The result is seen in fig. 2, which gives a better view of the form of the waves in this region than can be obtained from the smaller figure given by Prof. Wood. The waves are shown as proceeding from a mirror of aperture 52° lying to the left of the diagram; the complete forms are given by Prof. Wood in the paper already referred to. Each wave-front is a figure of revolution consisting of a

Fig. 2.



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* Phil. Mag. [5] vol. l. (July 1900).

saucer-shaped part in front, bounded by a circular cuspidal edge which is tracing out the caustic surface, and a trailing part behind, which has already passed the caustic.

The wave-fronts are given by the equations

$$\begin{cases} r=\left\{\mathrm{K}-a\left(\cos\theta+\dfrac{1}{2\cos\theta}\right)\right\}\sin 2\theta, \\ z=r\cot 2\theta-\dfrac{a}{2\cos\theta}, \end{cases}$$

where K is a parameter constant over any one wave-front, being, in fact, the optical distance from the incident wave-front through the centre of curvature. These equations, however, are not easily worked with.

Imagine the wire placed so as to meet only the trailing part of the wave-front ; as the curvature of the wave-front is not uniform, the trace left upon it by the wire will be distorted as the wave advances, and the shadow will be a single-branched curve on the same side of and farther from the axis. This curve will be concave to the axis in its central parts, and will have two points of inflexion and an asymptote parallel to the wire.

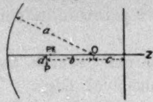
Let the wire be moved towards the axis until it just grazes the passing cuspidal edge. The trace will now consist of the branch previously considered and a conjugate point which, when the principal focus is passed and the wave-front completely unfolded, will be on the other side of the axis.

If the wire is moved still farther on it is met three times by the advancing wave-front : once by the saucer, next by the trail immediately behind, and finally by the tail part of the trail from the other side. The first two intersections, however, are continuous and meet upon the cuspidal edges forming a closed curve. The complete shadow now consists of an open branch, and a closed branch on the other side of the axis. If the wire intersects the axis the shadow passes into the ϕ form particularly noticed by Prof. Thompson. The circular part of this is due to a single point on the wave, the intersection of the trail with the axis (the normals to the wave-surface at this point forming a circular cone), and will of course be absent if the trail has not yet met the axis.

If the wire is placed so near to the mirror that the cusp has not yet begun to form (this stage is not shown in fig. 2), the shadow will be single-branched and open on the opposite side of the axis. It will also be convex towards the axis. This, however, becomes closed, and an open branch appears on the other side of the axis if the angular aperture of the mirror is increased. If, on the other hand, the wire is beyond the principal focus, the shadow will be of the form first described.

Fig. 4 is a reproduction of a series of drawings of the shadow curves. These were obtained in the following way:—Cylindrical coordinates were taken with the axis of the pencil for the axis of z and the origin at the centre of curvature, the positive direction of z being away from the mirror, and ϕ being measured from the plane of the paper. The wire lies in the plane $z=b$ at right angles to the paper and its distance from the axis is d. The shadow is considered in the plane $z=c$ (fig. 3). The equations of a ray are

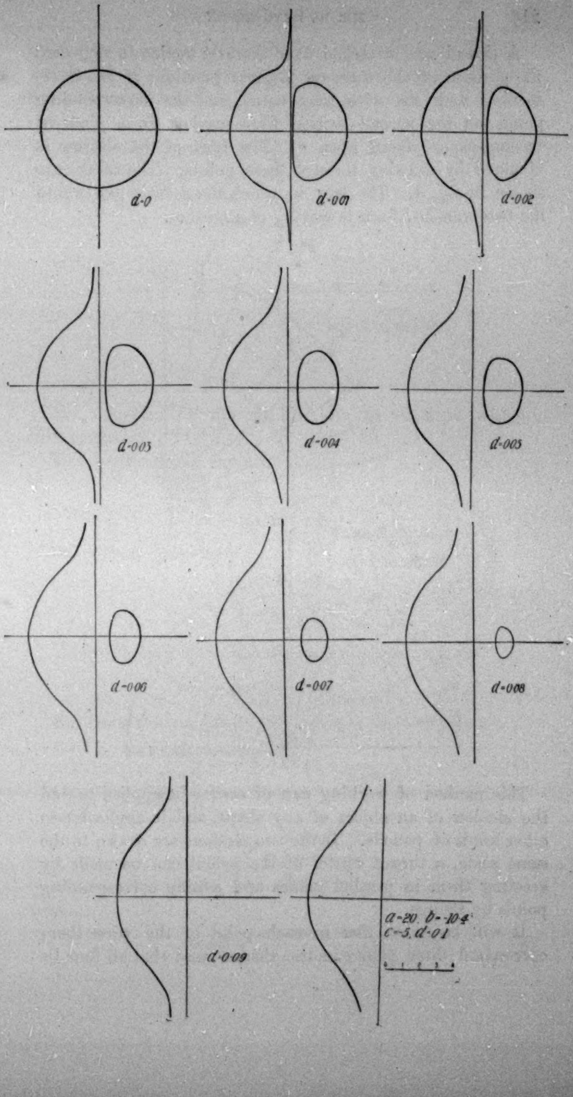
Fig. 3.



$$\begin{cases} \phi = \text{const.} \\ r = -\tan 2\theta \left(z + \dfrac{a}{2\cos\theta} \right). \end{cases}$$

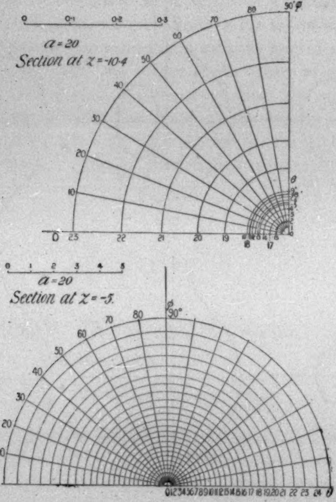
From the second of these equations values of r were computed for different values of θ, in the planes $z=b$ and $z=c$. Sections of the pencil by these two planes were obtained by drawing a number of concentric circles with the values of r for radii, crossed by radial lines at intervals of 5°. It will be obvious that corresponding points on the two sections lie on the same ray. The numerical values taken in the preparation of fig. 5 are: $a=20$, $b=-10{\cdot}4$, $c=-5$, while d has values rising from 0 to $\cdot 1$ by $\cdot 01$ at a time. Parts of the two sections obtained are reproduced in fig. 5. The section at $z=-10{\cdot}4$ was drawn to 20 times the scale of the other.

Fig. 4.


d=0
d=·001
d=·002
d=·003
d=·004
d=·005
d=·006
d=·007
d=·008
d=0·09
a=20, b=−10·4
c=−5, d=·01
0 1 2 3 4

A thread was stretched over the first section to represent the position of the wire, the angular positions of the intersections with the wire were noted, and the corresponding points on the second section were marked on a piece of tracing-paper placed upon it. The form of the shadow is obtained by drawing through these points. The results are shown in fig. 4. The way in which the ϕ form passes into the two-branched form is worthy of attention.

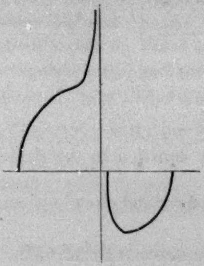
Fig. 5.


0 0·1 0·2 0·3
a = 20
Section at z = -10·4
10 20 30 40 50 60 70 80 90° φ
θ
0 23 22 21 20 19 18 17
0 1 2 3 4 5
a = 20
Section at z = -5
10 20 30 40 50 60 70 80 90° φ
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 θ

This method of working can of course be applied to find the shadow of an object of any shape, and is applicable to other kinds of pencils. If the two sections are drawn to the same scale, a thread model of the pencil can be made by erecting them in parallel planes and joining corresponding points by strings.

It will be noted that to each point of the wire there correspond three points in the shadow, and that all four lie in a plane passing through the axis. Thus if the wire terminates at the nearest point to the axis, the shadow will be as shown in fig. 6.

Fig. 6.



The equation of the shadow can be deduced without difficulty:—

The equations of a ray are:

$$\begin{cases}\phi = \text{const.}\\ r = -\tan 2\theta\left(z + \dfrac{a}{2\cos\theta}\right).\end{cases}$$

The wire is given by:

$$\begin{cases}z = b,\\ r = \dfrac{d}{\cos\phi}.\end{cases}$$

Where the ray meets the screen $z = c$, and we have

$$r = -\tan 2\theta\left(c + \frac{a}{2\cos\theta}\right). \quad . \quad . \quad . \quad . \quad (1)$$

But since the ray passes through a point in the wire,

$$\frac{d}{\cos\phi} = -\tan 2\theta\left(b + \frac{a}{2\cos\theta}\right). \quad . \quad . \quad . \quad (2)$$

From (1) and (2),

$$\frac{r\cos\phi}{d} = \frac{c + \dfrac{a}{2\cos\theta}}{b + \dfrac{a}{2\cos\theta}},$$

whence

$$\cos\theta = \frac{a(r\cos\phi - d)}{2(cd - br\cos\phi)} \quad . \quad . \quad . \quad . \quad , \quad (3)$$

We will now eliminate θ between (1) and (3). Squaring (1) we have

$$r^2 = \left(c + \frac{a}{2\cos\theta}\right)^2 \tan^2 2\theta$$
$$= \left(c + \frac{a}{2\cos\theta}\right)^2 \cdot \frac{4\cos^2\theta(1-\cos^2\theta)}{(2\cos^2\theta-1)^2}.$$

Substituting for $\cos\theta$ and simplifying:

$$\{a^2(2\cos\phi - d)^2 - 2(cd - br\cos\phi)^2\}^2$$
$$= \cos^2\phi a^2(b-c)^2\{4(cd-br\cos\phi)^2 - a^2(r\cos\phi - d)^2\}.$$

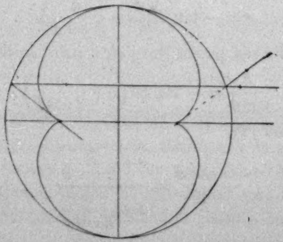
This is the polar equation to the shadow, and is of the fourth degree in r.

Substituting $r\cos\phi = x$ and $r^2 = x^2 + y^2$ we get the Cartesian equation

$$\{a^2(x-d)^2 - 2(cd-bx)^2\}^2(x^2+y^2)$$
$$= x^2a^2(b-c)^2\{4(cd-bx)^2 - a^2(x-d)^2\}.$$

This is of the sixth degree and is met in six points by a line through the origin. Two degrees, however, are accounted for by a conjugate double point at the origin, introduced in the transformation. We are left with four shadow points for each point of the object. Evidently, however, the analysis applies to the complete spherical mirror, and one or three of these points will be due to rays which would pass virtually through the object after reflexion at the missing convex surface. The condition for this will be seen on reference to fig. 7, which shows a section of the complete mirror and of the epicycloid, which is the caustic for reflexion at both

Fig. 7.



surfaces, the part to the right of the centre for the convex and that to the left for the concave surface. The rays which pass through any point are the tangents drawn through it to the epicycloid. Of these there are four if the point is outside the epicycloid; three from the concave and one from the convex surface of the mirror if the point is to the left of the centre, and three from the convex and one from the concave if it is to the right. If the point is inside the epicycloid there will be two rays only, one from each surface. Evidently, then, the virtual shadow passes through much the same forms as we have already seen for the real.

If the virtual shadow were drawn in fig. 4 it would be nearly straight, very close to the axis, and on the same side of it as the object.

Referring back to the Cartesian equation, we see that there are two asymptotes parallel to the axis of y given by

$$a^2(x-d)^2-2(cd-bx)^2=0\,;$$

or

$$x=\frac{a\pm c\sqrt{2}}{a\pm b\sqrt{2}}d\,;$$

so that we have always two open branches to the shadow, one real and the other virtual, and a closed branch which may be real or virtual, which contracts to a point when the object touches the caustic surface and vanishes when it does not meet it at all.

The Cartesian equation gives y^2 as a single-valued function of x and affords an alternative method for plotting the shadow forms.

If the pencil is not parallel before reflexion, or if we take a pencil rendered convergent by a lens with spherical surfaces, we get in general a less simple expression for the longitudinal aberration, and the equation of the shadow takes a more complicated form. The real shadows are, however, of much the same appearance. Fig. 8 (Pl. XII.) shows a series of photographs of shadows of a straight wire in a pencil produced by refraction at a plano-convex lens. The lens used was of about 10 cms. diameter and the radius of curvature of its spherical face was 7·8 cms. The disposition of the apparatus was as follows:— An image of the filament of a Nernst lamp was focussed on to a card screen pierced by a pinhole, so that the image actually crossed the pinhole. The plano-convex lens was placed 43 cms. beyond this with its convex side towards the incident light. A vertical wire was placed in the emergent beam about 18 cms. from the lens, and its shadow received on a photographic plate 15 cms. beyond. As the shadows were heavily fringed with colour, a coloured screen was placed before the lamp. This consisted of a parallel-sided glass cell 5 mms. thick, containing equal volumes of a 0·5 °/$_0$ solution of acid green in water and 0·1 °/$_0$ solution of tartrazin. This was recommended by Mr. A. J. Bull as transmitting only a short region in the blue-green of the spectrum, from about λ 4900 to λ 5100, and proved very satisfactory. The plates were Westerndorp and Wehner's Isochromatic. Fig. 9 (Pl. XII.) shows a shadow obtained by this method with the wire not completely crossing the pencil, and illustrates the collinearity of the shadows of a point.

There is another and a more convenient way in which these forms can be observed. If the eye takes the place of the illuminated pinhole, the disposition of the lens or mirror and the object-wire remaining the same, the form of the image of the wire that is seen is that of a section of the surface ruled by rays passing through the pupil and the object-wire, which is that of the virtual shadow on the surface of the lens or mirror, and practically identical with that of the real shadow on a plane beyond the wire. Thus the forms can be seen with great clearness by looking at the reflexion in a concave mirror of a straight wire held suitably between the mirror and the eye. On moving the eye from side to side the image passes through the same phases, as shown in fig. 4, while if the eye be moved nearer to or farther from the mirror, the effect of moving the wire in a direction parallel to the axis is imitated. If two eyes are used two of the phases are seen simultaneously. Similar results can be obtained by looking at a wire through a suitable lens ; the chromatic effects are more in evidence if the wire is replaced by the straight filament of an incandescent lamp, or by an illuminated slit, so that the shadow-forms are seen bright against a dark background. If the slit is replaced by a number of parallel slits, or by a rectangular grating, very complicated forms are seen, which are formed by the superposition of the curves due to the various straight slits. I have seen the same effect by looking through the globe of a sunshine recorder at the rulings on the record strip beneath.

This arrangement may also be used for photographing the shadows, the eye being replaced by a camera. The place of the pupil must be taken by the diaphragm of the lens, which must be stopped down to a small aperture.

Discussion.

Dr. W. **Watson** expressed his interest in the paper, and said that the diagrams shown proved that when dealing with problems similar to those considered in the paper, it was better to use wave-fronts rather than rays. In his opinion the subject of aberrations in lenses was one which should be more frequently studied by elementary students. It was possible to illustrate many important facts by the use of cheap and simple apparatus.

Dr. **Glazebrook** supported Dr. Watson in his remarks with reference to the advantages of dealing with wave-fronts instead of rays in the consideration of optical problems. He impressed upon the meeting the necessity of a more careful study of aberrations and image-formations, stating that papers similar to that read by Mr. Bennett could not fail to be of practical importance in the optical industry.

XXII. *On Normal Piling, as connected with* Osborne Reynolds's *Theory of the Universe. By* Prof. J. D. Everett, *F.R.S.*[*]

Prof. O. Reynolds's theory was set forth in an elaborate paper communicated to the Royal Society in Feb. 1902, and accepted for the Transactions. It is published as a separate volume of about 250 quarto pages, under the title "The Sub-Mechanics of the Universe." It is briefly summarised in the Rede Lecture for 1902, which is published as a small book,

[*] Read April 22, 1904.

with excellent pictorial illustrations, under the title "On an Inversion of Ideas as to the Structure of the Universe" (Camb. Univ. Press). Its fundamental supposition is that the universe consists of equal small and perfectly hard spherical grains, arranged for the most part in the manner usually adopted for piles of shot, hence called by Prof. Reynolds NORMAL PILING. As a knowledge of the structure of such piles is essential to an intelligent discussion of the theory, and is unfamiliar to most students of physics, I propose to commence by giving a full description of it.

When 4 equal spheres touch one another, their centres are at the 4 corners of a regular tetrahedron. The tetrahedron has 6 edges, each joining the centres of a pair of spheres; and if the 4 spheres are a portion of a normal pile, each of these pairs belongs to a continuous line of spheres in contact, extending all through the pile. The whole pile can thus be resolved in 6 different ways into parallel lines of spheres in contact. Every sphere in the interior of the pile belongs to 6 of these lines, and is therefore touched by 12 spheres, at the opposite ends of 6 diameters. Its centre is the meeting-point of the vertices of 8 of the tetrahedrons; and the gaps between these tetrahedrons would be exactly filled by 6 square pyramids; all the 8 edges of a square pyramid being of the same length as an edge of a tetrahedron.

The pile can be split up into planes of spheres, either parallel to the base of a square pyramid, or to any face of a tetrahedron. In the former case the spheres in the plane layer are in square formation, so that each touches 4. In the other case the spheres in the plane tier are in triangular formation and each touches 6, whose centres are at the corners of a regular hexagon. This last is the arrangement which gives the closest possible packing of spheres *in plano*. The tiers which have this formation are parallel to a face of the fundamental tetrahedron; and as a tetrahedron has 4 faces, we have our choice of 4 ways of splitting up a normal pile into tiers of closest formation.

A pile can be built up in the form of a regular tetrahedron, and each of its 4 faces will then be composed of spheres in triangular formation. Or it can be built up in the form of a square pyramid; in which case the 4 sloping faces will be in triangular formation, while the base and all planes of spheres parallel to the base will be in square formation. The vertical planes of spheres parallel to the diagonals of the square base will also be in square formation.

A normal pile can thus be split up into planes of spheres in 7 different ways, there being 3 sets of parallel planes containing spheres in square formation, and 4 sets containing spheres in triangular formation.

Perhaps the clearest idea of the relative position of spheres in normal piling is obtained by imagining a solid chequer built up of equal cubes alternately black and white, so that every section taken parallel to the cube-faces is like a chess-board. Then, if we imagine all the cubes of one colour (say all the blacks) to be annihilated, and each white cube to be replaced by a sphere having the same centre, and of diameter equal to the diagonal of a cube-face, these spheres will just fit together as they stand, and will constitute a normal pile. A section through cube-centres, taken parallel to a cube-face, is in square formation; and a section taken perpendicular to a body diagonal is in triangular formation.

If we think of a normal pile as placed with a set of squarely formed layers horizontal, each sphere rests on 4 in the layer below, and in its turn supports 4 in the layer above, these last 4 being vertically over the first 4. Besides touching these 8 in adjoining layers, it also touches 4 in its own layer, making 12 in all.

On the other hand, if we think of the pile as placed with one of its sets of triangularly-formed tiers horizontal, each sphere rests on 3 in the tier below, supports 3 in the tier above, and also touches 6 in its own tier, again making 12 contacts in all.

The following is another representation of normal piling:— Divide space into equal cubes by 3 sets of parallel planes. Then the centres of the spheres are to be at the cube-corners, and at the middle points of the cube-faces; the common sphere-diameter being made equal to the distance of the middle point of a face from a corner of the face.

Another representation (the one adopted by Prof. O. Reynolds) consists in dividing space into cubic compartments as before, with the same sphere-diameter, but placing the sphere-centres at the cube-centres, and at the points of bisection of the cube-edges. This representation has the advantage that each cube, considered by itself, shows the positions of the 12 spheres which touch a single sphere placed at the centre.

Another representation is obtained by dividing space into parallelepipeds whose edges are equal and parallel to three edges of the fundamental tetrahedron. The corners of the parallelepipeds will be the positions of the sphere-centres.

Coming now to the subject of closeness of packing:—It is possible to pack spheres in a box in a symmetrical arrangement in which each sphere touches 6 instead of 12, the points of contact being the ends of three diameters at right angles. The space thus occupied by a given number of spheres will be $\sqrt{2}$ times as great as that occupied in normal piling. It is also possible to adopt a symmetrical arrangement in which each sphere touches 8, the points of contact being given by drawing lines from the centre of the sphere to the corners of the circumscribed cube. This will give a density of packing intermediate between the other two.

It would, however, be a mistake to suppose that normal piling is the only system which gives the maximum of compactness. Every system of piling that has the maximum compactness must consist of parallel tiers in triangular arrangement; and each sphere in a given tier must touch 3 in each adjacent tier, besides 6 in its own; but these conditions allow, in adding each successive tier, a choice between two positions. In normal piling the choice is always made in the same way. In the system which comes next in order of simplicity, and which may conveniently be called *antinormal* piling, the choice is made alternately in one way and in the other. The departure from normal piling cannot begin till we come to the third tier: and if there are $2+n$ tiers the number of different arrangements is 2^n, all giving the maximum of compactness.

In normal piling the choice is governed by a fixed tetrahedron; any two spheres that touch must be in a line parallel to one of its edges. The alternative choice is represented by the edges of a second tetrahedron base-to-base with the first.

The base is parallel to the tiers, and must not be turned about; but as regards the 3 edges which meet in the vertex, the change is the same as if they were rotated through 60° in the plane of the tiers. Each sphere in a tier is surrounded by 6, giving 6 gaps 60° apart. Three spheres touching each other can be laid over 3 of the gaps, 120° apart; and the question is, which of the 2 sets of 3 thus available shall we choose.

In normal piling there are 6 directions of lines of touching spheres, 3 of them being in the tier planes, and parallel to the edges of the base of our tetrahedron of reference. The other 3 are parallel to the edges which meet in the vertex of the tetrahedron; and as soon as we depart from normal piling these lines of spheres are broken. Moreover, in normal piling there are, in addition to the assumed set of parallel tiers, 3 other sets parallel to the 3 faces of the tetrahedron that meet in the vertex. In departing from normal piling, we twist these faces alternately one way and the other through 60°, and thus destroy these 3 sets of parallel tiers.

As regards the 3 sets of parallel planes containing sphere-centres in square formation, they have all disappeared. In place of the regular equidistant planes of this kind, we have to draw planes 3 times as close, if we wish to include all the sphere-centres; the distribution of centres in each plane being in alternate stripes of greater and less closeness, the average closeness being of course one third of what it was in normal piling.

If, instead of attending to the centres of the spheres, we attend to the tangent planes at every point of contact of 2 spheres, we find that, in normal piling, each sphere is inscribed in a dodecahedron, whose faces are equal and similar rhombuses, the point of contact being in each case the centre of the rhombus. Space is thus partitioned into equal and congruent cells. The dodecahedron has 24 edges consisting of 4 sets of 6 parallels; and if we take a section across one of these sets, each of the two halves has the shape of a bees' cell. If we rotate one half through 60°, the halves will still fit together; and in their new position they form the dodecahedron in which each sphere is inscribed in anti-normal piling.

According to Prof. O. Reynolds's theory, the material universe (including æther as well as ordinary matter) consists of equal spherical grains, exactly alike, and infinitely hard, so that when they collide the rebound is instantaneous, without loss of kinetic energy.

For the most part, their free paths are infinitesimal compared with their diameters; and their packing differs only infinitesimally from normal piling, so that each grain is securely hedged in by its 12 neighbours, with extremely little room to disport itself between them. But though this is the prevailing condition, there are exceptional spots where there is a crack or loose joint in the piling, affording opportunity for grains to pass across it, and thus partially change their neighbours. "Surfaces of misfit" is a name frequently applied to these places of weakness; and according to the theory they are closed surfaces, approximating to the form of spherical shells, the enclosed nucleus of grains being in normal piling. The surface of misfit, together with the enclosed nucleus, is called a *negative inequality*; and the magnitude of the negative inequality is reckoned by the number of grains which are deficient.

According to the theory, these surfaces of misfit travel through the grains after the manner of solitary waves. The mode of propagation is, that the impulses propagated through external grains towards the loose joint are stronger from one side than from the other, and there is a preponderating transfer of grains across the gap in obedience to the stronger impulse, causing the gap to travel in the opposite direction, as a geometrical consequence; just as a bubble travels upwards through water in virtue of the water moving downwards.

Each of these negative inequalities is an atom of ordinary matter: what we call the momentum of moving matter being merely the manifestation of the real momentum of grains moving in the opposite direction. The apparent mass of an atom is measured by the number of grains that are wanting in its surface of misfit. Calculations are given to show that a moving surface of misfit tends to continue moving with a uniform velocity of translation and with unchanged magnitude.

Again, calculations are given to show that two of these surfaces of misfit have a tendency to approach each other. The argument is to the effect that their mutual approach increases the average compactness of arrangement, and is therefore promoted by the perpetual battering from outside which is equivalent to hydrostatic pressure. Furthermore, the calculation shows that the mutual force between the two atoms varies inversely as the square of their distance. Gravitation is thus explained as a partial relief of pressure in front, allowing a *vis à tergo* to prevail.

This calculation of attraction inversely as square of distance is based on a first approximation, which is sufficient when the distance is a large multiple of the diameter of an atom. It neglects the square of the small ratio of the diameter to the distance. But when the two atoms are very close, this square must be taken into account. At a certain small distance it gives much stronger attraction, thus explaining cohesion and surface-tension ; and for still closer approach it gives repulsion.

Again, electricity is explained by a particular kind of disarrangement called a *complex inequality*. It is conceivable that, in the rubbing together of portions of matter built up in the manner above described, a small cluster of grains may be torn away from its place, and shifted to another neighbouring place. Two opposite disarrangements are thus produced. At the place from which the cluster has been removed, surrounding grains are pushed inwards to fill the gap. At the place to which the cluster has been removed surrounding grains are pushed outwards to make room for it. Each of these two opposite disarrangements produces dilatation, by rendering the packing abnormal ; though Prof. Reynolds calls one of them a *positive inequality*, and uses language which conveys the impression that the intrusion of the cluster of grains produces an increase of compactness.

By reasoning which is difficult to follow, the attempt is made to prove that these two dissimilar centres of disturbance will exhibit mutual attraction. They are regarded as constituting the two opposite kinds of electricity.

Another kind of complex inequality is introduced to explain magnetism. Suppose a spherical cluster of grains to receive a forcible twist, large enough to make portions near its equator (where the motion is largest) change their external neighbours; while portions near the poles have not moved far enough to change their neighbours. These latter portions of the sphere will tend to return to their original position, while portions which have passed a little beyond the position of unstable equilibrium will tend to move further away. Rotational stresses will thus exist between the spherical cluster and the surrounding grains, the stresses being opposite-ways round in different portions. The strains accompanying these stresses involve dilatation; and attractions and repulsions will be exhibited, governed by the criterion that the tendency is for less space to be occupied. The reasoning is not worked out in detail; but the brief statement is made that permanent magnetism is thus explained.

The propagation of light is ascribed to transverse displacement of the granular medium by successive impacts; and Röntgen rays, to normal displacement, similarly propagated. Calculations are given to show that, when proper values are assigned to the diameters, masses, velocities, and mean free paths of the grains, the diminution of the energy of a wave, to the fraction $\frac{1}{e^2}$ of its original amount, will occupy some millions of years in the case of transverse waves, and less than a millionth part of a second for normal waves, the distance traversed by the normal waves in undergoing this diminution being 2800 metres.

The same assumptions, as to the grains, which give these results, give also the correct velocity of light, and the correct value of the gravitation constant.

Prof. Reynolds nowhere hints at the existence or possibility of closest piling other than the normal kind. I do not think I am going too far in saying that he identifies a tendency to closest packing with a tendency to that particular kind of closest packing which he names "normal piling."

It is necessary to face the question, whether this ignoring of other modes of closest piling vitiates the arguments by which the theory is supported.

It is legitimate, in framing a hypothesis as to the structure of the universe, to assume the particular structure which best answers the purpose of yielding an explanation; and where there is a choice between several hypotheses, the simplest is the best, provided it proves sufficient. Of all the possible modes of closest piling, normal piling is incomparably the simplest.

The theory does not expressly assert that the grains, coming together fortuitously, have settled down into this simple arrangement; but nevertheless the question does occur to an inquiring mind whether fortuitous clashing might be expected to lead, in the long run, to such a result. It is near akin to the question whether equal grains of shot, shaken together in a bag, will arrange themselves in normal piling. Prof. Reynolds gives experiments showing that shot so treated do arrange themselves in closest packing, but adduces no evidence as to the particular kind of closest packing that they assume.

Looking at the matter from a theoretical point of view, we have, on the one hand, the fact that, in the casual laying of 3 successive tiers in closest order, it is an even chance whether the piling is normal. In the casual laying of 4 tiers, the chance is only 1 in 4; for 5 tiers it is 1 in 8, and for $2+n$ tiers it is 1 in 2^n.

On the other hand, we have the fact that a normally piled cluster has 6 sets of parallel lines of balls, and these will form efficient battering-rams in collisions with other clusters which can only have 3.

If, instead of collision, we suppose a statical push, as in a football maul, there is still the same advantage. Balls ranged in parallel lines, pushing end on, will be better able to retain their formation than balls with only oblique support behind them.

Again, as has been pointed out by Mr. Barlow [*] in a different connexion, if we suppose that, in the first instance, the balls, under the influence of the tendency to compression, form themselves into small clusters in different kinds of closest piling, there will be misfits at the junctions between such clusters, involving a waste of space; and adjacent clusters will tend to unite and become continuous, so as to occupy less room. Such union is most easily effected when the tiers of the two clusters are nearly parallel to begin with.

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[*] W. Barlow, "A Mechanical Cause of Homogeneity, &c." Sci. Proc. Roy. Dublin Soc. vol. viii. pt. vi. (1897) p. 535.

If each of the two clusters has only one set of tiers (as in the case of a cluster not in normal piling), these tiers are very unlikely to be nearly parallel. If each of them has 4 sets of tiers (as in normal piling) the probability is immensely increased.

The tiers here spoken of are those in triangular formation. And the case is strengthened if we include tiers in square formation; for of these each normal cluster has 4, and the other clusters have none.

For these reasons, clusters in normal piling will unite together much quicker than other clusters.

They will thus become larger than other clusters, and their increased size will add to the effectiveness of their 6 sets of battering-rams.

With these advantages in the struggle for existence, it appears to me feasible to maintain that, in the fortuitous clashing of a universe of grains, with free paths gradually diminishing, normal piling would eventually become the prevailing system.

Prof. O. Reynolds's theory as it actually stands deals, not with remote history, but with the universe as it is; and it supposes the great majority of the grains to be at present locked in their places, with no opportunity for change except at the surfaces of misfit. At these surfaces, the motion is so restricted that the stability of the formation does not seem to be seriously threatened.

I have not made any attempt to verify the elaborate statistical calculations with which Prof. Reynolds's paper abounds; and in the parts of the paper that I have read carefully, several conclusions are drawn which to me are not obvious, but appear very questionable.

My present purpose is not controversy, but explanation; and the style of the paper is so excessively technical that a good deal of explanation seems necessary before intelligent controversy can begin. I have chiefly aimed at an explanation of the geometrical conditions which underlie the system supposed; thereby clearing the way for a more searching criticism, and helping towards the working out of the very fruitful suggestions which the theory contains.

XXIII. *Some Instruments for the Measurement of Large and Small Alternating Currents. By* W. DUDDELL, *Wh.Sc.*[*]

[Plate XIII.]

FOR the measurement of alternating currents over the range from 0·1 to 100 amperes, there are a large number of commercial instruments available which will give fairly accurate results and which are quite easy to use. As soon as one requires to measure a current much outside this range, say $\frac{1}{100}$ ampere or 1000 amperes, then more or less serious difficulties are encountered, even if a moderate accuracy of say 2 or 3 per cent. is required. I propose to describe in this paper some instruments which I have devised, by means of which very small alternating currents and voltages can be measured, and which are also suitable when shunted for very large currents.

In order that an instrument shall read R.M.S. or effective amperes, it is necessary that its deflexion shall be determined by the mean squared value of the current flowing. This leads to a difficulty when it is required to construct instruments for very small currents, which I will illustrate by an example.

Suppose an instrument gives a certain deflexion for 1 ampere and that it is required to alter it to give the same deflexion for a milliampere, then the available deflecting force, which depends on the square of the current, will be reduced in the ratio of one million to one. This is the reason why so little advance has been made in the measurement of very small alternating currents compared to that made in the measurement of direct currents.

The means at our disposal to measure alternating currents may be roughly divided under three heads according to which of the effects produced by the current is taken as a measure of it.

(1) *Electromagnetic Instruments.*—These may be further subdivided into two classes, namely: (*a*) Pure-air core-dynamometers, in which the current to be measured flowing

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[*] Read May 6, 1904.

through one coil reacts on the same current flowing through another coil, no magnetic material of any kind entering into the instrument. (*b*) Instruments containing iron either as a core to the two coils, or as the part reacted on by the magnetic force of one coil. In all these instruments the essential part is a coil or coils carrying the current to be measured which produce magnetic fields; and it is in the design of these coils that the main difficulties are encountered. If it is required to measure a small current, then the coils must be wound with fine wire and contain many turns, so that both their resistance and self-induction are high, which in many experiments is very undesirable, if not absolutely prohibitive.

I came across a very good example of the importance of the self-induction of dynamometers for measuring small currents in connexion with some experiments in which I wanted to measure 0·1 amp. at a frequency of 100,000 ∼ per second. A small reflecting dynamometer was available having 8 turns in fixed and 40 turns in moving coil, which gave about 100 scale-divisions deflexion for 0·1 ampere. On attempting to use it no appreciable deflexion could be obtained, although everything appeared to be working satisfactorily; the trouble was finally traced to the dynamometer behaving like an insulator owing to the great importance of its self-induction at this high frequency. Another effect which may occur at high frequencies with coils of many turns and layers, is that the capacity effect from wire to wire and layer to layer may shunt some of the current which should pass round the instrument, and so destroy any accuracy in the reading. In fact, given a good instrument, the evaluation of high-frequency alternating currents is of great difficulty owing to the errors introduced by the self-induction and capacity of the leads, resistances, &c. which may be used in the measuring circuit.

If the resistance and self-induction of the instrument are of no consequence and the frequency is not high, then about 0·001 ampere can be accurately measured by means of a dynamometer without much difficulty.

The measurement of large alternating currents by electromagnetic means offers quite another set of problems. To carry large currents large conductors are necessary, which have to be carefully stranded so as to avoid skin-effects. This stranding presents many serious practical difficulties, as in order to make the current divide equally between the strands it is necessary to make the self-induction and resistance of each strand the same, and also the mutual induction of all the other strands on it the same. The suitable arrangement of the strands leads to the use of large cables and large-sized coils, generally consisting of a single turn. There is a further very serious obstacle to be overcome, namely, eliminating the effect of the magnetic field surrounding the cables which connect the instrument to the supply. There seems no reason why all these difficulties should not be overcome except the large size and cost of the instrument and considerable loss of power in it.

It is true that transformers can be used to bring the current within the easily measured range, but only at the cost of extra power losses in them and uncertainty in the calibration unless calibrated for the particular frequency and wave-form on which they are to be used. In most cases, when it is required to measure very small currents step-up transformers cannot be used owing to the power lost in them disturbing the circuit in which it is required to measure the current. Dynamometers cannot in general be shunted to use them for large currents unless the drop on the shunt is at least ½ volt, owing to the necessity of using a considerable dead resistance in series with the dynamometer itself to swamp its self-induction and temperature-coefficient errors. Such a drop as ½ volt on the shunt corresponds to a considerable waste of power with large currents; thus with 2000 amperes to be measured the loss of power is 1000 watts, which requires a large shunt to dissipate it.

The above remarks, which refer more especially to pure-air core-dynamometers, apply also to iron core or attracted iron type of instruments, with the further disadvantage of this latter class of instrument being subject to frequency and wave-form errors.

(2) *Electrostatic Instruments.*—In this method the current is measured by measuring the P.D. between the terminals of a known non-inductive resistance through which the current

flows. Using carefully designed non-inductive resistances, this is probably one of the most accurate methods of measuring alternating currents of any value, frequency or wave-form ; its chief limitation lies in the fact that up to the present no electrostatic voltmeter has been put on the market which can be relied on to measure accurately P.D.'s much below 1 volt, and it is better to work with 4 or 5 volts. When measuring large currents a drop of 1 volt on the shunt is generally quite inadmissible ; and when measuring very small currents the P.D. in the whole circuit is often much less than 1 volt. It must also not be forgotten that at high frequencies electrostatic voltmeters take an appreciable current which must be corrected for.

(3) *Thermal Instruments.*—Probably instruments depending on the heating produced by the current are as much used as any others for measuring alternating currents. If the resistance of that part of the instrument which is heated by the current is constant, then the mean rate of production of heat is accurately proportional to the mean square of the current. If this rate of production of heat can be accurately measured, then we obtain an ammeter whose accuracy is entirely independent of wave-form or frequency, and whose calibration is the same for both direct and alternating currents, with the further advantage that the instrument is unaffected by magnetic fields, and can be constructed practically non-inductive.

There would seem, at first sight, to be a very large number of ways of accurately determining the rate of production of heat ; but this is not the case if the watts lost in the instrument are kept small, as this corresponds to a small rate of production of heat which is difficult to measure. In practice the methods of measuring the rate of production of heat have been reduced to four, viz.:—The linear expansion of the heated wire. The expansion of a gas or liquid near or surrounding the wire. The measurement of its change in resistance. The measurement of the temperature by means of a thermocouple.

It must be noted that in all these methods what is actually measured is a rise in temperature and not the rate of production of heat, which is really required, the assumption being that when the steady state is attained and the heat losses from the wire or other heated part equal the rate of production of heat, then the heated part will be at a definite temperature above some other part of the instrument. This fact that what is really measured is a difference of temperature, and not the rate of production of heat, is the cause of most of the defects of thermal instruments. The main defects are the following:—

Slowness of action caused by having to wait till the steady state is attained.

Creep of zero caused by change of temperature of surrounding bodies.

Comparatively large power required to work the instrument due to the great inefficiency of converting electrical energy into heat and then reconverting the heat into mechanical energy to move the pointer.

Temperature errors caused either by the resistance of the heated part varying with temperature, or by the means of measuring the rate of production of heat not being independent of temperature.

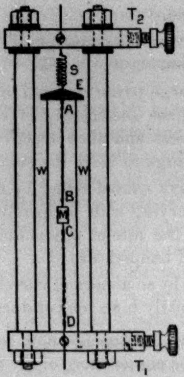
Easily destroyed by an abnormal current owing to the small mass and normally high temperature of the heated part.

The new thermal instruments now to be described were designed to get over one or more of the above difficulties for special purposes, and do not pretend to have overcome the whole of them. The first instrument was designed to be very quick in action for a thermal instrument, and was made to work with a small current so that it could be used as a quick-acting voltmeter. It is essentially a very delicate Ayrton-Perry twisted strip-ammeter which has been improved by the addition of a temperature-compensation device to minimize the zero-creep when the temperature of the whole instrument changes. The working parts of the instrument are shown in fig. 1. A B C D is the Ayrton-Perry twisted strip, of which the part A B is twisted in one direction and the part C D in the opposite. A mirror M and a very thin mica damping-vane are fixed at its centre. This strip is stretched in a frame formed of a brass block T_1 carrying one terminal and a piece of ebonite E, the sides of the frame being formed of wires W W.

This frame itself is stretched by means of a spiral spring S fixed to the other terminal block T_2.

The twisted strip A B C D and the wires W W are made of the same metal; in fact, the strip A B C D is actually flatted from the wire W W, so that the twisted strip and wires have the same temperature-coefficient of expansion. If the wires and strip rise in temperature equally, then the whole frame E W T_1 W simply gets longer and no twist of the mirror ensues. If, however, a current be sent from T_1 to T_2 through the strip, then it heats and twists up, rotating the mirror M. Owing to the fineness of the strip (0·001″ Pt Ag) which is heated by the current, the instrument is very quick in action. The mechanical periodic time is also very short, about $\frac{1}{15}$ second; so that it is able to follow with a fair accuracy currents which vary over a small range as rapidly as one or two cycles per second. The data of this actual instrument are:—

Fig. 1.



Resistance 20 ohms.

Current to give 25 cm. at 100 cm. scale-distance, 22×10^{-3} ampere.

P.D. to give 25 cm. at 100 cm. scale-distance, 0·44 volt.

So that taking 1 cm. as the smallest measurable deflexion and 0·1 mm. as the smallest detectable movement—

The smallest measurable current is $4{\cdot}4 \times 10^{-3}$ ampere.
„ detectable „ $0{\cdot}5 \times 10^{-3}$ ampere.
Watts required to produce smallest measurable deflexion, *i. e.* 1 cm., 387 micro-watts.

It is evident that this very simple instrument has many uses. It is very easily set up, requiring no careful levelling, and is quite robust. The present instrument, which I made three years ago, has often been carried about just as it is in the pocket. The self-induction of the wire is extremely small, and the temperature-coefficient of the resistance of the wire ("Pt Ag) is small also; so that the instrument can be used as a voltmeter to measure voltages down to about 0·1 volt, and I have used it in series with high resistances to measure voltages up to 10,000 volts. The instrument can of course be shunted to measure large currents; but if a good deflexion of say 25 cm. is required, the drop in volts on the shunt, 0·44 volt, is serious. Its main defect is that it requires screening from quick vibrations of the order of $\frac{1}{15}$ second, as the damping is not quite sufficient. I find that this can be easily done by standing the instrument on a heavy block which is suspended by means of wires and springs as in the Julius suspension for galvanometers.

The chief practical use to which I have put the instrument has been to observe the quick variations of the R.M.S. voltage in supply-stations produced either by cyclic irregularity of the engine or by phase-swinging between alternators, converters, &c. For this purpose I put the instrument in the place of my oscillograph in the ordinary film-camera recording apparatus, and photograph the movement of the spot.

It is easy by working to a false zero to obtain a sensibility of 1 cm. change in deflexion for 1 per cent. change in the P.D. By observations of this kind it is often possible to trace the causes of alternators not operating properly in parallel and to find which particular defect and which engine is the cause.

Figs. 2 and 3 (Pl. XIII.) are two records obtained on a circuit in which there was purposely produced a cyclic irregularity having a known wave-form. In each case the cyclic change on the voltage was about ±1 per cent. from the mean. In fig. 2 the voltage was changing sinusoidally about 121 ∼ per minute, and in fig. 3 it was changing along a square wave-form about 68·5 ∼ per minute. In this latter record the shape of the curve of growth and decay of deflexion and the consequent rounding off of the corners of the curves are well shown. In both records 1 cm. = 1 per cent. change in voltage and 10 cm. = 1 second.

The second instrument is much more delicate and sensitive. It was made primarily to measure very small voltages and currents even with very high frequencies up to 120,000 ∼ per second. At this high frequency the alternator I made would only give an extremely small output; it was therefore necessary, in order to carry out the experiment, to reduce the power required to work the instrument to its lowest possible value. I have called this instrument a Thermogalvanometer.

The principle of the thermogalvanometer is quite simple. It consists of a resistance which is heated by the passage of the current to be measured, and the radiant heat from which falls on the thermojunction of a Boys radio-micrometer *. As at first constructed, it consisted of a heating-resistance made of three or four turns of 0·001 inch diameter platinum-silver wire wound on a piece of mica, and placed as close as possible to the radiation receiving-plate of an ordinary Boys radio-micrometer, made by the Cambridge Scientific Instrument Company. This instrument was sensitive but exceedingly slow in action, taking over a minute to approximately attain the deflexion corresponding with the current flowing.

A new radio-micrometer was therefore constructed following the instructions given in Professor Boys' paper, but having a very much smaller suspended loop than the radio-micrometer as usually constructed. The radiation receiving-plate was also omitted in order to reduce the quantity of metal to be heated, and the heating resistance was placed directly underneath the lower thermojunction, so that the junction received heat from it both by radiation and convection. The arrangement of the instrument is shown diagrammatically

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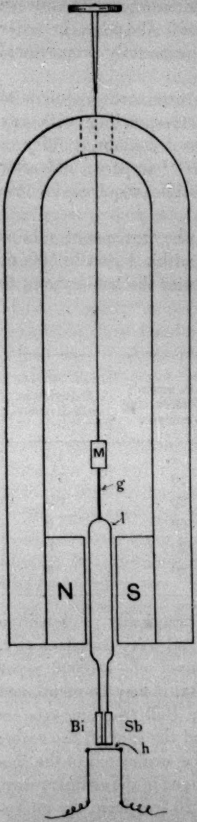
* Philosophical Transactions of the Royal Society, 1889, vol. clxxx. p 159.

in fig. 4, and a photo of the actual instrument with the case removed in fig. 5 (Pl. XIII.).

Fig. 4.



In the field between the pole-pieces NS (fig. 4) of a permanent magnet is suspended by means of a quartz fibre a single-turn coil or loop of wire *l*, to the lower ends of which is fixed the thermocouple Bi, Sb. This loop is surmounted by a glass stem *g* and a mirror M. Below the lower junction of the thermocouple is fixed the heating resistance *h*, one end of which was connected to the frame of the instrument to avoid electrostatic forces. The action of the instrument is as follows. The current to be measured flows through *h* and raises its temperature, causing the lower junction of the thermocouple to rise in temperature above the upper, thus producing a current round the loop, which is deflected by the magnetic field against the torsion of the quartz fibre.

The data of the instrument are:—

Resistance 18 ohms.

Current to give 25 cm. at 100 cm. scale-distance, 8×10^{-4} ampere.

P.D. to give 25 cm. at 100 cm. scale-distance, $14{\cdot}4 \times 10^{-3}$ volt.

So that, taking as before 1 cm. as the smallest measurable and 0·1 mm. as the smallest detectable deflexion,

The smallest measurable current is $1{\cdot}6 \times 10^{-4}$ ampere.

The smallest detectable current is $0{\cdot}2 \times 10^{-4}$ ampere.

Watts required to produce smallest measurable deflexion, 0·46 micro-watt.

The deflexions of the instrument were nearly proportional to the square of the current; assuming the instrument correct at 100 divisions deflexion, then higher deflexions required to be increased by amounts gradually increasing up to between 4 and 5 divisions at a deflexion of 400 divisions, in order that the corrected deflexion might be accurately proportional to the square of the current flowing.

To give an idea of the time the instrument requires to take up the permanent deflexion corresponding with any current, Table I. is given below of the deflexions at different times after starting a current of 8×10^{-4} ampere; and after interrupting the same, to show the time required for the instrument to return to zero.

From which it will be seen that the instrument attains after 10 seconds its full deflexion to within 1 part in 500, or as it is a square law instrument indicates the true current to within 0·1 per cent.

TABLE I.

Time in seconds after starting a current of 8×10^{-4} ampere.	Deflexion in scale-divisions.	Time in seconds after interrupting the current.	Deflexion in scale-divisions.
5	500·5	5	1·5
10	501·0	10	1·0
30	501·5	30	0·5
60	502·0	60	0
120	502·0	120	0
180	502·0	180	0

This instrument, which forms practically a sensitive alternate-current galvanometer, worked very satisfactorily, even with the highest frequencies used of 120,000 ~ per second. As examples of its sensibility, it may be mentioned that on making a suitable noise into a Bell *telephone-receiver* sufficient current is generated to send the spot off the scale; and that if the thermogalvanometer be connected to the line wires of a *microphone-transmitter* arranged in the ordinary way, whistling at a distance of from 15 to 20 feet from the microphone will cause deflexions of the instrument of several hundred scale-divisions. As the instrument will give a deflexion of over two scale-divisions for one millivolt alternating, it might in some cases replace, with advantage, the telephone in the Kohlrausch method of measuring the resistance of electrolytes, as it would avoid the difficulties attendant on estimating the position of a sound minimum.

It is perhaps of interest to compare the current taken by the thermogalvanometer with that which flows through an electrostatic voltmeter when both are used to measure the same P.D. at high frequencies. Thus at 100,000 ∼ per second a certain sensitive "Ayrton-Mather" electrostatic voltmeter, which had applied between its terminals a P.D. of 3·6 volts and gave a deflexion of 392 scale-divisions, was found to have an alternating current of $0{\cdot}12 \times 10^{-3}$ ampere flowing through it due to its capacity. In order to produce the same angular deflexion on the thermogalvanometer, a current of about $0{\cdot}8 \times 10^{-3}$ ampere was required, so that for the same sensibility, considered as voltmeters, the thermogalvanometer, a hot wire instrument, only required less than 7 times the current taken by a sensitive electrostatic voltmeter, at a frequency of 100,000 ∼ per second. Later modifications of the instrument have actually reversed the positions, so that the thermogalvanometer can be made to take less than one-third the current required by the electrostatic voltmeter to give the same deflexion.

The sensibility and the relative calibration depend on the resistance and exact position of the heater h. It is evident that if it is required to measure a very small current, then the heater must have as high a resistance as possible; conversely, to measure a low voltage h should have a low resistance, so that for any given purpose there is a best resistance for the heater.

The power dissipated by the heater to work the instrument does not depend much on the resistance of the heater.

Owing to the fact that the heater must be small in order that the instrument may be quick in action, I have usually used a straight wire about 3 or 4 mm. long, as shown at h, fig. 4, or sometimes a single loop. It was easy enough to make heaters of this type having a resistance under 5 ohms, but for higher values great difficulties were experienced. Straight heaters 3 mm. long, having resistances up to 20 ohms, were finally made by cutting very narrow strips out of gold-leaf. In this way I have made strips of gold 1 inch long having a resistance over 100 ohms. The manufacture of still higher resistances presented great difficulties, until lately I have succeeded in platinizing glass threads and soldering them in place. By this means it is quite simple to make heaters having resistances up to 1000 ohms for 3 mm. length and still higher values can be made. These resistances will easily carry sufficient current for the present purpose; a 2000 ohm heater stood 2 milliamperes without damage, but was destroyed by 6 milliamperes. It remains, however, to be seen whether they deteriorate with time; up to the present the results with heaters under 1000 ohms seem satisfactory.

A set of these heaters have been made and put in the above thermogalvanometer, and the results obtained are given in Table II.

The currents, P.D.'s, and power are those required to give a deflexion $\frac{1}{4}$ the scale-distance, which forms a very convenient basis on which to compare square law instruments. I have also included the Ayrton-Perry twisted strip instrument in the table for comparison.

TABLE II.

For deflexion of 25 cm. at 100 cm. scale-distance.

Instrument.	Resistance in ohms.	Current in micro-amperes.	P.D. in milli-volts.	Power in micro-watts.
Thermogalvanometer, gold heater	18	800	14·4	11·5
Thermogalvanometer, platinum on glass heater	103	346	35·6	12·3
Do. do.	202·5	275	55·6	15·3
Do. do.	363	231	84	19·4
Do. do.	1071	121	130	15·7
Do. do.	3367	88	296	26·0
Do. do.	13910	31	431	13·9
Ayrton-Perry twisted strip	20	22000	440	9680

The small power taken to deflect this instrument combined with the fact that the self-induction and capacity of the instrument are so extremely minute, and that it obeys a square law, should render it very serviceable for many measurements which I need not detail. One instrument can be easily used for various sensibilities by simply changing the heater; and should at any time a very excessive current be put through the instrument and destroy the heater, it can be readily replaced without touching the delicate suspended system.

With the highest resistance heater so far tested, namely 13910 ohms, a detectable deflexion of 0·1 mm. at one metre is given by a current of 0·6 micro-ampere. This resistance is not the highest which I think can be made, as I succeeded in making one of 25,000 ohms, which unfortunately got broken before the sensibility was tested. The radio-micrometer part can also be easily made more sensitive. I have, therefore, reason to hope that it may be possible to improve this thermogalvanometer so as to detect small fractions of a microampere with it.

The present high sensibility combined with the suitability for use with high-frequency currents, leads me to think that it might be applied to measure the current in the vertical receiving wire in spark telegraphy, and thus permit some quantitative measurements to be made which are so much wanting in the published literature on this subject.

The controlling force in the thermogalvanometer is extremely small, and it seemed to me for some time a very drastic procedure to pivot such an instrument and apply to it ordinary controlling springs so as to develop a switchboard instrument from it. Measurements of the power obtainable from different designs of thermojunctions and heaters, led me to conclude that a pivoted instrument might be constructed. I have now made two such instruments. In order to save time these two first instruments were made by converting ordinary moving-coil direct-current instruments. So that it is probable that improvements will be effected when the instruments are manufactured from the commencement specially for the purpose. These two first instruments have turned out very successful. Both instruments have ordinary pointers about 10 cm. (4 inches) long, and deflect through an angle of about 75° for their full scale-deflexion. No. 1 is cased in wood with a mirror under the pointer. No. 2 is iron-cased of the ordinary switchboard type. The works are the same except that one has a heater designed for a small current and the other for a small P.D.

The data of the instruments is as follows:—

	No. 1.	No. 2.
Current for full scale-deflexion	0·1 ampere	2·0 amperes.
P.D. ,, ,, ,,	2 volts	0·15 volt.

These instruments attain to within 1 per cent. of the proper deflexion in 10 seconds after starting the current, so that they are fairly quick in action. They have the property of reading slightly too high (about ½ per cent.) if the current be suddenly brought to its full value, when the instrument is quite cold and has been out of use for some time, the reading decreasing to within ¼ per cent. of the proper value during the first 30 to 40 seconds after starting the current.

The creep of zero is small; in an ordinary room it is practically nil. If the back of the iron-cased instrument where the thermo-junction is situated is heated in front of the fire till hot to the touch while the front remains quite cold, a zero creep of about 1° can be produced. If this great difference in temperature should occur in practice, which is highly improbable, it will produce an error in measuring the full current of under 1 per cent.

The power required to work the instrument, namely 0·2 to 0·3 watt, is comparatively low for pivoted thermal instruments. The low current, 0·1 ampere, taken by No. 1, and the low P.D., 0·15 volt, required by No. 2, compare very favourably with other similar instruments. If No. 2 is shunted for 1000 amperes the loss will be only 150 watts, and a comparatively small shunt will be required. The same applies to No. 1, if used in series with a high resistance or connected to a transformer as a voltmeter.

Temperature error.—In the two present instruments this error is about 1 per cent. for 15° F., the deflexion decreasing with increase of temperature. It is to be noted that this low value has not been obtained by making the moving-coil circuit have a resistance which is practically independent of temperature, but by so choosing the materials of the thermo-junctions that the thermoelectric power rises as the temperature rises, and so partially balances the increase in resistance of the coil. I have found materials which will make this balance complete, and so do away with the temperature correction, but practical reasons which will be referred to later have weighed against their use. If the instrument is required only to measure small currents and not to be shunted—this is the case of a voltmeter for, say, 50 volts or over—then the temperature error can be easily reduced to a still smaller value by suitably choosing the alloy of which the heater is made.

The iron-cased instrument No. 2 will stand a fairly large excess-current without being injured. It will carry twice the normal current required to give a full-scale deflexion for half-an-hour without being permanently injured or its calibration altered. It has also had 2½ times the normal current put through it for 20 seconds without injury. Of course, after such ill-usage, the calibration is temporarily altered a few per cent., but the instrument has so far always recovered itself in about ¼ hour.

The necessity of making the instrument to stand this large overload, limits very seriously the choice of materials and the design of the heater and thermo-junction. If I could be certain that the instrument would not be seriously overrun, the present small temperature error could at once be got over or more sensitive instruments could be built. For practical work, whether on the switchboard or in the laboratory, it is essential that ammeters shall stand safely a considerable excess-current without being injured, for it is better to have a little less accuracy than no ammeter at all at the critical moment when the reading is required.

I must now refer to an objection which is special to this type of instrument, namely, that its calibration is not exactly the same in the vertical as in the horizontal position. I think that this difficulty can be overcome, and I think that in any case too much importance should not be attached to it. Switchboard instruments can be calibrated in the vertical position, and are not likely to be used in any other.

Laboratory standard instruments will naturally be calibrated and used horizontally as usual, because the pivot friction is less in this position.

The heaters in these instruments are made of platinum alloys to stand the high temperature and not deteriorate. Each of the heaters is quite small, and would get easily into a space 6×6×0·1 mm. That in No. 1 consists of a short piece of wire bent zigzag, and that in No. 2 of a piece of foil. The self-induction of these heaters is therefore very small, probably much less than that of any leads that can be used between them and the circuit in which the instruments are used. The instruments can be used to measure very high frequency currents.

The question of the permanency of the calibration is of course a vital one. Judging from our knowledge of the permanency of ordinary moving-coil intruments and the great stability of the platinum alloys used for the heaters, I see no reason to expect any serious alteration in these with time. The constancy of the thermo-junction and its thermo-force is open to question, which can only be satisfactorily answered by time. Up to the present the instruments are keeping all right.

The design of sufficiently non-inductive shunts to carry several thousand amperes for use with these and other instruments is receiving attention from Mr. Mather and myself, and the matter is being experimentally investigated. I have no doubt that the shunts can be made, it is merely a question of the best design.

The advantage of being able to use a single instrument like No. 2, which can be easily and accurately standardised with direct current in connexion with a set of standard shunts giving 0·15 volt drop for all currents from 2 amperes to several thousand amperes, will be appreciated by all who have any quantity of testing work to carry out.

The ease with which one can measure any direct current with a single good moving-coil ammeter and a set of shunts, and the facility with which one can check the accuracy of the instrument and shunts, have always greatly appealed to me, and the present instrument is, I venture to think, a step forwards in this direction in the case of alternating currents.

DISCUSSION.

Prof. R. **THRELFALL**, in a letter to the Secretary, said he should like to congratulate Mr. Duddell on the instruments he had produced and trusted that they would prove as valuable as his oscillograph had done. Mr. Duddell had not told us what was the effect of the change of relative position in the heater and thermojunction during the operation of the instrument. It occurred to him that by taking advantage of this peculiarity, it should be possible to make an instrument having either a scale of equal parts or of any desired form. The great drawback to instruments working by means of heating was, in his experience, that the sensitiveness increased with the reading, but it seemed as if Mr. Duddell might arrange his instruments to have a reading sensitiveness even greater, *i. e.* a more open scale, at the lower readings.

Dr. P. E. SHAW, in a letter to the Secretary, said it was possible to measure the amplitude of vibration in the diaphragm of a telephone-receiver, and Mr. Duddell's instruments would enable us to determine the currents producing these vibrations.

Dr. D. K. MORRIS pointed out that the working forces in an electrostatic instrument could be increased by limiting the angular range without increasing the moment of inertia of the moving system. For instance, this could be effected in a quadrant electrometer with a wide needle by subdividing the needle into narrow fingers and subdividing the quadrants in a similar manner. Dr. Morris showed an instrument constructed on this principle capable of measuring 0·1 volt when used idiostatically and giving larger deflexions with telephone voltages. The limit to this process of subdivision is decided by the practical working clearance: the clearance being determined by the tilting of the needle due to unbalanced electrostatic forces. Such an instrument may be called a *multipolar electrostatic instrument*; and Dr. Morris has used the same idea in the construction of a *multipolar electrodynamic instrument*. Both of these instruments were exhibited at the meeting. Dr. Morris said he was glad of the opportunity of Mr. Duddell's paper for bringing them forward, and although they were well suited for use as zero instruments as detectors, they were not nearly so sensitive as the instruments designed by the author of the paper.

Prof. W. E. AYRTON said we had another example of the ingenuity of Mr. Duddell, and drew attention to the fact that he had not only designed the instruments but had also constructed them himself. He was pleased to see that good results had been obtained by using an Ayrton-Perry twisted strip. The use of such strips had hitherto been limited by the absence of contrivances for compensating temperature-change effects. With such compensation it is possible by using a twisted strip to combine simplicity of construction with constancy of zero. In expressing especial interest in the experiment on spark telegraphy, Prof. Ayrton referred to the fact that Mr. Duddell had designed his thermo-galvanometer for use in an important investigation, being unable to purchase a suitable instrument. The author had spoken about an alternator with a frequency of 120,000, and he hoped we would soon have some account of it.

Referring to the remarks of Dr. Morris, he said it was often difficult to calculate the sensibility of a quadrant electrometer with a divided needle. In order to do so it was essential that the edges of the needles should never be near to the edges of the inductors. Any arrangement to increase the sensitiveness of an electrostatic instrument was desirable, and if Dr. Morris had obtained large deflecting forces without bringing the needles and inductors very near together, he was to be congratulated. He asked the author if it would not be better in his thermogalvanometer to double the heater on itself, and thus obtain a greater heating effect with diminished self-induction.

[*Note received July* 25, 1904.—Since making these remarks Professor Perry has drawn Professor Ayrton's attention to the fact that he must have forgotten that temperature compensation devices were used by them in their instruments some sixteen years ago. Mr. H. B. Bourne, who was their assistant at that time, writes July 18th, 1904 :—" I do not remember any difficulties with the temperature compensation of twisted strip voltmeters ; I think that a tube, part iron and part brass, answered all right. I remember making a sagging wire voltmeter with the sides of the supporting frame constructed with the same wire as described by Mr. Duddell."]

Mr. A. CAMPBELL expressed his interest in the instruments shown. Some years ago, in connexion with the measurement of small alternating magnetic fluxes, he used a hot wire near a minute thermopile connected with an ordinary moving-coil galvanometer. One of the main difficulties with such arrangements was the slowness in reaching a steady deflexion and in settling back to zero. For some relative positions of the thermopile and heater the deflexion passes through its steady value and creeps back to it; for other positions the steady value is reached asymptotically. Even in Mr. Duddell's instruments the effect is quite noticeable, and it appears to want further elucidation. In order to render more uniform the calibration of the scale (which normally follows the square law), Mr. Campbell had sometimes mounted the heater so that by its expansion (or that of another hot wire) a spring drew it further from the thermopile as the temperature of the heater rose.

Mr. K. Edgcumbe congratulated the author, and said that as far as small alternating currents were concerned the instruments shown would be very useful. He could not, however, follow the author when he said that his pivoted form of thermogalvanometer would make a good switchboard instrument, because it was never necessary to measure small currents. In using Mr. Duddell's instruments it was necessary to have very good contacts. Mr. Duddell had spoken of the advantage of being able to remove one heater and put in another, but the speaker pointed out that it would be necessary to adjust the second heater with accuracy, or errors would be introduced. He was not a believer in hot-wire instruments, which were unsuitable for central-station work and gradually dying out in this country.

Mr. W. A. Price said he had recently been thinking of the possibilities of the instrument described by Dr. Watson in the discussion on Prof. Fleming's recent paper on a "Hot-wire Ammeter." He thought that in many respects this instrument would be more practical than those brought forward, although it would probably not be so sensitive.

Mr. W. Bennett asked how far the temperature correction in the twisted strip was perfect. It might be slightly affected by the fact that the twisted strip, though of the same material as the side wires, had been treated rather roughly in the process of flattening and twisting, the coefficient of expansion being possibly lessened. The fact that the strip and wires were at different temperatures might also affect the correction adversely on account of the temperature variation of elasticity. The sensitiveness of the radio-micrometer form of instrument would be increased if the efficiency of heat-transference between the heater and the thermojunction could be improved. If it were possible to make a heater that would nearly surround the junction this might be effected, and the variation of sensitiveness due to inaccurate replacement of the heater after removal would be lessened. He suggested using as a heater a thin film of platinum deposited on the inside of a small thin tube of glass or quartz.

Mr. R. S. **Whipple** said he would like to testify to the great patience of Mr. Duddell in overcoming the difficulties met with in the construction of his instruments. He was of the opinion that a good deal of the bad name of hot-wire instruments could be attributed to their inability to stand overload.

Mr. H. **Tinsley**, referring to Prof. Ayrton's remarks about the position of the needle in an electrostatic instrument, said he had recently tested the matter accurately. So long as the edges of the needle were well within the inductors, it was easily possible to calculate the deflexion which would be produced by any voltage to about one part in five hundred. Under such conditions the instrument accurately followed a square law, and its working constant could be obtained from one observation.

Mr. J. **Swinburne** said that a thermopile had been used many years ago in a similar way to that in which the thermojunction had been used by Mr. Duddell, but the sensitiveness obtained was not nearly so great as that of the instruments exhibited. With regard to electrostatic instruments, he did not see any advantage in being able to predict the deflexion produced by any voltage, according to a definite law, as it was always easy to calibrate such instruments. The shape of an instrument with an Ayrton-Perry twisted strip often rendered it unsuitable for many purposes.

Mr. W. **Duddell**, in reply, said that it was necessary that the heater and the thermojunction should be very small to obtain quickness in action and avoid creeping effects. The smallness of the heater in the thermogalvanometer made it very difficult to double it back on itself to reduce its self-induction, as suggested by Prof. Ayrton. The self-induction of the actual heaters used was extremely small, being only that of a loop 15 mm. long and 4 mm. wide. The drop back in deflexion referred to by Mr. Campbell might be due to two causes—the slow rise in temperature of the cold junction of the thermocouple, or to the increase of the resistance of the circuit by heating. By altering the design of the thermojunction these effects could be reduced. Referring to Mr. Edgcumbe's remarks, he pointed out that his pivoted instrument used less watts than any instrument on the market, and could be used if necessary in connexion with transformers.

XXIV. *On the ascertained Absence of Effects of Motion through the Æther, in relation to the Constitution of Matter, and on the FitzGerald-Lorentz Hypothesis.* *By* Prof. J. Larmor, *Sec.R.S.*[*]

In a recent paper by Prof. D. B. Brace (Phil. Mag. April 1904, p. 318) the author removes by very refined experimenting all trace of doubt from Lord Rayleigh's conclusion that motion of transparent solids through the æther does not induce any double refraction, even to the second order of the ratio of the velocity of the translation to that of radiation; but he infers from this the non-existence of the second-order deformation of the solid due to its translation, suggested by FitzGerald and by H. A. Lorentz to account for Michelson's earlier demonstrated absence of effect on optical interferences over long paths in free æther. As he remarks, it had previously been suggested by Lord Rayleigh that such an inference might possibly follow from this result. The object of this note is to explain that the inference in question is the opposite to that which I still hold to be the natural result of the theory of the motion of molecular aggregates through æther, as hitherto developed[†].

The argument of Prof. Brace proceeds on the basis that the *whole* effect of the convection through the æther is to *introduce new forces* between the molecules, causing the shrinkage aforesaid along the direction of convection; and it can be readily granted that if this were all, double refraction must result. But both the line of argument suggested as probable by Lorentz *, and the molecular analysis offered by me some years later †, proceed by comparing a system shrunk in the FitzGerald-Lorentz manner and convected through the æther, with the *same system* unshrunk and at rest, and finding a complete correspondence between them as regards the states and activities of the individual molecules. As the argument is somewhat complex and has been misunderstood, a brief re-statement of the result may prove useful.

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[*] Read May 27, 1904.

[†] 'Æther and Matter,' Camb. Univ. Press, 1900, chapter xi.

We are to compare the field of physical activity of a system of molecules at rest, with the field of the identically same configuration of molecules in uniform translatory motion through æther. If small quantities of the order of the square of the ratio of the velocity of convection to that of radiation (v/C) are neglected, the Maxwellian physical equations for the second system, referred of course to axes of co-ordinates moving along with it, can be reduced to the form belonging to the same system at rest, by the transformation first developed by Lorentz: namely, each point in space is to have its own origin from which time is measured, its "local time" in Lorentz's phraseology, and then the values of the electric and magnetic vectors

$$(f,\ g,\ h) \text{ and } (a,\ b,\ c),$$

at all points in the æther between the molecules in the system at rest, are the same as those of the vectors

$$\left(f,\ g-\frac{v}{4\pi\mathrm{C}^2}a,\ h+\frac{v}{4\pi\mathrm{C}^2}b\right) \text{ and } (a,\ b+4\pi vh,\ c-4\pi vg)$$

at the corresponding points in the convected system at the same local times. This correspondence can, in fact, be shown to locate the electrons at corresponding points in the two systems, and to make them equal; if, then, they are held in rigid connexion, or more generally if their states of orbital motion in the molecules are conserved, the effect of translatory motion of the system with velocity v is to transform the æthereal field around them and between them as here specified. The fields of æthereal activity are *not identical*, but where one vanishes at any point so does the other at the same point. This conclusion was reached by Lorentz, who pointed out that it carried with it a null result for all recognizable optical tests of convection in the system, up to the first order, with the one exception of the Doppler effect which is involved in the "local" time measurements, and which is only a partial exception because it refers to radiation coming from outside the system.

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* '*Versuch einer Theorie*,' 1895, §§ 91–2, translated in part in 'Æther and Matter,' p. 186.

† *Loc. cit.*

Does, however, the system of electrons need to be constrained in order to prevent change of configuration when being convected? The force acting on an individual electron e is thereby changed from

$$4\pi C^2 e\left(f,\ g-\frac{v}{4\pi C^2}c,\ h+\frac{v}{4\pi C^2}b\right) \text{ to } 4\pi C^2 e(f,\ g,\ h).$$

If there is a magnetic field (a, b, c) there will thus be alteration: if there is no sensible average magnetic field, even among the molecules, we may perhaps fairly assume, with Lorentz, that no constraint is needed in order to prevent change in molecular configuration in the system due to convection. Anyhow, the absence of recognizable optical result to the first order is certain, as the physical constants of the system in bulk must be unaltered to that order.

But the brilliant experimenting of Michelson and Morley had already led to the recognition of absence of optical result up to the second order of the ratio of the velocities. Thus the question was suggested whether the above correspondence between the resting and convected systems can be effectively extended up to the second order. It is, in fact, found that the Maxwellian circuital equations of æthereal activity, in the ambient æther, referred to axes moving along with the uniform velocity of convection v, can be reduced to the same form as for axes at rest, up to and including $(v/C)^2$, but not $(v/C)^3$, by adopting a local time $\epsilon^{-\frac{1}{2}}(t-vx/C^2)$ as before, but with a new unit $\epsilon^{-\frac{1}{2}}$, and also a reduced unit of length parallel to x equal to $\epsilon^{-\frac{1}{2}}$, where here and in what follows ϵ represents $1+v^2/C^2$, the units of length along y and z remaining unaltered. It is found that for two æther-fields, one referred to fixed axes and the other to moving axes, standing in this mutual correlation, the electrons, or poles, in approaching which the æthereal electric vector becomes infinite as er^{-1}, are situated at corresponding points and are of equal values: the relation, exact to the second order, is now that

$$(f, g, h) \text{ and } (a, b, c)$$

in the field belonging to the fixed system of poles correspond to

$$\epsilon^{\frac{1}{2}}\left(\epsilon^{-\frac{1}{2}}f,\ g-\frac{v}{4\pi \mathrm{C}^2}c,\ h+\frac{v}{4\pi \mathrm{C}^2}b\right)$$

and

$$\epsilon^{\frac{1}{2}}(\epsilon^{-\frac{1}{2}}a,\ b+4\pi vh,\ c-4\pi vg)$$

for the field belonging to the convected system; where ϵ is $1+v^2/\mathrm{C}^2$, as above, the factor $\epsilon^{\frac{1}{2}}$ being needed to make corresponding poles equal in value instead of merely proportional.

If each pole or electron is connected with a molecule possessing extraneous mass, and it may be having an extraneous field of gravitational and other force of its own, and thereby interacting with other molecules, we shall want to know the forces exerted on that molecule by the surrounding æther, in order to form its own equations of motion, which must be combined with those of the æther-field around it in order to constitute a complete system. But if such other forces are molecularly insignificant, or better, if the electron is a mere passive pole—nucleus of beknottedness in some way—in the æther, conditioned and controlled entirely by the æther around it, just as a vortex ring is conditioned by the fluid in which it subsists and is also carried along thereby, then, as in the familiar hydrodynamics of vortices, the motion of the æther determines the motion of the entirely passive electrons, and the idea of force acting between them and the æther is dispensed with.

If, then, matter is for physical purposes a purely æthereal system, if it is constituted of simple polar singularities or electrons, positive and negative, in the Maxwellian æther, the nuclei of which may be either practically points or else small regions of æther with internal connexions of pure constraint, the propositions above stated for the first order are extended to the second order of v/C, with the single addition of the FitzGerald-Lorentz shrinkage in the scale of space, and an equal one in the scale of time, which, being isotropic, is unrecognizable.

On such a theory as this the criticism presents itself, and was in fact at once made, that one hypothesis is needed to annul optical effects to the first order; that when these were found to be actually null to the second order another hypothesis had to be added; and that another hypothesis would be required for the third order, while in fact there was no reason to believe that they were not exactly null to all orders. Such a train of remarks indicates that the nature of the hypotheses has been overlooked. And if indeed it could be proved that the optical effect is null up to the third order, that circumstance would not demolish the theory, but would rather point to some finer adjustments than it provides for: needless to say the attempt would indefinitely transcend existing experimental possibilities.

As, then, the theory contains no further power of immediate adaptation, what are the hypotheses on which it rests, and how far are they gratuitous hypotheses introduced for this purpose alone? Up to the first order the electron hypothesis, that electricity is atomic, suffices by itself, as Lorentz was the first to show. Yet, even if the nature of the particles of the cathode discharge had never been made out, and the Zeeman effect had never been discovered, the facts known to Ampère and Faraday were sufficient to *demonstrate* that no other conception of electricity than the atomic one is logically self-consistent *.

Up to the second order the hypothesis that matter is constituted electrically—of electrons—is required in addition. For this there is no independent evidence except perhaps the general simplicity of the correlations of physical law. The circumstance that positive electrons have not yet been isolated naturally counts considerably on the other side; yet the theory puts no limit to the size and inertia and complexity of an electron, it only prescribes that it must be a collocation of æther poles connected together by some sort of pure constraint, but with no extraneous activities.

Any rival theory must on the threshold give an account of the Michelson null optical result, of Trouton's null electric result for convection of a charged condenser †, and of Rayleigh's absence of double refraction now rendered thoroughly secure by Brace ‡.

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* *Cf.* 'Æther and Matter,' p. 337.

† Phil. Trans. 1903.

‡ The null influence on optical rotation, observed by Rayleigh, counts here as a first-order effect.

As electrons are already held to be a reality on various grounds, theoretical and experimental, it would appear therefore that there is much to be said for a benevolent attitude to the proposition that all the interactions of matter, so far as the laws of physics and chemistry extend, are to be described as phenomena occurring in and through the æther, and thus differentiated from the more recondite world of vital growth and change which they make manifest to our senses. This principle does not yet, so far as one can see, stand in the way of any other branch of physical science, while it accounts for the very remarkable absence of influence of the earth's motion through space on the most sensitive phenomena, and is almost led up to thereby.

It is pertinent to the present subject to refer to Mr. Sutherland's recent remarks (Phil. Mag. April, p. 406) on the magnetic effect of electric convection, in relation to the mysterious action of a dielectric varnish that has been announced by Crémieu and Pender. The discrepancy in the conservation of energy, there described, applied to the domain of electric polarization, is too startling to have been overlooked by the current theory*; and accordingly closer consideration gets rid of the difficulty. When an electron e is transferred in an electric field from a place where the potential is V_1 to a place where it is V_2, the force acting on it, being e multiplied by the gradient of V, does work equal to $e(V_1 - V_2)$. When, however, the electron is embedded in a piece of dielectric matter which is so transferred, the force acting on the electron itself is diminished by the presence of the surrounding polarized matter, and so the work done on the electron is less than before: but now the electric polarization induced by the electron in this surrounding matter is also acted on by the electric field, and if we add the work done on it during the movement, we shall get the same total work as before for the system that is moved, and there will be no discrepancy to be otherwise explained.

Cambridge, April 7, 1904.

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* *Cf.* Phil. Trans, 1897 A, p. 248, and 'Æther and Matter,' 1900, Appendix A.

XXV. *On Coherence and Recoherence.*

By P. E. SHAW, *B.A.*, *D.Sc.*, *and* C. A. B. GARRETT *.

WHEN electromagnetic waves fall on a loose contact of two metallic or other conducting surfaces, coherence occurs between the surfaces whether there be a completed metallic circuit or not. The resistance of the contact falls, and the surfaces cling together and can only be sundered by forces which are small, but measurable. The best proof that coherence has taken place is, not that the P.D. between the surfaces has fallen (for this always occurs when the pressure between the surfaces is great), but that definite force is necessary to sunder them, always provided pseudo-coherence is avoided †.

In this paper coherence is tested and measured solely by the sundering forces. In a former paper ‡ one of us showed how, by taking the simplest case of two surfaces only, it is possible to measure the forces necessary to sunder them when cohered. It was also shown that the surfaces recohere, *i. e.* show a power of clinging when brought together after coherence has been broken, and that this clinging power is comparable with that occurring in the original coherence.

In the present paper the experimental investigation is carried further than before, and an attempt is made at explanation.

Apparatus.

Two forms are used.

1. *The Boom Apparatus* (fig. 1).—Similar to an apparatus described in the former paper. The suspended coil C of a D'Arsonval galvanometer is suitably wound, and to its frame is soldered a boom A of copper wire. The boom projects a short distance and is perpendicular to the coil. A similar wire B is vertical and is long enough to make contact with A as shown. Coherence is produced at the single contact where A and B meet. M is a mirror by which a spot of

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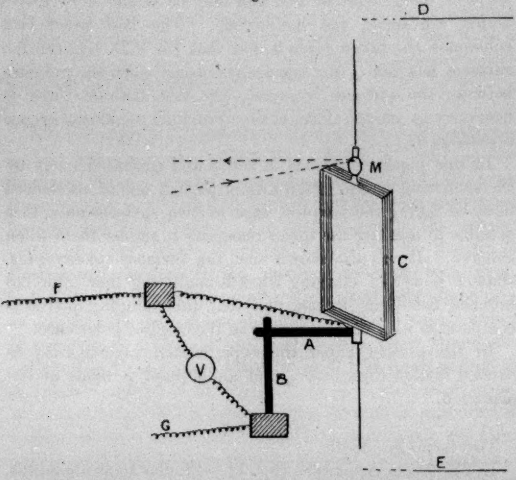
* Read May 27, 1904.

† See Shaw, Phil. Mag. March 1901, p. 271.

‡ Shaw, Phil. Mag. March 1901.

light is made to indicate the position of the boom. The suspensions of the coil are of thin strip phosphor-bronze, and the connexion between F and A is a long loose piece of thin phosphor-bronze. D and E are leads for the coil. The circuit is completed through a battery and resistances (a dial pattern rheostat is used to give slow gradational changes in resistance). F and G are the leads for the contact-circuit.

Fig. 1.

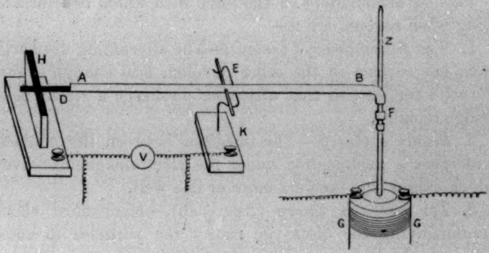


In this circuit we have a battery, resistances, galvanometer, and as shown a voltmeter V as shunt to the contact. The voltmeter, reading to $\frac{1}{100}$ volt, is a convenient indicator. It shows whether there is (1) full coherence, (2) partial coherence, (3) pressure between A and B without current passing, or (4) pressure between A and B with current passing.

This apparatus is very sensitive, but has the disadvantage that it is difficult to clean or remove the contact attached to the coil without breaking the delicate strips of phosphor-bronze of the suspension. The following device, though less sensitive, has no delicate suspension and is more handy.

The Balance Apparatus (fig. 2).—Here DH is the variable contact. AB is a tube made of thin sheet aluminium. At B it is continued in a vertical helix F through which an ordinary sewing-needle Z is thrust. Into the end A of the tube a copper wire D is thrust. Through the point E another fine needle is pushed, the two ends of the needle resting on two curved supports of the wire K.

Fig. 2.



By pushing D in or out of the tube, the beam can be balanced about E, and by pushing the needle up or down in F the beam can be rendered *just* stable. The beam is obviously very light, so that in AB we have all the conditions required in a sensitive balance, and the wire D can be made to rest with any small pressure on the wire H which crosses it at right angles. A coil GG is placed below the magnetized needle, so that by passing a suitable current in GG the needle is sucked into GG and the contact DH is broken.

It is easy to see how by varying the current in GG the pressure between D and H can be changed at will.

The voltmeter V is shown shunting the contacts as before. Whether we use Apparatus I. or II., it must be placed on a table supported by long rubber springs, to avoid extraneous vibration. This is indispensable, since even a small shock breaks the coherence.

Each apparatus has to be roughly calibrated for force, so that knowing what current is put on in the actuating coil

GG (fig. 2) we may convert it into a known thrust or tension felt at the contact, according as the current is direct or reverse. In the former paper the method of calibrating the boom apparatus was described. For the balance apparatus calibration is easy: we put a rider of known weight on the balance itself at such a place that its effect is equal to that of the current used in GG.

Coherence.

Possible explanations of the force with which two surfaces cling when cohered are :—

1. *The fusion theory* (Lodge).—The alternating discharge from one surface to the other develops heat enough to fuse the solid surfaces, so that after the discharge a strong solid bridge remains.

2. Branly's *theory.*—The medium between the contacts undergoes a modification under the electromagnetic waves, which causes it to conduct more or less well.

3. *The adhesion theory* (Auerbach).—Mechanical shock accompanying the discharge causes the particles to come into and remain in more intimate contact.

4. *The ionic theory.*—Either a film condensed from the atmospheric gases or else an oxide film is ionized by the passage of the discharge; so that positive and negative charges will exist in or on the opposed films causing them to attract each other electrostatically, and also causing the resistance of the contact to fall.

It is quite possible that there may be complications, *i. e.* two or more of the above effects may occur at the same time; thus both adhesion and ionization may occur and conspire to produce coherence.

One point must be borne in mind in these contact phenomena, viz., the part played by the films on the contact surfaces. The contact between the two metal surfaces cannot pass until these films have been penetrated. The researches of Blondel[*], Van Gulik[†], Sundorph[‡], Guthe & Trowbridge[§], Shaw[‖], all show this point clearly.

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[*] *Éclairage Électrique*, 1892.

[†] Wied. *Ann.* 1898.

[‡] Wied. *Ann.* 1899.

[§] Phys. Rev. July 1900, p. 22.

[‖] Phil. Mag. March 1901.

It is not easy to say, in a general case, how much of these effects is due to each of the following: (*a*) condensed air, (*b*) condensed water-vapour, (*c*) oxide on the metal. For this reason, an uncleaned surface often behaves in a most contradictory way. If the movable surface meets the other impulsively, the films are penetrated and a current passes; whereas a gentle approach results in the films remaining intact and no current passes; and even if the surfaces be pushed together by reversing the current in the coil (fig. 2), it requires a great steady force to bring the metals into contact. In one case, using the balance apparatus, the force was measured, and found to be about 20 dynes. This is a very large force, since ½ dyne is the force with which the surfaces are usually pushed together. It is hard to believe that any film other than a solid one could offer such resistance. Again, the voltmeter reading is not so directly related to the strength of coherence as would be expected if metal and no film existed at the contact.

By scraping and drying the surfaces we can get rid of the films of condensed water-vapour and of oxide, and it is then found that, for several minutes at least, the effects are quite simple.

In order to assist us in deciding between the rival theories, we will give two crucial experiments.

(*a*) Suppose the two contact surfaces, in either of the above forms of apparatus, are brought together gently so that the films are not penetrated by an impulsive blow: *no current will pass*. When a spark is produced near them, coherence occurs: this shows that the surfaces are not more than about 12×10^{-6} cms. apart*. It might be supposed that the P.D. for the steady current between the surfaces, being small, will cause a spark to pass if the film separating them is so thin. But the P.D. is sometimes only 1 volt, and never more than 6 volts. Until recently, the discharge distance for less than 40 volts had not been measured; but one of us has been able to show that for 6 volts the distance is about 6×10^{-6} cms., and for 1 volt is 1×10^{-6} cm. These results show us what is the superior limit to the thickness of film existing when current passes through the contact. Suppose the P.D.

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* *Loc. cit.* p. 291.

between the contacts is 6 volts, then, as shown, the film between the contacts must be more than 6×10^{-6} cm. thick, and when discharge occurs the electric waves burst over this interval.

(*b*) Now sunder the surfaces; clean and dry them. Bring them together gently as before: *the current now passes readily.* On producing a spark, it is found that coherence will either be slight or non-existent. Thus a high resistance of the contact is a necessary antecedent condition of coherence. We may conclude either (1) that the film acts as a flux assisting the fusion of the two surfaces, or (2) that the ionization of the film into positive and negative charges causes attraction of the surfaces, or (3) that there must be some gap between the surfaces; this gap provides a large resistance so that enough heat is developed by the passage of the waves to fuse the surfaces. The last theory can be proved experimentally to be correct, for it is possible *after cleaning the surfaces* to bring them together with such slight pressure that the current passing is small. On passing a spark near the coherer, coherence readily occurs, as shown below.

TABLE I.

	Force in dynes to sunder surfaces, F.	Voltmeter Reading, V.
Before spark	·5 dyne	0·25
After „	1·12×·5 „	0·10
Before spark	·5 „	0·30
After „	1·12×·5 „	0·10
Before spark	·5 „	0·30
After „	1·10×·5 „	0·12

Thus coherence occurs even when no film existed previously, provided there is a gap between the surfaces. In the former paper * it was shown, by using the electric micrometer, that when coherence is broken the surfaces part with a "snap" which sounds in the telephone like the sundering of a solid body, and that when the surfaces are pressed together again, the ends of the bridge *ab* (formed when the surfaces were severed) are found and can be easily pressed back flat.

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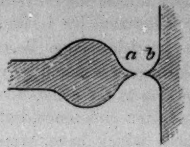
* *Ibid.* p. 292.

These facts show that coherence is a solid effect, and dispose of the ionization theory and also of Branly's theory.



It was shown in the former paper[*] that the strength and resistance of the bridge proves it to have about the tenacity of the solid metal (for copper $T = 2 \times 10^8$ C.G.S.). This again supports the fusion theory; and though it does not confute the adhesion principle, it requires that the forces of adhesion should be at least as great as if the surfaces were fused and solidly attached to each other. But a simple calculation of the heat developed at the contact shows that there is ample to fuse the metal. This disposes of the simple adhesion theory; and leaves only the fusion theory, which seems in entire accord with all facts known.

Recoherence.

After the cohered surfaces have been sundered, if they are brought together again they often cohere with considerable, though much less, force than in coherence. This can be repeated sometimes once or twice (without further sparking), but the power to recohere soon leaves the surfaces, and they return to the normal state of inability to cling together. This is shown in three distinct cases below. F is the force in dynes required to sunder the surfaces before coherence; F_1 is the force directly after coherence; F_2, F_3 . . . , are successive values, showing recoherence.

TABLE II.

	Case 1.	2.	3.
F	·5 dyne	·5 dyne	·5 dyne
F_1	5 × ·5 „	4·5 × ·5 „	13 × ·5 „
F_2	3·7 × ·5 „	4·5 × ·5 „	8 × ·5 „
F_3	4·6 × ·5 „	1·1 ·5 „	·5 „
F_4	·5 „	·5 „	·5 „
F_5	·5 „	·5 „	·5 „

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[*] *Ibid.* p. 282.

Thus after a few times the surfaces become normal and show no signs of clinging. This is quite a different effect from coherence, for there is now no rapidly alternating current passing through the contact causing coherence by fusion. There seems evidence here that the particles have been oriented by the powerful currents at the contacts when they were cohered. The orientation must be due to the coherence and not to the sundering, for it occurs after coherence, but not after the sundering of two surfaces which have been carrying a current in the ordinary way. The current-density at the bridge between the contacts is enormous, and it may be assumed that the particles composing the bridge would be orientated in some way so as to give maximum conduction. This orientation is seen for a short time, but is unstable (just as are magnetic and other orientations) and disappears after a few makes and breaks.

The Nature of Recoherence.

Various tests applied :—

(1) Recoherence is, like coherence, a solid effect, for if the surfaces be brought together gently at once after sundering, they recohere ; but if an interval, say of half a minute, elapses, they refuse to recohere ; no doubt due to the growth of a film on the surfaces.

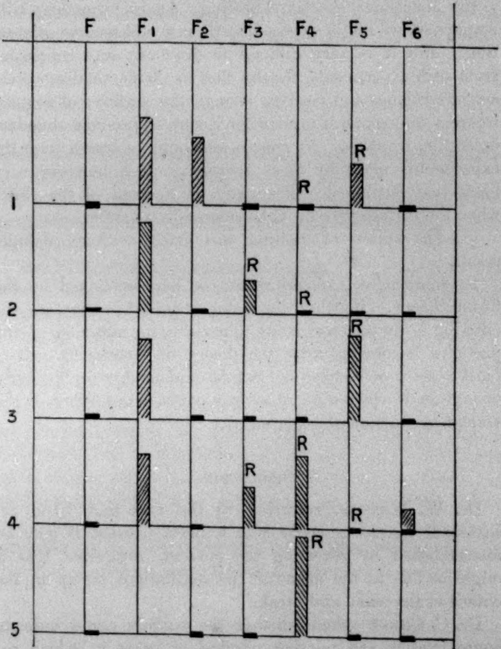
(2) Suppose recoherence no longer occurs for a direct current, if the current passing through the contacts be reversed recoherence often occurs and is renewed once or twice more by reversals. This is shown below in five distinct cases.

TABLE III.

	1.	2.	3.	4.	5.
	dyne	dyne	dyne	dyne	dyne
F	·5	·5	·5	·5	·5
F_1.........	8 ×·5	8·2×·5	7·4×·5	6·7×·5	
F_2.........	6·1×·5	·5	·5	·5	·5
F_3.........	·5	R 3 ×·5	·5	R 3·8×·5	·5
F_4.........	R ·5	R ·5	·5	R 6·7×·5	R 12×·5
F_5.........	R 4 ×·5	·5	R 8 ×·5	R ·5	·5
F_6.........	·5	·5	·5	1·8×·5	·5

Here R stands for reversal of current; otherwise the current is direct. In the fourth case, after 3 trials with direct current, we have 3 with reverse and then one with direct; each change develops recoherence (see fig. 3).

Fig. 3.



(3) If the cohered wires are sundered and left touching with no current passing for many hours, they are sometimes found to recohere readily. There is no question of film since the surfaces have been in contact throughout the long interval. In the fifth case in the above column, coherence

was produced and broken overnight and the wires left in contact. In the morning the values F_1, F_2 ... were found.

(4) Reversal of current. By analogy with magnetism, it might be possible to "wipe out" orientation by rapid reversal of current of gradually decreasing strength. This was tried with no result.

(5) Mechanical shock and heating. Again, by analogy with magnetism we might expect that these would remove orientation. But it is very difficult to draw any sure conclusion from such experiments, for the first is liable to disturb the delicate balance and perhaps change the surfaces of contact, whereas the second if carried far is sure to produce abundant oxide-film troubles. No conclusion could be drawn from the experiments made by these agencies. It is, however, very likely that recoherence dies away on account of the shock when the surfaces are brought repeatedly together.

(6) The action of radium was tried without definite result.

To summarize, Coherence can be best explained by the fusion theory. Recoherence is due to residual coherence, the clinging of the surfaces being apparently an adhesion of the particles (somewhat after the theory of Auerbach). It is hard to see how recoherence can be explained, except by some change in or orientation of surface particles occurring in the violent commotion of coherence.

DISCUSSION.

Dr. W. WATSON, referring to the fact that when recoherence ceases to occur with a direct current it may be re-established by reversing the current, suggested that it might be due to the effect of the oscillations set up in the circuit at the make and break.

Dr. C. CHREE asked whether the authors could indicate approximately the relative number of cases in which recoherence failed when the current was not reversed, but occurred when the current was reversed. He suggested that the fragmentary remains of the bridge which the authors described as existing during coherence might sometimes be so shaped as to make it easier for the current to pass in one direction than the other.

Prof. TROUTON asked if recoherence would occur without a current running through the contact. He suggested that one side of the contact might be a point and the other side approximately plane, so that the current might flow more easily in one direction than the other, in analogy with the well-known fact in connexion with spark direction in similar cases.

Mr. R. S. WHIPPLE asked if the effect depended upon the materials. In the case of a Callendar recorder, with a boom contact, the contact-pieces being made of platinum, it was found that the sensitiveness of the instrument depended upon the direction of the current.

Dr. R. S. CLAY asked if the reversal of the current which established recoherence took place when the substances were in contact.

Dr. P. E. SHAW, in reply to various questions, said that mere reversal of current would not produce the recoherence phenomenon. It only occurs as a sub-permanent effect following coherence, and cannot therefore be attributed to sparking caused by rapid reversals. The power to recohere disappears when contact has been made and broken a few times. As to the action of platinum and other materials as contact-surfaces, he found platinum singularly apt at coherence, but recoherence has not been looked for except in a copper-copper contact. Any explanation offered for recoherence must take account of the observed effects, viz.:—that it is an effect only found after coherence, is unstable to various influences, and arises most often by reversal of current but sometimes without such reversal.

XXVI. *A new Automatic Gas-Pump.*

By C. E. S. Phillips*.

This apparatus is constructed upon a plan which enables the pump, when once set in operation, to continue automatically and to produce as perfect a Torricellian vacuum as is possible.

It has been devised with a view to providing a comparatively portable machine suitable for special laboratory work or for researches requiring prolonged pumping, and consists of three distinct parts, viz. :—a small motor-driven mechanical pump, a four-way control valve, and a modified Toepler apparatus by which the final vacuum is obtained. The action is as follows :—A vessel (say a Röntgen-ray lamp) having been sealed on to the Toepler pump, the motor is set in motion, and a vacuum produced in the whole apparatus equivalent to within about ·5 inch of the barometric column.

The mercury now fills the lower chamber L (fig. 1) of the Toepler and completes an electrical circuit, which includes an electromagnet capable of so moving the controlling valve that it allows air at atmospheric pressure to enter and drive the mercury slowly into the upper chamber U. The Toepler pump operates in the usual manner. At the instant the quantity of mercury, moving as described, diminishes in L, the circuit just completed is broken, but the mercury soon accumulating in the trap T "makes" a second circuit which moves the slider of the valve back to its original position. The chamber is thus connected with the mechanical pump, the mercury falls, and the cycle is repeated automatically.

The gas removed by the Toepler would accumulate in the trap T were it not that the mechanical pump is periodically connected directly therewith. This is achieved by the control valve, when permitting air to enter L also placing the mechanical pump in communication with the trap. The pressure in T is thus kept fairly constant, and the difference between it and that within the Toepler-pump head may at any moment be observed by measuring the height of mercury in the capillary-fall tube C.

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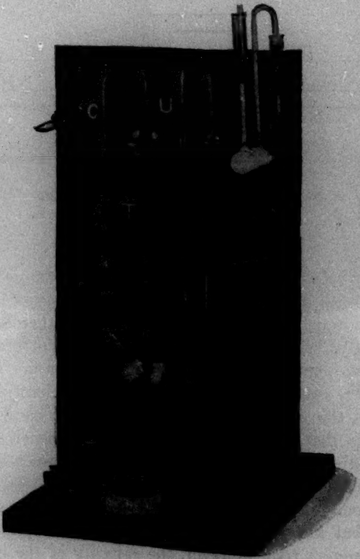
* Read April 22, 1904.

The control device consists essentially of a hollow slider moving over a smooth flat bed, and covering or uncovering

Fig. 1.

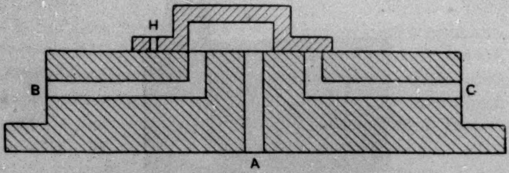


ports in the same manner as an ordinary slide-valve, but since the travel of this slider S (fig. 2) is only one-eighth of an inch many devices are possible for actuating it; an electromagnet appears very suitable.

Two relays are introduced, one into each circuit, in order that the current which operates the control-valve may be broken at external mercury cups.

In the circuit, the completion of which moves the slider into the position for allowing air to enter L, the action of the relay is retarded, for the purpose of enabling the mercury in the Toepler pump to remain in the lower chamber during a distinct interval of time before ascending.

Fig. 2.



A communicates with the mechanical pump.
B ,, ,, lower chamber L.
C ,, ,, the trap T.

A small hole H bored through the slider-flange ensures that the inrush of air into the lower chamber may be checked, and the upward movement of the mercury therefore take place slowly. The ideal conditions for efficiently working the Toepler pump are therefore attained, for the mercury rises slowly, sweeping out before it very efficiently the gas which fills the upper chamber; on the other hand, it falls rapidly (so that no time may be lost) and remains in the lower chamber for a fixed period before again ascending.

Experiment has shown that the apparatus is fairly rapid in its action. Careful tests under specific conditions are in progress, but not yet sufficiently complete for publication. In a preliminary trial, however, the capacity of the chamber U being 160 c.c., 5 lbs. of mercury were raised three times per minute, with the result that a Röntgen-ray bulb of 200 c.c. capacity was exhausted in half-an-hour. The glass work is easily removable for cleaning or repairs, and the wood supports fitted with adjustable brackets. The height of the complete apparatus is 18 inches.

FIG. 8.

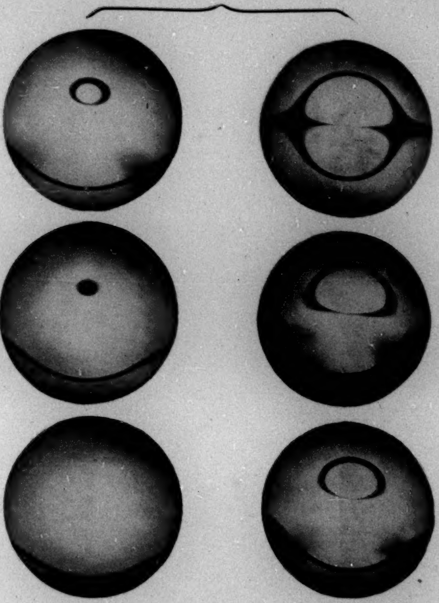


FIG. 9.

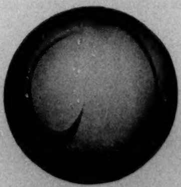
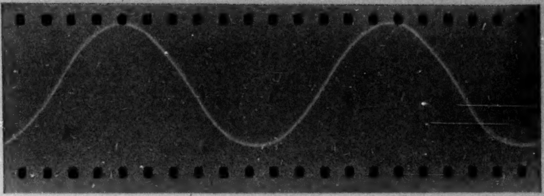
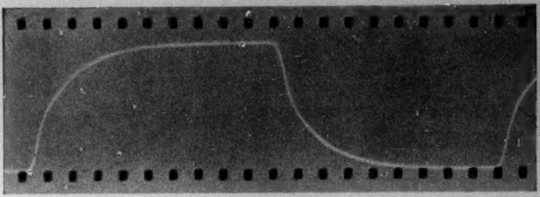


Fig. 2.



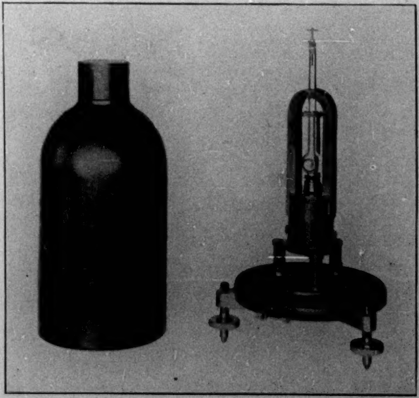
← - - - - - - - - - - - - - - - - 1 second - - - - - - - - - - - - - - - - →

Fig. 3.



← - - - - - - - - - - - - - - - - 1 second - - - - - - - - - - - - - - - - →

Fig. 5.



Thermogalvanometer.

XXVII. *The Law of Action between Magnets and its bearing on the Determination of the Horizontal Component of the Earth's Magnetic Force with Unifilar Magnetometers.* By CHARLES CHREE, *Sc.D., LL.D., F.R.S.* (*From the National Physical Laboratory.*)*

CONTENTS.

§§ 1–3. Fundamental formulæ; P, Q, and R coefficients.
4–9. Effect of errors in the setting of the deflexion-magnet, or in the deflexion distances.
10–11. Effect of errors in deflexion-angles.
12–13. Consequences of the neglect of the higher coefficients.
14–20. Theoretical expressions for the coefficients and their numerical evaluation in particular cases.
21–23. Observational results for coefficients, and evidence of change in individual magnets.
24–25. Departures from the conditions assumed in the ordinary theoretical treatment of the deflexion experiment, and their consequences.
26–31. Experiments on the pole-distance in magnets.

§ 1. WHEN a freely suspended magnet is deflected out of the magnetic meridian by a second magnet, the two being at right angles to one another in the first Lamont's position, the following relation holds:—

$$\frac{2m}{r^3}\left(1-\frac{2\mu}{r^3}-qt-q't^2\right)\left(1+\frac{P}{r^2}+\frac{Q}{r^4}+\frac{R}{r^6}+\dots\right)=\mathrm{H}\sin u. \quad (1)$$

Here m is the magnetic moment, at 0° C., of the deflecting magnet, μ its temporary induction coefficient, q and q' its temperature coefficients, r the true distance between the centres of the two magnets, H the horizontal component of the earth's magnetic force, u the deflexion-angle, and P, Q, R ... constants depending on the dimensions of the two magnets and their distribution of magnetism.

In 1899† I pointed out that the assumption usually made that all the higher constants Q, R . . are negligible did not seem fully justified for the ordinary Kew pattern magnetometer. Before investigating this point further, I shall explain how Horizontal Force observations may be carried out when Q is not negligible.

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* Read May 27, 1904.

† Roy. Soc. Proc. vol. lxv. p. 411.

§ 2. Let us for shortness put

$$\tfrac{1}{2}r^3 \sin u\left(1-\frac{2\mu}{r^3}-qt-q't^2\right)^{-1}=\mathrm{W}, \quad . \quad . \quad (2)$$

and employ suffixes 1, 2, 3 to refer to deflexions at three distances r_1, r_2, r_3; then from observations at these distances we have

$$\left.\begin{aligned}(m/\mathrm{H})(1+\mathrm{P}r_1^{-2}+\mathrm{Q}r_1^{-4})=\mathrm{W}_1,\\(m/\mathrm{H})(1+\mathrm{P}r_2^{-2}+\mathrm{Q}r_2^{-4})=\mathrm{W}_2,\\(m/\mathrm{H})(1+\mathrm{P}r_3^{-2}+\mathrm{Q}r_3^{-4})=\mathrm{W}_3.\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

The induction and temperature coefficients are supposed known, the angle u and the distances r are observed, thus the only unknowns in the above relations are m/H, P, and Q. Our primary object is to determine m/H. To do so, we eliminate P and Q in the usual way, and find

$$m/\mathrm{H}=\begin{vmatrix}\mathrm{W}_1 & r_1^{-2} & r_1^{-4}\\ \mathrm{W}_2 & r_2^{-2} & r_2^{-4}\\ \mathrm{W}_3 & r_3^{-2} & r_3^{-4}\end{vmatrix} \div \begin{vmatrix}1 & r_1^{-2} & r_1^{-4}\\ 1 & r_2^{-2} & r_2^{-4}\\ 1 & r_3^{-2} & r_3^{-4}\end{vmatrix},$$

or

$$m/\mathrm{H}=\mathrm{A}_1\mathrm{W}_1+\mathrm{A}_2\mathrm{W}_2+\mathrm{A}_3\mathrm{W}_3, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\mathrm{A}_1=r_1^4(r_3^2-r_2^2)$

$$\div\{(r_1-r_2)(r_2-r_3)(r_3-r_1)(r_1+r_2)(r_2+r_3)(r_3+r_1)\}, \quad (5)$$

and A_2, A_3 may be written down from symmetry.

In reality, the distances r_1, r_2, r_3 vary slightly with the temperature; but assuming the deflexion-bar uniform, the distances all alter in the same proportion. Thus the coefficients A, being of dimensions 0 in length, are absolutely independent of temperature, and once calculated cause no further trouble.

If we were to work out a value for m/H from (4), employing values actually found for W_1, W_2, and W_3 on an individual occasion, slight errors in these values might produce undesirably large errors in the result. The above method, in short, though theoretically perfect, cannot be recommended for the separate treatment of individual day's observations. Let us suppose, however, that a large number of observations have been made at one station, with an instrument whose magnets may be regarded as of constant moment, and that the mean values of the W's so observed are denoted by $\overline{W}_1$, $\overline{W}_2$, and $\overline{W}_3$. Let us put

$$\left.\begin{array}{c} A_1\overline{W}_1+A_2\overline{W}_2+A_3\overline{W}_3=N, \\ \bar{A}_1=N/3\overline{W}_1,\ \bar{A}_2=N/3\overline{W}_2, \bar{A}_3=N/3\overline{W}_3. \end{array}\right\} \quad . \quad . \quad (6)$$

If now W_1, W_2, W_3 refer to any individual set of observations,

$$(m/H)_1=3\bar{A}_1W_1,$$
$$(m/H)_2=3\bar{A}_2W_2,$$
$$(m/H)_3=3\bar{A}_3W_3$$

are three values for m/H given by the deflexions at the three distances r_1, r_2, r_3, after applying a correction for P and Q based on the whole series of observations, and

$$m/H=\bar{A}_1W_1+\bar{A}_2W_2+\bar{A}_3W_3 \quad . \quad . \quad . \quad . \quad (7)$$

the arithmetic mean of the three values may be accepted.

If the coefficient R in (1) also required to be taken into account, a similar procedure would apply, observations being taken at four instead of three distances.

§ 3. There is always a certain advantage in observing at two distances, as this leads to the detection of errors of setting and reading which otherwise might escape notice. But observations at a third distance, whilst adding materially to the labour, provide little additional security against observational errors. It is thus desirable to consider whether it is really necessary always to observe at three distances when Q requires to be allowed for. Obviously this is not necessary if the magnets and the apparatus employed remain in an absolutely constant state, for then P and Q and $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ are absolute constants, and once determined remain available for all future use. It is also conceivable that whilst changes in P may be sufficiently large to be taken into account, there is no such necessity in the case of Q, in which event observations at two distances might suffice.

As to the actual formulæ for P and Q, if we suppose R negligible we easily find from (3)

$$P=\frac{W_1r_1^4(r_3^4-r_2^4)+W_2r_2^4(r_1^4-r_3^4)+W_3r_3^4(r_2^4-r_1^4)}{W_1r_1^4(r_2^2-r_3^2)+W_2r_2^4(r_3^2-r_1^2)+W_3r_3^4(r_1^2-r_2^2)}, \quad . \quad . \quad . \quad (8)$$

$$Q=\frac{r_1^2r_2^2r_3^2\{W_1r_1^2(r_2^2-r_3^2)+W_2r_2^2(r_3^2-r_1^2)+W_3r_3^2(r_1^2-r_2^2)\}}{W_1r_1^4(r_2^2-r_3^2)+W_2r_2^4(r_3^2-r_1^2)+W_3r_3^4(r_1^2-r_2^2)}. \quad (9)$$

Here W_1, W_2, W_3 may represent results from a single observation, but it is highly desirable they should represent means from a large number of observations. Only those days' results should be used on which observations have been taken at all three distances.

It will be noticed that P being of dimensions r^2, and Q of dimensions r^4, they are not strictly speaking independent of the temperature. Thus, theoretically, there is an objection both to combining observations taken at different temperatures, and to treating the functions of the distances in (8) and (9) as constants. In practice, however, the contributions from P and Q to the value of H are so small that these objections may be disregarded.

The most satisfactory way of investigating the degree of variability of P and Q under ordinary conditions, would be to take observations at three distances at frequent intervals during several years, with the same instrument, and compare the results deduced by formulæ (8) and (9) from different groups of the observations. We shall presently consider existing evidence as to the variability of P and Q, and the question how these quantities are related to the dimensions and other peculiarities of the magnets employed. But before doing so we shall discuss the uncertainties in the determination of P and Q, and in the value of H, arising either from errors of observation or in the assumed values of the deflexion distances.

Effect of Errors in the Setting of the Deflecting Magnet or in the Values accepted for the Deflexion Distances.

§ 4. The temporary induction correction is not absolutely unaffected by errors in the deflexion distances, but assuming ordinary care any error so introduced is too small to require consideration. Thus, for the purposes of the present inquiry, we may leave induction and temperature corrections out of account. As will be seen later, the coefficient R is seldom likely to be of practical moment in the case of English magnetometers, so to avoid undue length in the formulæ and discussion I shall in general neglect it.

Combining the vibration and deflexion experiments, omitting temperature and induction corrections, we find

$$1+Pr^{-2}+Qr^{-4}=\tfrac{1}{2}(H^2T^2/\pi^2 I)r^3 \sin u,$$

where T denotes the time of a half vibration (rest to rest), and I the moment of inertia of the collimator magnet. The other letters have the same signification as before.

Assuming T and u correctly measured, we may for the present inquiry put

$$(T^2/2\pi^2 I)\sin u = B, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

and treat B as a constant. Also if B_1, B_2, B_3 refer to the three observation distances, we have as a first approximation (since Pr^{-2} and Qr^{-4} are always small compared with unity)

$$B_1 r_1^3 = B_2 r_2^3 = B_3 r_3^3 = H^{-2}. \quad . \quad . \quad . \quad . \qquad (11)$$

We can use (11) in *small* terms, *after* differentiation.

From three equations of the type

$$H^2 B - Pr^{-5} - Qr^{-7} = r^{-3},$$

we find

$$H^2 = (r_2^2 - r_3^2)(r_3^2 - r_1^2)(r_1^2 - r_2^2) \div D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

$$-P = \{r_1^7(r_3^4 - r_2^4)B_1 + r_2^7(r_1^4 - r_3^4)B_2 + r_3^7(r_2^4 - r_1^4)B_3\} \div D, \quad . \quad . \qquad (13)$$

$$Q = r_1^2 r_2^2 r_3^2 \{r_1^5(r_3^2 - r_2^2)B_1 + r_2^5(r_1^2 - r_3^2)B_2 + r_3^5(r_2^2 - r_1^2)B_3\} \div D, \qquad (14)$$

where

$$D \equiv r_1^7(r_3^2 - r_2^2)B_1 + r_2^7(r_1^2 - r_3^2)B_2 + r_3^7(r_2^2 - r_1^2)B_3. \quad . \qquad (15)$$

Supposing r_1 alone to vary, we find from (12) and (15)

$$\frac{2}{H}\frac{dH}{dr_1} = \frac{2r_1}{r_1^2 - r_2^2} + \frac{2r_1}{r_1^2 - r_3^2}$$
$$- \frac{7r_1^6(r_3^2 - r_2^2)B_1 + 2r_1 r_2^7 B_2 - 2r_1 r_3^7 B_3}{r_1^7(r_3^2 - r_2^2)B_1 + r_2^7(r_1^2 - r_3^2)B_2 + r_3^7(r_2^2 - r_1^2)B_3}.$$

In this use $B_1 : B_2 : B_3 :: r_1^{-3} : r_2^{-3} : r_3^{-3}$ from (11), and after reduction we find

$$\frac{2}{H}\frac{dH}{dr_1} = -3r_1^3 \div \{(r_1^2 - r_2^2)(r_1^2 - r_3^2)\}.$$

This will sufficiently illustrate the mathematical operations necessary in obtaining the following results.

Supposing δr_1, δr_2, δr_3 small independent increments in r_1, r_2, r_3, we find the corresponding increments in H, P, and Q given by

$$\begin{aligned}\delta \mathrm{H}/\mathrm{H} = (3/2\Delta)\{(\delta r_1/r_1)r_1^4(r_2^2-r_3^2) + (\delta r_2/r_2)r_2^4(r_3^2-r_1^2)\\ + (\delta r_3/r_3)r_3^4(r_1^2-r_2^2)\}, \quad . \quad (16)\end{aligned}$$

$$\begin{aligned}\delta \mathrm{P} = (3/\Delta)\{\delta r_1/r_1)r_1^4(r_2^4-r_3^4) + (\delta r_2/r_2)r_2^4(r_3^4-r_1^4)\\ + (\delta r_3/r_3)r_3^4(r_1^4-r_2^4)\}, \quad . \quad (17)\end{aligned}$$

$$\begin{aligned}\delta \mathrm{Q} = -3(r_1^2r_2^2r_3^2/\Delta)\{(\delta r_1/r_1)r_1^2(r_2^2-r_3^2) + (\delta r_2/r_2)r_2^2(r_3^2-r_1^2)\\ + (\delta r_3/r_3)r_3^2(r_1^2-r_2^2)\}, \quad . \quad (18)\end{aligned}$$

where

$$\Delta \equiv (r_1^2-r_2^2)(r_2^2-r_3^2)(r_3^2-r_1^2). \quad . \quad . \quad (19)$$

We shall suppose r_1 the smallest, and r_3 the largest of the three distances.

If δr_1, δr_2, δr_3 are the corrections required to the values accepted for r_1, r_2, and r_3, then δH, δP, and δQ are the consequent corrections necessary to the values calculated for H, P, and Q. Considering, for example, the case where a correction is necessary to only one of the distances, say r_1, we find

$$\begin{aligned}\delta \mathrm{H}/\mathrm{H} = \tfrac{1}{2}\delta \mathrm{P}/(r_2^2+r_3^2) = -\tfrac{1}{2}\delta \mathrm{Q}/r_2^2r_3^2\\ = -\tfrac{3}{2}(\delta r_1/r_1)r_1^4 \div \{(r_1^2-r_2^2)(r_1^2-r_3^2)\}. \quad (20)\end{aligned}$$

The corrections to H, P, and Q thus depend not merely on the size of δr_1 and on r_1, but also—and that most materially—on the size of r_2 and r_3. In particular, if either or both of the quantities $r_1 \sim r_2$ and $r_1 \sim r_3$ is small, a slight error in r_1 may be decidedly serious.

§ 5. To see the full significance of this result, let us consider the formulæ which take the place of (16) when

(i.) P and Q are both negligible and observations are taken at the single distance r_1;

(ii.) Q is negligible, and P is eliminated by observations at the two distances r_1 and r_2.

The formulæ in question are

$$\text{for (i.)} \quad \delta \mathrm{H}/\mathrm{H} = -(3/2)\delta r_1/r_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\text{(ii.)} \quad \delta \mathrm{H}/\mathrm{H} = \frac{3}{2}\left\{\frac{\delta r_1}{r_1}\frac{r_1^2}{r_2^2-r_1^2} - \frac{\delta r_2}{r_2}\frac{r_2^2}{r_2^2-r_1^2}\right\}. \quad (22)$$

Comparing (21) and (22) with (16), we see that the errors in H answering to the same error in r_1 stand to one another in the following ratio:—

P and Q negligible.		Q negligible.		Neither negligible.
1	:	$r_1^2/(r_1^2-r_2^2)$	:	$\{r_1^2/(r_1^2-r_2^2\}\{r_1^2/(r_1^2-r_3^2)\}$.

When we observe at a single distance, as (21) shows, an error of 1 part in 10,000 in r entails an error of 1·5 parts in 10,000 in H, whatever r may be.

This relation is so simple that it is often convenient to make it the point of departure when considering the effect of errors in the distance when P, or P and Q, are retained, applying multipliers answering to the ratios considered above.

The following results for specified cases will serve for illustration.

TABLE I.

Distances one, two, three of the following: 22·5, 30, 40 cms.

For given error in	Observations at						
	Single distance.			Two distances.			Three distances.
	22·5	30	40	22·5 & 30	22·5 & 40	30 & 40	22·5, 30 & 40 cms.
22·5...	1	...	...	−1·29	−0·46	...	+0·60
30·0...	...	1	...	+2·29	...	−1·29	−2·94
40·0...	...	...	1	...	+1·46	+2·29	+3·34

The Table is to be interpreted as follows:—

If a certain error in the value accepted for the distance 22·5 cms. produces an error of e[*] × 1 in the value of H when P and Q are negligible, and observations are made at 22·5 cms. only, then if observations are taken at two distances, and combined in the usual way, the error is $-1{\cdot}29e$ or $-0{\cdot}46e$, according as the distances are 22·5 and 30, or 22·5 and 40 cms., while if P and Q are eliminated from observations at 22·5, 30, and 40 cms. the error is $+0{\cdot}60e$. As a concrete instance, suppose a correction of +·003 cm. necessary to the value assigned the 30 cms. distance, the local value of H being ·186 C.G.S. Then the value deduced for H by employing the erroneous value for the 30 cms. requires to be *diminished* by

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[*] When we observe at a single distance the errors in the values accepted for the distance and calculated for H are really opposite in sign.

$$(3/2) \times (0{\cdot}003/30) \times {\cdot}186 = 2{\cdot}79\gamma \text{ (where } 1\gamma \equiv 1 \times 10^{-5} \text{ C.G.S.)},$$

when observations at 30 cms. suffice, P and Q being negligible. When, however, Q only is negligible, the value requires to be

diminished by 2·29 × 2·79, or 6·4γ, when observations are made at 22·5 and 30 cms.,

increased by 1·29 × 2·79, or 3·6γ, when observations are made at 30 and 40 cms.

When neither P nor Q is negligible, and they have to be eliminated by observations at 22·5, 30, and 40 cms., the value of H requires to be *increased* by 2·94 × 2·79 or 8·2γ.

TABLE II.

Distances one, two, or three of the following :—

For given error in	Single distance.			Two distances.			Three distances.
	26·25	35	45	26·25 & 35	26·25 & 45	35 & 45	26·25, 35, & 45
26·25.	1	...	...	−1·29	−0·52	...	+0·66
35 ..	...	1	...	+2·29	...	−1·53	−3·50
45 ...	...	...	1	...	+1·52	+2·53	+3·84
	25	35	45	25 & 35	25 & 45	35 & 45	25, 35, 45
25 ...	1	...	...	−1·04	−0·45	...	+0·47
35 ...	...	1	...	+2·04	...	−1·53	−3·13
45 ...	...	...	1	...	+1·45	+2·53	+3·66
	30	35	40	30 & 35	30 & 40	35 & 40	30, 35, 40
30 ...	1	...	...	−2·77	−1·29	...	+ 3·56
35 ...	...	1	...	+3·77	...	−3·27	−12·31
40 ...	...	...	1	...	+2·29	+4·27	+ 9·75

§ 6. Other likely distances are dealt with in Table II., which is to be interpreted on similar lines to Table I. All distances are in centimetres.

The results in Tables I. and II. depend in reality only on the *ratios* of the distances, not on their absolute values.

The last combination dealt with in Table II. exemplifies what happens when the distances $r_1 \sim r_2$ and $r_2 \sim r_3$ are comparatively small. It clearly cannot be recommended when observations are to be taken at three distances. Unless Q were exceptionally large, it is clear that a trifling error in the 35 cms. distance might more than neutralize the advantages to be derived from not neglecting Q.

§ 7. When observations are made at three distances, and more than one of them is in error, the following Table will be found convenient. In it δH, δr_1, δr_2, δr_3 are all corrections, or else all errors.

Table III.

Values of			Value of $\delta H/H$.
r_1 cms.	r_2 cms.	r_3 cms.	
22·5	30	40	$-0·0397\ \delta r_1 \quad +·1469\ \delta r_2 \quad -·1254\ \delta r_3$
25	35	45	$-0·0279\ \delta r_1 \quad +·1340\ \delta r_2 \quad -·1220\ \delta r_3$
26·25	35	45	$-0·0379\ \delta r_1 \quad +·1500\ \delta r_2 \quad -·1279\ \delta r_3$
27	36	48	$-0·0331\ \delta r_1 \quad +·1224\ \delta r_2 \quad -·1045\ \delta r_3$
30	35	40	$-0·178\ \delta r_1 \quad +·528\ \delta r_2 \quad -·366\ \delta r_3$

In Table III. the unit of length is 1 cm., the unit of force is immaterial.

At a particular station it would be simplest to use an approximate value for H, and it might be most generally convenient to take ·001 cm. for unit of length, and 1γ for unit of force. This is done in Table IV., applicable to a station where H is approximately ·185 C.G.S.

TABLE IV.

Values of			Value of δH. (H=·185 C.G.S.) (·001 cm. unit of length, 1γ unit of force.)		
r_1 cms.	r_2 cms.	r_3 cms.			
22·5	30	40	$-0·73\ \delta r_1$	$+2·72\ \delta r_2$	$-2·32\ \delta r_3$
25	35	45	$-0·52\ \delta r_1$	$+2·48\ \delta r_2$	$-2·26\ \delta r_3$
26·25	35	45	$-0·70\ \delta r_1$	$+2·77\ \delta r_2$	$-2·37\ \delta r_3$
27	36	48	$-0·61\ \delta r_1$	$+2·26\ \delta r_2$	$-1·93\ \delta r_3$
30	35	40	$-3·29\ \delta r_1$	$+9·76\ \delta r_2$	$-6·76\ \delta r_3$

At a station where H=·370 C.G.S. the coefficients of δr_1, δr_2, δr_3 in Table IV. would be simply doubled.

§ 8. If in (16) we suppose

$$\delta r_3/r_3 = \delta r_2/r_2 = \delta r_1/r_1, \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

or if in (22) we suppose

$$\delta r_2/r_2 = \delta r_1/r_1,$$

we alike find

$$\delta H/H = -\tfrac{3}{2}\,\delta r_1/r_1.$$

Thus if all the distances have errors proportional to the true lengths—as might, for instance, occur through an error as to the coefficient of thermal expansion—the simple relation (21) applies irrespective of the number of points at which deflexions are made.

§ 9. The corrections δP, δQ to the values of P and Q, when corrections δr_1, δr_2, δr_3 are applied to the distances r_1, r_2, r_3 at which deflexions are made, are given in Table V. for the same combinations of three distances as appear in Table IV.

TABLE V.

(Unit of length 1 cm.)

r_1 cms.	r_2 cms.	r_3 cms.	δP.			$10^{-3}\ \delta$Q.		
22·5	30	40	$-\ 198\ \delta r_1$	$+\ 619\ \delta r_2$	$-\ 353\ \delta r_3$	$114\ \delta r_1$	$-\ 238\ \delta r_2$	$+114\ \delta r_3$
25	35	45	$-\ 181\ \delta r_1$	$+\ 710\ \delta r_2$	$-\ 452\ \delta r_3$	$138\ \delta r_1$	$-\ 339\ \delta r_2$	$+187\ \delta r_3$
26·25	35	45	$-\ 246\ \delta r_1$	$+\ 814\ \delta r_2$	$-\ 490\ \delta r_3$	$188\ \delta r_1$	$-\ 419\ \delta r_2$	$+216\ \delta r_3$
27	36	48	$-\ 238\ \delta r_1$	$+\ 743\ \delta r_2$	$-\ 423\ \delta r_3$	$197\ \delta r_1$	$-\ 411\ \delta r_2$	$+197\ \delta r_3$
30	35	40	$-1006\ \delta r_1$	$+2638\ \delta r_2$	$-1554\ \delta r_3$	$698\ \delta r_1$	$-1520\ \delta r_2$	$+806\ \delta r_3$

Here again we see the large effect of even small errors in the three distances when the intervals are only 5 cms. When $r_1/r_2=r_2/r_3=3/4$, the coefficients of δr_1 and δr_3 in the expression for δQ are absolutely equal.

A check on the results in Table V. and analogous Tables is supplied by the fact that if

$$\delta r_1/r_1=\delta r_2/r_2=\delta r_3/r_3,$$

then we ought to have

$$\delta P=0=\delta Q.$$

This is obvious from the general formulæ (17) and (18) applicable to three distances (and it is equally true for deflexions at two distances only, when Q is negligible, though this case is not covered by Table V.).

Additional interest attaches to this simple result from the fact pointed out in a recent paper *, that in ordinary English magnetometers the increments in the distances r_1 &c. due to bending of the deflexion-bar are nearly proportional to the distances themselves. The neglect of the bending effect thus entails fortunately but little error in the determination of P or Q.

Effect of Errors in Angles.

§ 10. When a small error—whether due to reading or to defective graduation—has been made in a deflexion-angle u, the consequent small error δH is given by the following formulæ :—

(i.) Observation at single distance r_1, P and Q negligible,

$$\delta H/H=-\tfrac{1}{2}\cot u_1\,\delta u_1\,; \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(ii.) Observations at two distances r_1, r_2, Q negligible,

$$\delta H/H=\tfrac{1}{2}\{r_1^2\cot u_1\,\delta u_1-r_2^2\cot u_2\,\delta u_2\}/(r_2^2-r_1^2)\,; \quad . \quad (25)$$

(iii.) Observations at three distances r_1, r_2, r_3, neither P nor Q negligible,

$$\delta H/H=\frac{1}{2\Delta}\{r_1^4(r_2^2-r_3^2)\cot u_1\,\delta u_1+r_2^4(r_3^2-r_1^2)\cot u_2\,\delta u_2$$
$$+r_3^4(r_1^2-r_2^2)\cot u_3\,\delta u_3\}\,; \quad . \quad . \quad . \quad (26)$$

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* Phil. Mag. Jan. 1904, p. 39.

where Δ is given by (19), and the suffixes distinguish the three distances.

These formulæ are deducible from the formulæ for the effects on H of errors in the distances (*e.g.* (16)) by writing $\cot u_1 \, \delta u_1$ for $3\delta r_1/r_1$ &c. Thus the way in which error in an angle modifies the value of H according as observations are made at one, or at two, or at three distances, is exactly the same as in the corresponding case of error in a distance, and Tables I. and II. apply equally to the case of errors in angles.

A complication arises, however, when we come to consider the *absolute* size of the error in H. This depends partly on the size of the angle, *i.e.* on the strength of the deflecting magnet. The stronger this magnet, the larger the deflexion-angle, and the smaller the effect of a given error in reading or graduation. Tables VI., VII., and VIII. give some numerical results for two different values of H, and two different strengths of collimator magnet. The error in the angle is supposed to be 1′, the consequent error in H is given in terms of 1γ as unit. The sign + or − is attached according as the corrections to be applied to the angle and to H agree or differ in sign. It should be noticed that the value of a deflexion-angle is usually based on four readings, so that it would require an error of 4′ in a single reading to produce an error of 1′ in the angle.

TABLE VI.

Error of 1′ in deflexion-angle. Effect on H (unit 1γ).
Observation at a single distance, P and Q negligible.

H	*m*	Distances 22·5	25	26·25	30	35	40	45 cms.
C.G.S.	C.G.S.	Sign ...... —	—	—	—	—	—	—
·185	1000	....	2·8	3·6	6·2	10·3	15·7	22·5
	750	.... 2·7	4·4	5·4	8·5	14·0	21·1	30·1
·370	1000	.... 10·0	14·6	17·2	26·3	42·3	63·4	90·6
	750	.... 14·1	20·1	23·4	35·4	56·7	84·8	120·8

When the magnetic moment is 1000 C.G.S., 22·5 cms. is too short a distance unless H exceeds ·185.

TABLE VII.

Error of 1′ in deflexion-angle. Effect on H (unit 1γ).
Observations at two distances, Q negligible.

H C.G.S.	m C.G.S.	Distances	22·5	30	25	35	26·25	35	30	40	35	45 cms.
		Sign	+	−	+	−	+	−	+	−	+	−
·185	1000	....	...	...	2·9	21·1	4·6	23·6	7·9	35·9	15·8	57·0
	750	....	3·4	19·5	4·6	28·5	6·9	31·9	11·0	48·1	21·4	76·3
·370	1000	....	12·8	60	15·2	86	22·1	97	33·9	145	...	...
	750	....	18·2	81	20·9	116	30·1	130	45·6	194	...	...

TABLE VIII.

Error of 1′ in deflexion-angle. Effect on H (unit 1γ).
Observations at three distances, neither P nor Q negligible.

H C.G.S.	m C.G.S.	Distances	22·5	30	40	25	35	45	26·25	35	45	30	35	40
		Sign	−	+	−	−	+	−	−	+	−	−	+	−
·185	1000	....	...	...	...	1·3	32	82	2·4	36	86	22	127	153
	750	....	1·6	25	70	2·1	44	110	3·6	49	116	30	172	205
·370	1000	....	5·9	77	212	6·8	132	332	11·4	148	348	...	...	...
	750	....	8·4	104	283	9·3	177	442	15·5	198	464	...	...	...

As an illustration of how to read the Tables, suppose that when a deflexion is made at 30 cms. with a magnet of moment 750, at a station where H is ·185, the angle is taken 1′ too high; then the *error* in the value calculated for H is:

—8·5γ when P and Q are negligible, and the 30 cms. observation is alone used;

−19·5γ when Q is negligible, and observations are taken at 30 cms. and 22·5 cms.;

+11·0γ when Q is negligible, and observations are taken at 30 cms and 40 cms.;

+25γ when neither P nor Q is negligible, and observations are taken at 22·5, 30, and 40 cms.;

−30γ when neither P nor Q is negligible, and observations are taken at 30, 35, and 40 cms.

So long as the errors are small, the effect on H is proportional to the size of the error, and the results in Tables VI., VII., and VIII. have only to be multiplied by the error (in minutes) to give the error in H in terms of 1γ as unit.

§ 11. Tables VI., VII., and VIII. give results only for two specified values of H and of m. Approximate values are, however, easily deducible for other values of H and m from the following considerations: For any given combination of distances, the relation between the error in u and the consequent error in H is of the form

$$\delta \mathrm{H} = f(r) \delta u \,.\, \frac{\cos u}{\sin u} \,.\, \mathrm{H},$$

where $f(r)$ depends only on the distances. Also, as a first approximation, we have

$$\sin u = 2m/\mathrm{H}r^3 \;;$$

and so in the above equation

$$\delta \mathrm{H} = \tfrac{1}{2} r^3 f(r) \delta u \,.\, \cos u \,.\, \mathrm{H}^2/m \,. \quad . \quad . \quad . \quad (27)$$

Now, in practice, the departure of $\cos u$ from unity is seldom great. For $u = 20°$ we have $\cos u = 0{\cdot}94$, and except at the shortest distance of deflexion, angles as large as 20° are unusual. Thus, treating δu as given, we may accept

$$\delta \mathrm{H} \propto \mathrm{H}^2/m$$

as giving a good approximation at all the larger distances of deflexion, and a fair approximation even at the least distance, unless H is exceptionally low.

According to this approximation, the corrections to H in Tables VI., VII., and VIII., for a given value of H, when $m = 1000$ and when $m = 750$ should be in the proportion 3 : 4; and the corrections to H, for a given value of m, should be four times as great when H = ·370 as when H = ·185. At the larger distances in the Tables, it will be seen that these relations are very nearly satisfied indeed.

§ 12. When one or more of the higher constants Q, R is not really negligible, the value found for P in the usual way from observations at two distances applies in reality to a different quantity, which we may call P′. As to the relation between P and P′, let us suppose that observations are taken at the two distances r_1, r_2, and that P, Q, R are the only constants requiring to be taken into account.

We clearly have from (1)

$$1+P'r_1^{-2}=1+Pr_1^{-2}+Qr_1^{-4}+Rr_1^{-6},$$
$$1+P'r_2^{-2}=1+Pr_2^{-2}+Qr_2^{-4}+Rr_2^{-6};$$

whence

$$P'=P+Q(r_1^{-2}+r_2^{-2})+R(r_1^{-4}+r_1^{-2}r_2^{-2}+r_2^{-4}). \quad (28)$$

Table IX. gives the form taken by (28) for a number of combinations of distances.

TABLE IX.

r_1 cms.	r_2 cms.	Expression for P'.
18	24	$P+4822Q\times10^{-6} + 1790R\times10^{-8}$
22·5	30	$P+3086Q\times10^{-6} + 733R\times10^{-8}$
22·5	40	$P+2600Q\times10^{-6} + 553R\times10^{-8}$
25	35	$P+2416Q\times10^{-6} + 453R\times10^{-8}$
25	45	$P+2094Q\times10^{-6} + 359R\times10^{-8}$
26·25	35	$P+2268Q\times10^{-6} + 396R\times10^{-8}$
26·25	45	$P+1945Q\times10^{-6} + 307R\times10^{-8}$
30	40	$P+1736Q\times10^{-6} + 232R\times10^{-8}$
35	45	$P+1310Q\times10^{-6} + 131R\times10^{-8}$

§ 13. When Q and R have wrongly been assumed negligible, the consequent effect on H may be found as follows:—

Assuming coefficients of higher terms negligible, we have

$$H^2=A(1+Pr^{-2}+Qr^{-4}+Rr^{-6}), \quad . \quad . \quad (29)$$

where r is a distance of deflexion, and A is independent of P, Q, or R. Supposing δH an increment to H, answering to increments δP, δQ, and δR, then

$$2\delta H/H=(r^{-2}\delta P+r^{-4}\delta Q+r^{-6}\delta R)/(1+Pr^{-2}+Qr^{-4}+Rr^{-6}).$$

Unless r is unduly small, or the constants exceptionally large, we may replace the denominator on the right-hand side of the equation by unity. If, then, H has been originally determined from observations at r_1 and r_2, we have

$$2\delta H/H=r_1^{-2}\delta P+r_1^{-4}\delta Q+r_1^{-6}\delta R=r_2^{-2}\delta P+r_2^{-4}\delta Q+r_2^{-6}\delta R,$$

whence, eliminating δP,

$$2\delta H/H=-\delta Qr_1^{-2}r_2^{-2}-\delta Rr_1^{-2}r_2^{-2}(r_1^{-2}+r_2^{-2}). \quad (30)$$

If we now suppose H to be the value found when Q and R were neglected, H + δH to be the correct value when Q and

R are assigned their true values, then we have to substitute Q for δQ and R for δR in (30), and so find

$$\delta H/H = -\frac{1}{2}\frac{Q}{r_1^2 r_2^2} - \frac{1}{2}\frac{R}{r_1^2 r_2^2}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right). \quad . \quad (31)$$

Here δH is the *correction* which is necessary to the value of H obtained in the usual way by neglecting Q and R. It is assumed that the correction is but a small fraction of H.

The result (31) may also be obtained by consideration of the quantities P and P′.

Nature of Coefficients P, Q, R.

§ 14. The law of action between two magnets in Lamont's first position—the position met with in deflexion experiments—was investigated by Lamont himself, and more recently with greater completeness by Dr. Börgen *. Let us first suppose the cross sections of the magnets negligible, and that their axes lie in the same horizontal plane, and are strictly perpendicular to one another. Let $d\mu$, $d\mu'$ be quantities of "magnetic matter" in the deflecting and deflected magnets, at distances x, x' from their centres, and let r be the distance between the centres. Then the couple L arising from the action of the one magnet on the other is given by

$$L = \iint (r-x)x'\{(r-x)^2 + x'^2\}^{-\frac{3}{2}} d\mu\, d\mu',$$

where the integration extends throughout both magnets.

Using the Lamont-Börgen notation

$$\int x d\mu = M, \qquad \int x' d\mu' = M',$$
$$\int x^i d\mu = M_i, \qquad \int x'^i d\mu' = M_i',$$

Börgen finds

$$L = 2MM'r^{-3}\Big[1 + \frac{1}{r^2}\left(2\frac{M_3}{M} - 3\frac{M_3'}{M'}\right) + \frac{1}{r^4}\left(3\frac{M_5}{M} - 15\frac{M_3}{M}\frac{M_3'}{M'} + \frac{45}{8}\frac{M_5'}{M'}\right)$$
$$+ \frac{1}{r^6}\left(4\frac{M_7}{M} - 42\frac{M_5}{M}\frac{M_3'}{M'} + \frac{105}{2}\frac{M_3}{M}\frac{M_5'}{M'} - \frac{35}{4}\frac{M_7'}{M'}\right) + \dots\Big] \quad (32)$$

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* Terrestrial Magnetism, vol. i. 1896, p. 176; *Archiv der Deutschen Seewarte*, 1891, &c.

M and M′ represent the magnetic moments of the magnets, and so M is equivalent to the more ordinary notation *m* used earlier in the paper. The coefficients of r^{-2}, r^{-4}, and r^{-6} inside the square bracket represent the P, Q, R of (1). If the magnets are replaceable by poles, at distances 2λ, 2λ′ apart respectively, then according to Börgen

$$M_i/M=\lambda^{i-1}, \quad M'_i/M'=\lambda'^{i-1}.$$

§ 15. When the areas of the cross sections are not neglected small additional terms are introduced. These must partly depend on the distribution of magnetism throughout the thickness of the magnets, which is unknown. In thin-walled circular cylinders the uncertainty cannot, however, be large. Taking the result which Börgen approves for this case *, we have in full

$$P=2\lambda^2-3\lambda'^2+(3/4)\{3(a'^2+b'^2)-2(a^2+b^2)\}, \quad . \quad (33)$$

where a, b are the radii of the outer and inner surfaces of the deflecting magnet, while a', b' are the corresponding radii for the deflected magnet. The value of P when the a, b, a', b' terms are neglected I shall denote by $P(\lambda, \lambda')$. Similarly, $Q(\lambda, \lambda')$, $R(\lambda, \lambda')$ will represent those parts of the complete expressions for Q and R which remain when the areas of the cross sections are neglected. Complete expressions for Q and R would contain terms involving a, b, a', b', but the influence of Q and R on the value of the deflecting couple is ordinarily so small that it seems desirable to avoid the complication which the retention of these small terms would entail. Omitting them, we have in terms of λ and λ′

$$\left.\begin{array}{l} Q(\lambda, \lambda')=(3/8)(8\lambda^4-40\lambda^2\lambda'^2+15\lambda'^4), \\ R(\lambda, \lambda')=(1/4)(16\lambda^6-168\lambda^4\lambda'^2+210\lambda^2\lambda'^4-35\lambda'^6) \end{array}\right\} . \quad (34)$$

Table X. gives the numerical value of the contribution to P from the terms in a, b, a', b', in some actual cases. The diameters $2a$, &c. are in cms.

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* Terrestrial Magnetism, *l. c.* p. 189.

TABLE X.

	2a.	2b.	2a′.	2b′.	$(3/4)\{3(a'^2+b'^2)-2(a^2+b^2)\}$
Jones's magnets :—					
Magnetometer i.	1·00	·735	·76	·50	−·11
„ ii.	·975	·74	·77	·50	−·09
„ iii.	·955	·665	·73	·435	−·10
Cooke magnets	1·02	·76	·70	·52	−·18
Dover „	1·035	·82	·825	·515	−·12
Elliott „	0·98	·77	·75	·40	−·18

The Jones's magnets—a very old form—are more variable in diameter than recent patterns, but absolute uniformity can hardly be claimed for any pattern, and in most cases the tube-walls are not quite uniform in thickness. Most collimator magnets, in fact, have shallow screws cut on the inner surface at the ends, a circumstance introducing some slight additional uncertainty, both as to the values of b and as to the degree of accuracy of Börgen's formula.

According to the figures in Table X. the contribution to P from the terms in a, b, &c. is negative in each case. It would be reduced numerically by a reduction in the wall thickness of the mirror magnets—keeping their external diameters unchanged or slightly increasing them—which would have the further recommendation of making these magnets more nearly than at present exact copies of the collimator magnets on a uniformly reduced scale.

There are, as will appear in § 24, other possible contributions to the value of the couple L, but as these arise from instrumental or observational defects it is desirable to postpone their consideration.

§ 16. There has been considerable difference of opinion as to the ratio between the pole-distance 2λ and the length l of an ordinary magnet. Kohlrausch found to a close degree of approximation

$$2\lambda/l = 5/6 = 0{\cdot}8\dot{3}.$$

Börgen concludes that the average value of the ratio is ·805, deducing this as a mean from numerous experiments based on a theoretical investigation, which appears more complete than Kohlrausch's, though not absolutely free from assumptions. Börgen finds that as the magnetic moment of a magnet is reduced, the pole distance tends normally to increase, until in a very weak magnet $2\lambda/l$ may rise to ·815. Whether he regards this as an absolutely limiting value is not quite clear, but he strongly combats the idea that it can ever approach unity, a result which he attributes to Mr. Blakesley * in a paper read before this Society in 1890.

§ 17. As the two magnets of an English magnetometer are fairly similar in type, there are grounds even apart from Börgen's experiments for regarding the assumption

$$2\lambda/l = 2\lambda'/l' = p \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

as likely to be at least approximately true. Adopting this hypothesis, provisionally, for the average magnetometer of any particular make, and for shortness writing z for λ'/λ, we have

$$\mathrm{P}(\lambda, \lambda') = p^2 l^2 (1/4)(2-3z^2) = p^2 l^2 f_2(z), \text{ say,} \quad . \quad . \quad . \quad . \qquad (36)$$

$$\mathrm{Q}(\lambda, \lambda') = p^4 l^4 (3/128)(8-40z^2+15z^4) = p^4 l^4 f_4(z), \quad . \quad . \quad . \qquad (37)$$

$$\mathrm{R}(\lambda, \lambda') = p^6 l^6 (1/256)(16-168z^2+210z^4-35z^6) = p^6 l^6 f_6(z). \qquad (38)$$

Table XI. gives the values of the above functions of z for values of z increasing by ·1 from 0 to 1, including the range encountered in practice.

Table XI.

$z=$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$10^4 f_2(z)=$	+5000	+4925	+4700	+4325	+3800	+3125	+2300	+1325	+ 200	−1075	−2500
$10^4 f_4(z)=$	+1875	+1782	+1506	+1060	+ 465	− 249	−1044	−1875	−2685	−3412	−3984
$10^5 f_6(z)=$	+6250	+5602	+3755	+ 998	−2206	−5243	−7382	−7819	−5734	− 351	+8984

$f_2(z)$, and so $\mathrm{P}(\lambda, \lambda')$, vanishes when $z=·8165$. It is positive for smaller, and negative for all larger values of z; it diminishes algebraically continually as z increases from 0.

$f_4(z)$, and so $\mathrm{Q}(\lambda, \lambda')$, vanishes when $z=·4667$ or $1·565$. It diminishes algebraically as z increases from 0 to 1·155, where there is an (algebraical) minimum. It is negative when z lies between ·4667 and 1·565, otherwise it is positive.

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* Phil. Mag. March 1891, p. 281.

§ 19. If we suppose the pole-distance in each magnet to be ·8 of its length, and neglect terms involving the radii of the cross sections, except in the value of P, we obtain the following expressions for the couple between the two magnets (moments m and m') :

Jones, or Cambridge Instrument Company.	$2mm'r^{-3}(1+0{\cdot}15r^{-2}-872r^{-4}-9180r^{-6})$,
Cooke	$2mm'r^{-3}(1+7{\cdot}96r^{-2}-529r^{-4}-13130r^{-6})$,
Dover	$2mm'r^{-3}(1+7{\cdot}08r^{-2}-528r^{-4}-11890r^{-6})$,
Elliott....................	$2mm'r^{-3}(1+7{\cdot}78r^{-2}-534r^{-4}-13030r^{-6})$.

The relative importance of the different terms in the above formulæ is shown in Table XIII. for the typical Jones and Cooke magnetometers.

TABLE XIII.

Deflexion distance cms.	Jones Magnetometer.				Cooke Magnetometer.			
	Principal term. +	P term. +	Q term. −	R term. −	Principal term. +	P term. +	Q term. −	R term. −
22·5	100000	30	340	7	100000	1572	206	10
26·25	100000	22	184	3	100000	1155	111	4
30	100000	17	108	1	100000	884	65	2
35	100000	12	58	0	100000	650	35	1
40	100000	9	34	0	100000	497	21	0

The results for the typical Elliott instrument are practically the same as for the Cooke; and those for the Dover instrument are practically the same in the case of the Q term, though some 10 per cent. less for the P and R terms.

In the above typical case, $p=0{\cdot}8$, the corrections (all +) necessary to the value of H when Q and R are neglected,—observations being taken only at two distances,—are given in Table XIV.

The effect of an increase in p would be to raise the corrections. For instance, if we assumed $p={\cdot}86$, then for

TABLE XIV.

	Jones or Camb. Inst. Co.			Cooke or Elliott.			Dover.		
Distances cms.	$10^5 \frac{\delta H}{H}$	δH (unit 1γ). H=·185	H=·37	$10^5 \frac{\delta H}{H}$	δH (unit 1γ). H=·185	H=·37	$10^5 \frac{\delta H}{H}$	δH (unit 1γ). H=·185	H=·37
22·5 & 30	99	18·3	36·6	63	11·6	23·1	62	11·5	22·9
26·25 & 35	53	9·8	19·6	33	6·1	12·2	33	6·1	12·2
30 & 40	31	5·7	11·4	19	3·5	7·1	19	3·5	7·1

H=·185, and observations at 30 and 40 cms., we should raise the value of δH to 7·6γ for the Jones, 4·8γ for the Cooke, and 4·7γ for the Dover magnetometer.

§ 20. In any individual case the values of $2\lambda/l$ and $2\lambda'/l'$ may one or both differ from the mean, and if they do differ, they may differ in opposite directions. Table XV. gives the changes in P, Q, and R due to change of p from ·80 to ·81. A change from ·80 to ·79 has a nearly equal effect, but in the opposite direction.

TABLE XV.

Effect of change in p from ·80 to ·81.

Magnet in which p increased.	P Collr.	P Mirror	Q Collr.	Q Mirror	R Collr.	R Mirror	P′(30 & 40 cms.). Collr.	P′(30 & 40 cms.). Mirror
	+	−	−	−	−	+	+	−
Jones (or Camb. Inst. Co.).	·704	·697	18·9	24·3	1444	750	·668	·737
Cooke (and practically Elliott).	·692	·487	4·6	21·5	1050	60	·682	·524
Dover..............	·671	·490	5·5	20·6	994	98	·659	·526

Increments in p in the collimator and mirror magnets always influence P in opposite directions, and a given increment in p has a very much greater effect when occurring in one than when occurring in both of the magnets. Small uncertainties in the pole-distance have comparatively little influence on Q, but materially affect P and P′.

§ 21. It has been usual when testing at Kew to observe at least twice with each magnetometer, so as to check the values of the constants and the general behaviour. Deflexions are made at 30 and 40 cms., and the value of P′ is thence determined. In the case of Dover magnetometers the following are the results:—

mean value of P′ from all instruments examined $+6{\cdot}20$
" " " last 13 " " $+6{\cdot}43$.

Applying the correction $-{\cdot}12$ from Table X. to the values of $P'(\lambda, \lambda')$ in Table XII., we have for the average Dover magnetometer:

$p=$	·80	·81	·82
Calculated value of $P'=$	$+6{\cdot}13$	$+6{\cdot}26$	$+6{\cdot}40$

Thus the mean from all the Dover magnetometers would accord practically identically with the value ·805 assigned to p by Börgen; but the mean from the last 13 instruments would answer to $p={\cdot}82$.

From the last 18 pairs of Cooke magnets, I find

mean value of P′ (at 30 and 40 cms.) $= +8{\cdot}00$.

Allowing a contribution of $-{\cdot}18$ from Table X., we should infer from Table XII. that p slightly exceeds ·86.

From the last dozen pairs of Elliott magnets, I find

mean value of P′ (at 30 and 40 cms.) $+6{\cdot}77$.

Allowing a contribution of $-{\cdot}18$ from Table X., we deduce $p={\cdot}80$.

For the Cambridge Instrument Co's. magnets the average P′ (at 30 and 40 cms.) has been $-1{\cdot}50$; which allowing for the correction in Table X. answers to $p={\cdot}81$.

The above mean values accord pretty closely with Börgen's value ·805 except in the case of the Cooke magnets. In their case the collimator has a central transverse groove which affords a grip for a screw determining the position of the magnet in its stirrup. This groove presumably exercises some influence on the pole-distance, and a very small influence would suffice to explain the observed high value of P′. The combination, for instance, of the value ·815 in the p of the collimator magnet with ·805 in that of the "mirror" magnet would raise P′ to 8·0.

§ 22. To test the influence of the magnetic moment, *m*, I divided each maker's instruments into two equal groups, containing respectively the strongest and the weakest collimator magnets. The mean results appear in Table XVI.

TABLE XVI.

	Cooke.		Dover.		Elliott.	
	m.	P′.	*m*.	P′.	*m*.	P′.
Group with largest *m*	977	+7·76	884	+5·88	1031	+6·04
„ „ smallest *m* ...	907	+8·24	788	+6·89	888	+7·49

Nothing is known of the strength of the mirror magnets, but we may reasonably suppose that a strong mirror magnet is at least as likely to accompany a strong as a weak collimator magnet. Assuming the average mirror magnet the same for the two groups, a larger P′ means a larger *p*—*i. e.*, an increased pole-distance—in the collimator magnet. Thus the results seem all in favour of Börgen's law that the pole-distance is greater the weaker the magnet. There are grounds, however, for some reserve of judgment. To account for the differences between the mean values of P′ in Table XVI. would require changes in *p* varying from ·007 to fully ·02. Now ·01 seems the limiting departure from the mean allowed by Börgen for the weakest magnets, whereas the difference between the stronger and weaker magnets in Table XVI. is comparatively small.

§ 23. Table XVI. gives but an inadequate idea of the apparent variability of P′ in different magnetometers. How far this apparent variability is real is somewhat doubtful, in view of observational and other uncertainties. We are, however, on surer ground when we consider changes in the mean annual values found for P′ at a fixed observatory, where numerous deflexions have been taken with the same two magnets, at the same two distances, in a single magnetometer.

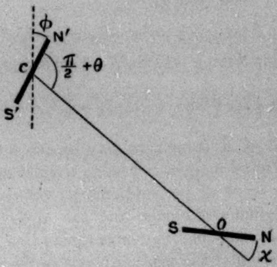
At Kew, the same instrument and magnets have been in use for a generation. During the last twenty years, the extreme values found for P′ (at 30 and 40 cms.) from a year's observations have been −1·899 and −0·603, the mean value for the whole period being −1·238. The difference between the extremes might be accounted for by changes of ·01 in opposite directions in the values of p for the two magnets. There is, however, no reason to think that either magnet has increased in strength, and whilst the moment of the collimator magnet has gradually diminished, it altered by less than 2 per cent. between 1889 and 1892 when the extreme values of P′ presented themselves. If the cause lay with the mirror magnet, whose moment has not been recorded, its p must have altered nearly ·02. As a matter of fact, the variation of P′ has been so irregular as to suggest that the principal cause must have been something other than change of magnetic moment.

At Batavia Observatory, values were calculated for the P′ of a magnetometer from each two months' observations, from 1885 to 1893. A table recording the mean for the year, and the greatest and least of the 2-monthly values, appears in the Batavia "Mag. & Met. Observations" for 1893. The extreme 2-monthly values are −5·073 and +1·638, while the mean annual values vary from −0·920 to −3·786. The magnets are, I think, of English make, and the figures calculated here for the old Jones magnetometers will probably apply fairly. The view expressed at Batavia is that "the values of the constant P...show rather large differences which are to be ascribed to variation in the momentum (magnetic moment) of the deflected magnet; since 1891 this momentum has been kept as constant as possible." If this explanation be correct, we should infer, even from the annual means, a variation in the p of the mirror magnet amounting to nearly ·04.

According to the records of the Royal Alfred Observatory, Mauritius, the mean annual values of P′ for two English magnets 24_A and 24_C changed from −5·059 in 1890 to −4·236 in 1893, and then to −5·006 in 1897. This might be accounted for by a change of little over ·01 in the p of one of the magnets.

§ 24. In general, the ideal conditions of the deflexion experiment cannot be strictly fulfilled. The magnets are not only of finite section, but their axes may be neither strictly horizontal nor perpendicular to one another, while their centres may differ in level. Having already, § 15, considered the finiteness of the cross section, we shall in treating the other defects suppose the magnets linear. Also, as a small inclination in the deflected magnet reduces practically in the same proportion the couples acting on it in the horizontal plane, whether due to the deflecting magnet or the earth's field, we shall, to avoid undue complexity, suppose this magnet strictly horizontal. In fig. 1—which is practically Börgen's* figure—N′S′ represents the axis of this magnet, C its centre. The deflecting magnet if properly adjusted has its axis horizontal when suspended, but unless this axis happens to coincide with the mechanical axis it will ordinarily in the deflexion experiment be slightly inclined to the horizontal. In fig. 1, NS (middle point O) is the projection of the magnetic axis of the deflecting magnet on the horizontal plane containing N′S′. The notation employed—which differs from Börgen's—is as follows:—

Fig. 1.



ϕ = inclination of N′S′ to magnetic meridian,

$\frac{\pi}{2}+\theta$ = „ „ N′S′ to CO,

χ = „ „ NS to CO,

ψ = „ „ deflecting magnet to horizon,

z = height of centre of deflecting magnet above O,

r = CO.

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* *Archiv der Deutschen Seewarte*, 1891, no. 2, p. 2.

We shall suppose θ, χ, ψ, and z/r all small quantities, as they ought to vanish with perfect adjustment.

The general expressions given by Börgen are complicated, and considering several sources of uncertainty I shall not go beyond the terms in λ^2 and λ'^2, and shall retain only the principal corrective terms. The results were mostly obtained independently, but have been checked by reference to Börgen's formulæ. The deflexion equation is

$$\begin{aligned}
\mathrm{H}\sin\phi = {} & 2mr^{-3}\cos\psi\cos\chi\cos\theta(1-\tfrac{1}{2}\tan\theta\tan\chi)\times \\
& \Big[1-3(z^2/r^2)(1+\tfrac{1}{4}\tan\theta\tan\chi) \\
& +2(\lambda^2/r^2)(1-\tan\theta\tan\chi-\tfrac{5}{2}\sin^2\psi-\tfrac{5}{2}\sin^2\chi-\tfrac{15}{2}z^2/r^2) \\
& -3(\lambda'^2/r^2)(1+\tfrac{1}{4}\tan\theta\tan\chi-\tfrac{15}{2}\sin^2\theta+\tfrac{5}{2}\sin\theta\sin(\theta-\chi)-\tfrac{15}{4}z^2/r^2) \\
& +\tfrac{3}{2}(z/r)\tan\psi\Big\{1+\tfrac{1}{2}\tan\theta\tan\chi+\tfrac{1}{2}\tan^2\chi-\tfrac{5}{2}z^2/r^2+10\frac{\lambda^2}{r^2}-\tfrac{5}{2}\frac{\lambda'^2}{r^2}\Big\}\Big] \qquad (39)
\end{aligned}$$

A departure of θ from zero may be due to the magnetic axis of the deflected magnet not being normal to the attached mirror, or to a want of perpendicularity between the axis of the mirror magnet's telescope and the deflexion-bar. An examination of some old mirror magnets (see § 26) showed that the inclination of the magnetic axis to the normal to the mirror may be as much as 1° or 2°, if not more. The necessity of providing for accuracy in this adjustment seems to have been somewhat overlooked.

A departure of ψ from zero may arise from want of level of the deflexion-bar, or want of parallelism between this bar and the magnetic axis of the deflecting magnet, due for instance to defect in the magnet carriage or to difference between the magnet's magnetic and mechanical axes.

A departure of χ from zero may be due to want of coincidence between the horizontal projections of the magnetic and mechanical axes of the deflecting magnet, to want of straightness in the deflexion-bar or to its not lying in the same vertical plane as the centre of the deflected magnet.

For the same instrument θ, ψ, and χ should be practically constants, at least for a considerable period of time; z, on the other hand, will vary from day to day. If the axis of the sighting-tube when in the carriage takes the same position as that ordinarily occupied by the magnetic axis of the deflecting magnet, positive and negative values of z are equally probable, so that the last line in (39) should not appreciably influence the mean value of H from a large number of observations. Observational bias and other causes —*e.g.* stretching of the suspending fibre, or slipping of its support—may of course lead to a preponderance of values of z of one sign.

§ 25. To get some idea of the probable importance of the different terms, let us suppose, as a concrete case, that deflexions are being taken at 30 cms., while

$$\lambda=4,\ \lambda'=3,\ z=0{\cdot}3\text{ cm.},$$
$$\theta=\psi=\chi=1^\circ.$$

Then we may take $\tan 1^\circ = {\cdot}0175$,

$$\tan\theta\tan\chi=\sin^2\chi=\sin^2\psi=\sin^2\theta=\tan^2\chi={\cdot}0003.$$
$$z^2/r^2=1\times10^{-4},\ \lambda^2/r^2={\cdot}0178,\ \lambda'^2/r^2={\cdot}0100.$$

Writing in the terms in the precise order they occupy in (39) we have

$$\begin{aligned}\text{H}\sin\phi = 2m\,r^{-3}\times 0{\cdot}99939\times\\ [1-{\cdot}0003(1+{\cdot}0001)\\ +{\cdot}0356(1-{\cdot}0003-{\cdot}0008-{\cdot}0008-{\cdot}0007)\\ -{\cdot}0300(1+{\cdot}0001-{\cdot}0021-{\cdot}0004)\\ +{\cdot}00026\{1+{\cdot}0002+{\cdot}0002-{\cdot}0002+{\cdot}178-{\cdot}025\}].\end{aligned}$$

The most important correction is the introduction of ·99939 in place of unity in the coefficient of $2m\,r^{-3}$. This represents a change of 6 per 10,000 in the value of H/m, or of 3 per 10,000 in the value of H. If the factor were, as is usual, assumed to be unity, H would be overestimated to this extent.

The next most important correction arises from the term $-3z^2/r^2$ in the first line of the expression in square brackets. This term, it should be noticed, contributes to the value of P, thus showing how variations in the height of the deflected magnet may partly account for differences found for P on different days. In the concrete case the departure of the coefficient of $-3z^2/r^2$ from unity is quite negligible. The subsidiary terms in the coefficients of $2\lambda^2/r^2$ and $3\lambda'^2/r^2$ are also very small, but might come in. At all events they cannot safely be assumed negligible. The terms depending on the angles and on z inside the bracket in the last line are wholly negligible, but clearly the last line is not safely negligible as a whole, unless somewhat greater care is taken about the level of the suspended magnet than is supposed in the concrete case. It is, however, mainly the fault of the observer if the last line is not negligible.

Experiments on Pole-Distance.

§ 26. Let us suppose that two magnets 1, 2, closely identical in size and pattern, can deflect each the other in the same magnetometer. Let the suffix 1 distinguish the properties of the first magnet, and let P_1, Q_1 &c. answer to the case when it is the deflector; similarly, let the suffix 2 relate to the second magnet. Then we have

$$\left.\begin{aligned} 2\lambda_1^2-3\lambda_2^2 &= P_1(\lambda_1, \lambda_2) = P_1'-f(a_1, a_2)-AQ_1-BR_1, \\ 2\lambda_2^2-3\lambda_1^2 &= P_2(\lambda_2, \lambda_1) = P_2'-f(a_2, a_1)-AQ_2-BR_2, \end{aligned}\right\} \quad (40)$$

where $f(a_1, a_2)=(3/4)\{3(a_2^2+b_2^2)-2(a_1^2+b_1^2)\}$,

$$A=r_1^{-2}+r_2^{-2}, \quad B=r_1^{-4}+r_1^{-2}r_2^{-2}+r_2^{-4},$$

r_1 and r_2 being the two deflexion-distances.

Differences between the values of p for the pole-distances of the two magnets are, as we have seen, generally much less effective in altering Q and R than in altering P, and the contributions from Q and R in the above equations are relatively small. Thus a close approximation may be expected from the formula

$$5(\lambda_1^2-\lambda_2^2)=P_1'-f(a_1, a_2)-\{P_2'-f(a_2, a_1)\}, \quad . \quad (41)$$

which neglects the difference between Q_1 and Q_2 and between R_1 and R_2.

I have found in this way approximations to the differences between pole-distances in two sets of mirror magnets. The first set consisted of three very old magnets AN, Na, and Ne by Jones, practically identical in length, and differing only slightly in diameter. Their great similarity led one to anticipate a very close approach to equality in their pole-distances. This expectation was fulfilled for two of the magnets, but not for the third, Ne. At first I was disposed to look for an explanation in a difference between the directions of the magnetic axis and the normal to the attached mirror (see § 24). This does not seem, however, to be the true explanation, as may be seen from the following results. The notation is the same as in § 24; λ refers in each case to the deflecting, λ' to the deflected magnet.

TABLE XVII.

Deflecting magnet.	Deflected magnet.	θ.	χ.	True formula for $P(\lambda, \lambda')$.
AN	Na	$-60'$	$+25'$	$2\lambda^2-2{\cdot}996\lambda'^2$
Na	AN	-123	$+85$	$1{\cdot}999\lambda^2-2{\cdot}987\lambda'^2$
AN	Ne	$+49$	$+25$	$2\lambda^2-2{\cdot}996\lambda'^2$
Ne	AN	-123	$+61$	$2\lambda^2-2{\cdot}985\lambda'^2$
Na	Ne	$+49$	$+85$	$1{\cdot}996\lambda^2-2{\cdot}995\lambda'^2$
Ne	Na	-60	$+61$	$1{\cdot}999\lambda^2-2{\cdot}998\lambda'^2$

The differences from the normal form $2\lambda^2-3\lambda'^2$ are too trifling to be of real moment, and have been neglected in the subsequent calculations.

The second set of magnets were also mirror magnets, No. 60 by Elliott Bros., Nos. 138 and 140 by Dover.

Observations were made at two pairs of distances 18 and 24 cms., and 22·5 and 30 cms. The dimensions of the magnets were as follows, all in centimetres.

TABLE XVIII.

Magnet	AN.	Na.	Ne.	60.	138.	140.
Length l..................	7·60	7·60	7·60	6·385	6·42	6·385
External diameter $2a$...	·725	·76	·77	·74	·82	·81
Internal „ $2b$...	·43	·50	·50	·405	·52	·53

The diameters appeared in all the magnets to be very nearly uniform throughout the entire length.

§ 27. Some fifty deflexion experiments were made by the Chief Assistant of the Observatory Department, Mr. T. W. Baker, of whose observational skill I was fortunately able to

TABLE XIX.

Deflecting magnet.	Deflected magnet.	$f(a, a')$.	$P' - f(a, a')$ for		The $\lambda^2 - \lambda'^2$ involved.	Value of $\lambda^2 - \lambda'^2$ from observations at		
			18 & 24 cms.	22·5 & 30 cms.		18 & 24 cms.	22·5 & 30 cms.	Mean.
		+	−	−		+	+	+
AN Na	Na AN	·20 ·09	13·90 13·87	13·63 12·98	$\lambda_2^2 - \lambda_1^2$	·006	·130	·068
AN Ne	Ne AN	·21 ·09	18·44 12·20	16·80 11·12	$\lambda_3^2 - \lambda_1^2$	1·248	1·136	1·192
Na Ne	Ne Na	·16 ·16	18·02 12·46	16·81 11·65	$\lambda_3^2 - \lambda_2^2$	1·112	1·032	1·072
60 138	138 60	·26 ·05	9·75 8·78	9·12 8·75	$\lambda_2^2 - \lambda_1^2$	·194	·074	·134
60 140	140 60	·26 ·05	9·20 9·54	8·46 9·24	$\lambda_1^2 - \lambda_3^2$	·068	·156	·112
138 140	140 138	·17 ·18	8·36 10·45	8·56 9·93	$\lambda_2^2 - \lambda_3^2$	·418	·274	·346

avail myself. The results appear in Table XIX. λ_1 refers to AN or 60, λ_2 to Na or 138, λ_3 to Ne or 140.

After the observations were completed, it was noticed that through the employment of a sighting-tube intended for a larger deflecting magnet, the level of the suspended magnet had been in each case slightly too high. A small correction ·12, answering to the mean error ·2 cm. in the level, has been applied in the Table to the observed values of P'.

§ 28. To determine the value of, say, $\lambda_2-\lambda_1$, we put

$$\lambda_2^2-\lambda_1^2=(\lambda_2-\lambda_1)(\lambda_2+\lambda_1),$$

and as $\lambda_2-\lambda_1$ is small, we may for a first approximation take

$$\lambda_2+\lambda_1=\cdot 8l=6\cdot 1 \text{ cm. for AN, Na, and Ne,}$$
$$=5\cdot 1 \quad ,, \quad ,, \quad 60,\ 138, \text{ and } 140.$$

In this way we find

TABLE XX.

For AN, Na, and Ne.		For 60, 138, and 140.	
cms.		cms.	
$\lambda_2-\lambda_1=\cdot 011$	$2(\lambda_2-\lambda_1)/l=\cdot 003$	$\lambda_2-\lambda_1=\cdot 026$	$2(\lambda_2-\lambda_1)/l=\cdot 008$
$\lambda_3-\lambda_1=\cdot 195$	$2(\lambda_3-\lambda_1)/l=\cdot 051$	$\lambda_1-\lambda_3=\cdot 022$	$2(\lambda_1-\lambda_3)/l=\cdot 007$
$\lambda_3-\lambda_2=\cdot 176$	$2(\lambda_3-\lambda_2)/l=\cdot 046$	$\lambda_2-\lambda_3=\cdot 068$	$2(\lambda_2-\lambda_3)/l=\cdot 021$

If theory and experiment were perfect, the results for $\lambda^2-\lambda'^2$ from the two pairs of distances in Table XIX. should be identical, and in Table XX. the value of, say, $\lambda_3-\lambda_2$ found directly should agree with that deduced from the values of $\lambda_1-\lambda_2$ and $\lambda_3-\lambda_1$. This cannot be claimed, but on the whole there is a very satisfactory agreement, considering that a complete elimination of the effects of variation of horizontal force and temperature could only be hoped from a largely increased number of observations.

The two magnets AN and Na behave practically alike, but differ notably from Ne, though to the eye that magnet seems an almost exact duplicate of Na. As these three magnets are practically equal in length, Table XX. implies that the value of p must be about ·05 larger in Ne than in the other two.

In the case of 60, 138, and 140, the agreement in pole-distance is closest between 60 and 140, though by different makers. The excess, however, of pole-distance apparent in 138 is partly accounted for by its excess of length, about ·035 cm., over the other two. If we allow for this, we find that the value of p in 138 exceeds that in 60 by only about ·003. The value obtained directly for $\lambda_2-\lambda_3$ is probably slightly in excess, and the excess of p in 138 over 140 is probably about ·01.

§ 29. Having now proved a close similarity in pole-distance between AN and Na on the one hand, and between 60, 138, and 140 on the other, we occupy a favourable position for determining an approximate value for p by the method of § 21. The necessary comparison between calculated and observed quantities is made in Table XXI. In the case of 60, 138, and 140, I have assumed a mean value 6·40 for l and combined all the observations. To the present degree of approximation, that should give a satisfactory mean result.

TABLE XXI.

Magnets.	p.	Calculated values of					Observed mean value of P′(λ,λ′).	
		P(λ,λ′).	$(r_1^{-2}+..)Q$ $+(r_1^{-4}+..)R$.		P′(λ,λ′).			
			18 & 24 cms.	22·5 & 30 cms.	18 & 24 cms.	22·5 & 30 cms.	18 & 24 cms.	22·5 & 30 cms.
		—	—	—	—	—	—	—
AN and Na	·84	10·19	3·08	2·00	13·27	12·19	13·88	13·30
	·86	10·68	3·38	2·19	14·06	12·87		
	·88	11·18	3·70	2·40	14·88	13·58		
60, 138, 140	·84	7·23	1·57	1·01	8·80	8·24	9·35	9·01
	·86	7·57	1·72	1·11	9·29	8·68		
	·88	7·93	1·88	1·22	9·81	9·15		

By interpolation we deduce for the value of p:

From observations	at 18 & 24 cms.	at 22·5 & 30 cms.	Mean.
Mean for AN and Na	·855	·872	·86
Mean for 60, 138, and 140...	·862	·874	·87

The results can be readily checked by reference to the fact that when λ and λ' are equal

$$P(\lambda, \lambda') = -\lambda^2 = -p^2l^2/4.$$

§ 30. The excess of these values for p over Börgen's limiting value ·815 may arise possibly from some instrumental peculiarity, or some defect in the theory; but it is difficult to imagine that any such cause can account for the apparent difference between Ne and its companion magnets. To obtain, however, a further check, I got Mr. Baker to carry out a series of observations in which AN, Na, and Ne were deflected in turn by the same collimator-magnet KC1 in the Kew magnetometer at the three distances 22·5, 30, and 40 cms. Values were calculated for P′ from the observations at 22·5 and 30 cms., and from those at 30 and 40 cms. The mean results appear in Table XXII.

TABLE XXII.

Magnet deflected.	$f(a, b)$.	Observed $P' - f(a, b)$.		$P(\lambda, \lambda')$ allowing for Q and R.		
		22·5 & 30 cms.	30 & 40 cms.	22·5 & 30 cms.	30 & 40 cms.	Mean.
	—	—	—			
AN	·10	3·23	1·92	−0·33	−0·31	−0·32
Na	·03	2·67	2·51	+0·23	−0·90	−0·33
Ne	·03	6·84	4·78	−3·94	−3·17	−3·55

In allowing for Q and R, p was taken as ·81, and the observations were not numerous, so that high accuracy cannot be claimed; but the difference between Ne and the two other magnets is conspicuous. Again, taking the mean observed values obtained above for $P(\lambda, \lambda')$, and employing 2λ to denote the pole-distance of the magnet KC1, we have

$$\left.\begin{aligned} 2\lambda^2 - 3\lambda_1^2 &= -\ \cdot 32 \\ 2\lambda^2 - 3\lambda_2^2 &= -\ \cdot 33 \\ 2\lambda^2 - 3\lambda_3^2 &= -3{\cdot}55 \end{aligned}\right\}\text{ whence }\begin{aligned} \lambda_3^2 - \lambda_1^2 &= 1{\cdot}08 \\ \lambda_3^2 - \lambda_2^2 &= 1{\cdot}07. \end{aligned}$$

The agreement with the values 1·19 and 1·07 obtained for $\lambda_3^2 - \lambda_1^2$ and $\lambda_3^2 - \lambda_2^2$ in Table XIX. is very satisfactory.

§31. During the deflexion experiments the magnetic moments of the magnets, in C.G.S. units, were approximately as follows:

AN.	Na.	Ne.	60	138	140
250	310	217	333	285	222

Of the Jones magnets Ne thus possessed the lowest moment as well as the greatest pole-distance; but amongst the other three magnets 140 had both the lowest moment and the smallest pole-distance. Also the difference between the moments of Ne and AN is less than the difference between AN and Na, which seemed practically identical so far as pole-distance is concerned. The facts thus throw no certain light on the question whether the strength influences the pole-distance.

The Jones magnets, as already stated, are very old. They were magnetized first, so far as is known, some fifty years ago, and are not known to have been remagnetized since.

The magnets 140, 138, and 60 vary in age from five to twenty years. They too are supposed never to have been remagnetized. It would seem worth inquiry whether mere age, as apart from change of strength, may not tend to increase the pole-distance.

Experiments with a set of collimator magnets, following the same lines as those with the six mirror magnets, would have been of fully greater interest; and the combination of the results obtained for the collimator magnets alone, and for the mirror magnets alone, with results obtained by employing the collimators to deflect the mirror magnets, would have formed a promising though laborious line of research. Ordinary magnetometers, however,—except those by Messrs. Cooke of the India Office pattern—do not possess a deflexion-chamber sufficiently long to take a collimator magnet; and no opportunity of carrying out deflexion experiments on collimators has presented itself.

Discussion.

Dr. W. Watson said that the author had referred in the course of his remarks to a statement which he (Dr. Watson) had made at a former meeting of the Society, to the effect that the limit of accuracy in the setting of the magnet-carriage upon the deflexion-bar was 0·1 mm. Although it might be possible, when working under favourable conditions at an observatory, to make the setting with greater precision, he thought that 0·1 mm. was as much as could be expected or obtained in ordinary work in the field. He pointed out that the number of settings in a deflexion experiment was a safeguard against inaccuracies in individual settings, and that it was probable that there was no source of constant error. He expressed his interest in the author's experiments on pole-distances, and said that when the magnetometers were being designed for the Indian Survey he had expressed the opinion that it might be advantageous to make the coefficient Q equal to zero by choosing suitable pole-distances for the magnets. Probably Dr. Chree's experiments will enable us to settle whether it is sufficiently accurate to make Q as small as possible and work with a value of P only, determined in the ordinary way.

Dr. P. E. **Shaw** pointed out that brass, which is used for the deflexion-bar, has a specially large thermal expansion, and if 50 cms. of it rises 3° C. the increase in length would be about 25 microns, which Dr. Chree stated to be the error in setting the magnet-carriage on the scale. Possibly some alloy, *e. g.* some form of invar, would be preferable to brass.

Prof. H. L. **Callendar** said that some time ago he made some experiments upon the distance between the poles of magnets, and the conclusion he came to was that it was impossible to express the pole-distance as a definite fraction of the length of the magnet. He had found that among other things the pole-distance depended upon the shape of the magnet, the intensity of magnetization, and the kind of steel of which the magnet was composed. There was also a variation of the pole-distance with time.

Mr. **Skinner** said it might be possible to fix the position of the poles by using magnets similar to those used by Mr. Searle in many of his experiments. These magnets consisted of long thin needles with steel balls at the ends, and it was found that the poles were very close indeed to the centres of the balls.

Dr. **Chree** said that the limit of accuracy in the setting of the magnet-carriage which he had mentioned, 0·03 mm., was that which was obtainable when working under observatory conditions. With regard to the material of the deflexion-bar, he pointed out that a sufficiently accurate temperature correction was always employed to correct for thermal expansion. There was no invar which was sufficiently non-magnetic to be used in accurate work with a magnetometer. With regard to the relation between pole-distance and length of magnet, he said that Dr. Börgen had carried out experiments on different magnets under different conditions, and had concluded that the pole-distance was a nearly constant fraction of the length of the magnet; but this was not altogether borne out by the present investigation.

XXVIII. *A Model illustrating the Propagation of a Periodic Electric Current in a Telephone Cable, and the Simple Theory of its Operation.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* *.

[Plate XIV.]

The propagation of alternating currents of electricity along linear conductors has been the subject of much mathematical investigation. Its importance in reference to the practical operation of submarine cables, telephone-lines, and more recently of the radiating antennæ in Hertzian wave telegraphy, is well understood. Nevertheless, practical telegraph and telephone engineers have not always shown readiness to assimilate the ideas elaborated by theorists, and in some cases unsound theories of the phenomena have been promulgated. The assistance which practical men derive from considering the operation of a working model when appropriating physical ideas is considerable, and the author has found that for teaching purposes the model here described is of great use in making plain the meaning of mathematical expressions.

The simple theory of it may first be given. The object is to explain the propagation of a periodic or alternating electric current in a telegraph or telephone cable having resistance, inductance, capacity, and leakage, or insulation conductance of a certain value per mile or knot.

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* Read June 10, 1904.

The discussion necessarily proceeds on the assumption that the currents and electromotive forces with which we are concerned are of simple harmonic form, and accordingly the only mathematical symbolisation necessary is that required for representing a simple harmonic motion or simple sine function and its time-rate of change. This is most readily achieved by the use of the complex quantity to denote the maximum value and phase of the periodic quantities with which we are concerned. The representation of a vector quantity by the symbol $a+jb$, where a and b are lengths of lines and j is the *sign of perpendicularity* denoting that b is at right-angles to a is now so universally understood that no further explanation of it is necessary.

Consider, then, in the first place, an electric cable indefinitely extended in one direction, and at the terminated end a simple periodic electromotive force applied. Let the cable have a capacity C farads, an inductance L henrys, a copper resistance R ohms, and a dielectric or insulation resistance r ohms, all per unit of length.

If we take any element of length in the run of the cable, then there is a difference between the potential and the current at the two ends of this element. Let δv and δi be this difference, v and i being the potential and current at the generator end of the element. Then the symbolical expression for the manner in which this difference of potential and of current is created, is given by the familiar equations :

$$\mathrm{L}\delta x\frac{di}{dt}+\mathrm{R}\delta x\, i=\delta v, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\mathrm{C}\delta x\frac{dv}{dt}+\mathrm{K}\delta x\, v=\delta i, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where K is written for $1/r$. Hence

$$\mathrm{L}\frac{di}{dt}+\mathrm{R}i=\frac{dv}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\mathrm{C}\frac{dv}{dt}+\mathrm{K}v=\frac{di}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If i and v both vary in a simple harmonic manner, and if I and V are their maximum values during the period; then, since the maximum values of di/dt and dv/dt considered as vectors are at right angles to the maximum values of i and v, we may write (3) and (4) in the form

$$\frac{dV}{dx}=(R+jpL)I, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dI}{dx}=(K+jpC)V,. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where j is the sign of perpendicularity, and $p=2\pi$ times the frequency n of the current.

Separating the variables by differentiation, we have

$$\frac{d^2V}{dx^2}=(R+jpL)(K+jpC)V, \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2I}{dx^2}=(R+jpL)(K+jpC)I. \quad . \quad . \quad . \quad . \quad (8)$$

The above are the well-known vector equations for the propagation of a simple periodic current in a cable having the four constants L, R, C, K.

If we write P for $\sqrt{R+jpL} \, . \, \sqrt{K+jpC}$, we have the equations

$$\frac{d^2V}{dx^2}=P^2V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{d^2I}{dx^2}=P^2I, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

as the differential equations for the potential and current at any point in the cable.

An obvious solution of the above equations is

$$V=a\epsilon^{+Px}+b\epsilon^{-Px}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$I=\frac{P}{R+jpL}\left\{a\epsilon^{+Px}-b\epsilon^{-Px}\right\}, \quad . \quad . \quad . \quad (12)$$

where a and b are constants of integration, and ϵ is the base of Napierian logarithms. The quantity P is clearly a complex quantity and can be expressed in the form $\alpha+j\beta$.

Hence if $\sqrt{R+jpL}\sqrt{K+jpC}=\alpha+j\beta$, we must have :—

$$\alpha^2+\beta^2=\sqrt{R^2+p^2L^2}\,\sqrt{K^2+p^2C^2}$$

and

$$\alpha^2-\beta^2=RK-p^2LC,$$

or

$$2\alpha^2=\sqrt{(R^2+p^2L^2)(K^2+p^2C^2)}+(RK-p^2LC), \quad (13)$$

$$2\beta^2=\sqrt{(R^2+p^2L^2)(K^2+p^2C^2)}-(RK-p^2LC. \quad (14)$$

The above equations are well-known expressions. We shall call α and β the secondary constants of the cable.

Returning, then, to the case of the semi-infinite cable, we shall take the origin of coordinates at the end at which the simple periodic electromotive force of maximum value E is applied.

Hence for $x=0$ we have $V=E$, and for $x=\infty$ we have $V=0$.

Since, however, in obtaining the original differential equations, we reckoned x in the direction in which potential and current *increase* in value, we have to write $-x$ for x in the general solutions (11) and (12) before we can apply them to the above particular case. Making this change and determining the constants a and b from the terminal conditions, we arrive at the particular solutions of (9) and (10) applicable to the case of a semi-infinite cable having a periodic electromotive force E applied at one end. The potential V and the current I at any point in the cable at a distance x from the generator end is then given by the equations

$$V=E\epsilon^{-Px}, \quad \ldots\ldots\ldots \quad (15)$$

$$I=E\epsilon^{-Px}\frac{\sqrt{K+jpC}}{\sqrt{R+jpL}}. \quad \ldots\ldots \quad (16)$$

It should be clearly understood that in speaking of the potential or current at any point in the cable we mean the maximum value of the potential or current regarded as a vector.

Since $P=\alpha+j\beta$ and $\epsilon^{\pm j\beta x}=\cos\beta x\pm j\sin\beta x$, we can write (15) and (16) in the form

$$V=E\epsilon^{-\alpha x}(\cos\beta x-j\sin\beta x), \quad \ldots\ldots \quad (17)$$

$$I=E\frac{\sqrt{K+jpC}}{\sqrt{R+jpL}}\epsilon^{-\alpha x}(\cos\beta x-j\sin\beta x). \quad . \quad (18)$$

Let us notice the physical signification of the above expressions. Equation 17 shows us that the potential V in magnitude and direction at any point at a distance x in the cable, reckoned from the generator end, is obtained by multiplying the magnitude of the terminal applied voltage E by two factors, (i.) by the attenuation factor ϵ^{-ax}, and (ii.) by the phase factor $(\cos\beta x - j\sin\beta x)$. At any distance x from the origin, V is less than E in magnitude in the ratio of ϵ^{-ax} to 1 and is shifted backwards through an angle βx relatively to E as regards phase.

This is easily seen to be the case if it is remembered that $\cos\theta + j\sin\theta$ is a rotating operator, and when it operates on a vector $a+jb$ it gives as a result a vector of the same size, viz. $\sqrt{a^2+b^2}$, but shifted forwards through an angle θ. The size of the vector $(a+jb)(\cos\theta+j\sin\theta)$ is $\sqrt{a^2+b^2}$, but its slope ϕ is such that $\phi = \tan^{-1}\frac{b}{a} + \theta$ where $\tan^{-1}\frac{b}{a}$ is the slope of the original vector $a+jb$. Hence the expression (16) merely means that the potential at any point in the cable whose distance from the origin is x is less than E in magnitude in an assigned ratio, and is shifted backward in phase by an assigned angle.

If x' is the distance from the generator end at which the maximum voltage has fallen to half the numerical value at the origin, then x' is given by the equation

$$\tfrac{1}{2} = \epsilon^{-ax'},$$

or

$$x' = \frac{0{\cdot}7033}{a}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Again, since $\cos\beta x = \cos(\beta x + 2\pi)$ it is obvious that the wave-length of the potential wave is $2\pi/\beta$, and therefore the velocity of propagation of the potential wave is p/β.

These facts may be presented graphically as follows:—Take a line OX (see fig. 1, Pl. XIV.) to indicate the cable, and set up a perpendicular OE to represent in magnitude and direction, the voltage at the generator end. Then at equidistant points draw other lines decreasing in length in geometrical progression, and each shifted backwards or forwards in direction relatively to the preceding line by an equal angle. If we suppose these lines to revolve with equal angular velocities round their ends as centres situated at equidistant intervals on the line OX ; then their projections at the same instant on vertical lines drawn through their centres will represent at that instant the actual voltage at these points in the cable. The periodic change with time and distance may be represented by a working model made in the following manner :— On a long steel axle AA′ are fastened a number of eccentric pulleys E_1, E_2, E_3, &c. (see fig. 2, Pl. XIV.). The eccentricities of these wheels decrease in geometric progression, and each eccentric is set in phase backward behind its preceding neighbour by an equal angle. These wheels are embraced by cords C_1, C_2, C_3, &c. attached to balls or blocks of metal P_1, P_2, P_3, &c. sliding on vertical rods R_1, R_2, R_3, &c. placed below each eccentric wheel. These cords are all of the same length.

When the axle carrying all the eccentrics is revolved by a handle W, the blocks P_1 &c. will rise and fall with a nearly simple harmonic motion, and at any instant all the blocks will be situated on a sinuous curve of continually decreasing amplitude. As the eccentric axle revolves the motion of the balls will depict the progression of a wave of potential along a cable having capacity, inductance, resistance, and leakance.

Returning, then, to the expressions (13) and (14) for the value of α and β, we may give these more convenient forms.

Taking the formulæ (13) and (14), add to, and subtract from, the quantity under the radical the term $2p^2\text{CLKR}$, and after rearranging terms we obtain

$$2\alpha^2=\sqrt{(\text{KR}+p^2\text{LC})^2+p^2(\text{LK}-\text{CR})^2}+(\text{KR}-p^2\text{LC}) \quad (20)$$

$$2\beta^2=\sqrt{(\text{KR}+p^2\text{LC})^2+p^2(\text{LK}-\text{CR})^2}-(\text{KR}-p^2\text{LC}) \quad (21)$$

If the cable constants have such relation that $\text{LK}-\text{CR}=0$ or $\text{L}/\text{R}=\text{C}/\text{K}$, we have Mr. Oliver Heaviside's distortionless cable, and then obviously when this is the case

$$\alpha=\sqrt{\text{KR}}, \quad \beta=p\sqrt{\text{CL}}. \quad . \quad . \quad . \quad . \quad (22)$$

Hence we have for the abscissa of half voltage x' and the potential wave-velocity w the values

$$x'=\frac{0{\cdot}7033}{\sqrt{\mathrm{KR}}}; \quad w=1/\sqrt{\mathrm{CL}}. \quad . \quad . \quad . \quad (23)$$

Both these quantities are independent of the periodicity, and hence for this cable all potential waves travel at the same rate and are attenuated in the same ratio.

If, therefore, such a cable could be practically constructed with a small enough value for $\sqrt{\mathrm{KR}}$, telephonic speech would be possible through it with results not possible in the case of non-distortionless cables.

The model shows clearly that the attenuation of the simple periodic wave depends on the value of α, that is on the variation of eccentricity of the wheels, and that the wave-length is determined by the value of β.

It is not then difficult even for a non-mathematician to comprehend that if a complex periodic electromotive force acts at one end, which may be resolved into the sum of a number of simple periodic electromotive forces, the result in general will be to produce corresponding simple periodic waves which travel at different speeds and degrade at various rates. Hence, in a comparatively short distance the initial complex wave form is so altered that it becomes unrecognizable. Herein, therefore, resides the cause of the practical limitations of telephony. The remedy for it rests in the alteration of the primary constants of the cable so as to bring about the necessary changes in the secondary constants.

There are, however, necessary practical limits to these changes. In overhead lines where capacity is relatively small, it is not difficult to introduce sufficient inductance (as Prof. Pupin has done) in the form of separated inductance-coils, and provided these are sufficiently close together within the limits of a wave-length the result is equivalent practically to continuous inductance. The same remedy for distortion could no doubt be applied to underground telephone trunk-lines, although the writer is not aware that it has been tried.

On the other hand, an addition to the inductance is not easily made in the case of a submarine cable. Any interference with the continuity of size or great increase in bulk or weight would hardly be permitted.

Hence, in the case of cables the favourite prescription for the defect has been that of creating a large dielectric conductance by artificial leaks. The effect of any such proposed added leakage can easily be foretold.

Taking the expressions for $2\alpha^2$ and $2\beta^2$ in (13) and (14) we may write them in the following form:

$$2\alpha^2=p\mathrm{L}\left(1+\frac{\mathrm{R}^2}{p^2\mathrm{L}^2}\right)^{\frac{1}{2}}p\mathrm{C}\left(1+\frac{\mathrm{K}^2}{p^2\mathrm{C}^2}\right)^{\frac{1}{2}}+(\mathrm{RK}-p^2\mathrm{LC}) \quad (24)$$

$$2\beta^2=p\mathrm{L}\left(1+\frac{\mathrm{R}^2}{p^2\mathrm{L}^2}\right)^{\frac{1}{2}}p\mathrm{C}\left(1+\frac{\mathrm{K}^2}{p^2\mathrm{C}^2}\right)^{\frac{1}{2}}-(\mathrm{RK}-p^2\mathrm{LC}). \quad (25)$$

If we employ any such frequency as 100 to 200, then p is a number of the order of 1000. In all actual well-insulated submarine or underground cables, the quantity $\mathrm{K}^2/\mathrm{C}^2$ per mile is a small quantity of the order of 10^{-4}, whilst $\mathrm{R}^2/\mathrm{L}^2$ is a large number of the order of 10^4 or 10^6. Hence, if the frequency employed is anything of the order of that of the waves of the speaking voice, the quantity $\mathrm{K}^2/p^2\mathrm{C}^2$ can be neglected, whilst the value of $\mathrm{R}^2/p^2\mathrm{L}^2$ may approximate to unity, or say to a number such as 2, 3, or 4 near to unity. Accordingly, in practice we might have for the value of $\left(1+\frac{\mathrm{R}^2}{p^2\mathrm{L}^2}\right)^{\frac{1}{2}}$, reckoned per mile or per knot, a number of the order of, say, 2.

Hence, in this case, the values of α and β per mile would reduce to:—

$$2\alpha^2= p^2\mathrm{LC}+\mathrm{RK} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$2\beta^2=3p^2\mathrm{LC}-\mathrm{RK} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Again, in the case of all well-insulated cables the value of RK is very small compared with $p^2\mathrm{LC}$ for such frequencies as above mentioned, and hence as a first rough approximation we can calculate the values of α and β for most actual submarine cables from the expressions

$$\alpha=\frac{p\sqrt{\mathrm{LC}}}{\sqrt{2}}, \quad \beta=p\sqrt{\frac{3\mathrm{LC}}{2}}.$$

In this case α is a function of the frequency, but the wave-velocity p/β is independent of the frequency. Accordingly there is attenuation depending on frequency. For any well-insulated cable, however, we have the value of α nearly given by the equation,

$$2\alpha^2 = p^2LC\left(1+\frac{R^2}{p^2L^2}\right)^{\frac{1}{2}} + RK - p^2LC; \quad . \quad . \quad (28)$$

and from this equation combined with (19) we can always calculate the distance for drop to half initial voltage or current, which may be roughly said to represent the limiting distance of good telephony.

Suppose, for instance, that we consider as an example any long submarine cable, say the Ireland-Newfoundland (1880) cable. The length (l) is 1423 knots $= 264{\cdot}2 \times 10^6$ centims. The resistance per knot $(R) = 4{\cdot}161$ ohms, the capacity per knot $(C) = 0{\cdot}315$ microfarad, the insulation resistance per knot $(1/K) = 400$–500 megohms*.

The value of the inductance per knot (L) is not so definitely known. The copper core weighs 300 lbs. per knot, and is equivalent in section to a round wire of 0·34 centim. in diameter. As a first approximation, if we calculate the total inductance L from the formula for the inductance of a straight single wire,

$$L = 2l\left(\log_e \frac{4l}{d} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

we find $L = 11{\cdot}5$ henrys, from which we deduce the average inductance per knot (L) to be 0·008 henry.

But the iron armour may increase this value 100 or at most 200 times. Hence the value of L/R per knot may be as low as 0·002, or perhaps as great as 0·02 or 0·4. On the other hand, the value of C/K per knot is 150 nearly, and hence the cable is very far from being distortionless.

If we calculate α corresponding to a value of $n = 170$, $p = 1000$, and an inductance 0·008 henry per knot, then we have $\alpha = 0{\cdot}012$, and the distance for drop to half initial voltage is 60 knots. If we attempted to cure distortion by decreasing the dielectric resistance per knot to 1/150 of a megohm, or about 6000 ohms, then we could make α independent of wave-length, but it would be increased in magnitude to a value 0·021, and the attenuation would be worse. On the other hand, if we could increase the inductance per knot to 0·8 of a henry, the dielectric resistance being only reduced to 2/3rds of a megohm per knot, the distortion would be cured, and the value of α reduced to 0·0021.

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* The above data are taken from Munro and Jamieson's 'Pocket-book of Electrical Rules and Tables,' 4th ed. p. 286. The term *naut* is used by some writers to denote a nautical or sea mile, a length of 2029 yards, whilst others employ the term *knot* for the above length. Strictly speaking a knot is the name for a speed, viz. one nautical mile per hour.

The distance for decay to half initial voltage would be raised to 350 knots, and a great improvement effected. This is only an illustration of the importance of inductance on which Mr. Oliver Heaviside has insisted for so many years. On the other hand, the difficulty is to apply the theoretical remedy practically in the case of submarine cables, and it is not easy to see how it is to be done.

Suppose, however, that we could make the resistance per knot 3 to 4 ohms, the capacity per knot 0·3 microfarad, the insulation resistance 300 megohms per knot, and the inductance say 300 henrys. We should then have $L/R = C/K = 100$. Hence we should have $\alpha = 0{\cdot}0001$ and $\beta = 10^5$ corresponding to $p = 1000$. Accordingly the distance for decay to half voltage would be some 7000 knots. Such an achievement may be quite beyond the reach of practical work, but an approximation to it is perhaps not impossible. It cannot be done, however, merely by increasing the iron armour or bringing it nearer to the copper, because in that case the capacity is increased very considerably. The annulment of distortion by creating lateral leakage either through inductive or non-inductive leaks always leads to large values of α. In other words, to great attenuation. Hence, although the distortion of the complex potential or current wave may be cured, the decay of amplitude would extinguish audibility as far as telephonic transmission is concerned within quite moderate distances, and far less than that which could be obtained by the right application of inductance.

As a small contribution to this practical consideration of the problem, the model here described and method of discussing the theory may perhaps be useful.

Note added July 9th, 1904.

In reference to the above method of discussing the problem, it has been represented to the author that it would assist some mathematical readers if the steps by which the usual algebraical expressions for the potential and current at any point in the cable can be deduced from equations (17) and (18), were set out more at length.

The equations (17) and (18) are expressions showing the manner in which the potential and current at any distance x along the cable, considered as vectors, are related to the vectors representing the potential and current at the origin at the same instant.

If we desire to ascertain the state of affairs at any time t we have to consider these vectors as rotating round their extremities with a periodic time T. Accordingly the proper expression for the vector representing the potential at the origin at any time t from the beginning of the epoch will then be $\mathrm{E}\epsilon^{jpt}$, where $p=2\pi/\mathrm{T}$. Hence the corresponding expression for the vector representing the potential at any distance x along the cable at the same instant will be obtained by multiplying equation (17) by ϵ^{jpt}, and will therefore be given by the equation

$$\mathrm{V}=\mathrm{E}\epsilon^{jpt}\,\epsilon^{-(\alpha+j\beta)x}$$

or by $$\mathrm{V}=\mathrm{E}\epsilon^{-\alpha x}\left\{\cos(pt-\beta x)+j\sin(pt-\beta x)\right\}. \qquad (17')$$

If, therefore, the numerical value of the potential at the origin at any instant is represented by the real part of $\mathrm{E}\epsilon^{jpt}$, viz. by $\mathrm{E}\cos pt$, then the numerical or algebraic value of the potential at the same instant but at the distance x along the cable will be given by the real part of (17′), viz. by

$$v=\mathrm{E}\epsilon^{-\alpha x}\cos(pt-\beta x).$$

The above equation is the ordinary algebraic expression for the potential in a semi-infinite cable at any time t and any distance x from the origin. Under the above conditions the time t would be reckoned from the instant when the potential at the origin has its maximum value.

The last equation is the usual expression for a wave of decreasing amplitude and wave-length $2\pi/\beta$ moving with velocity p/β.

Similar expressions can be obtained for the numerical or algebraic value of the current in the cable at any place and time.

Discussion.

Mr. Thomas H. Blakesley congratulated Prof. Fleming upon the interesting result contained in his paper, that under a certain critical relationship among the properties of the cable relatively distortionless waves might take place whatever the frequency. But he thought some other of the consequences of this relationship might be more simply exhibited in the forms for Potential and Current :—

$$V = Ee^{-\alpha x}\cos(\beta x - pt),$$

$$I = \frac{Ee^{-\alpha x}}{q}\cos(\beta x - pt + m),$$

where $$q^2 = \frac{R}{K}\left\{\frac{1+\left(\frac{Lp}{R}\right)^2}{1+\left(\frac{Cp}{K}\right)^2}\right\}^{\frac{1}{2}},$$

$$\tan 2m = \frac{(LK-RC)p}{RK+LCp^2}.$$

q is of the nature of resistance, and m is the angle of lag of the current behind the potential and is constant for all points of the cable.

It will be plainly seen that this lag angle disappears when the condition $LK-RC=0$, but from the expression for q^2 it will be clear that this quantity will be independent of the frequency if $\frac{L}{R}=\frac{C}{K}$, which is again the same condition. Thus the real degradation of current would be the same for all tones, and the conjunction of current and potential for all points, a very important matter, secured by this condition. But, as Prof. Fleming points out, it would not be easy in practice to secure the fulfilment of this condition. There is, however, something to be said for keeping both L and K small, as in fact they are in practice. For, looking at the expression for q^2, it will be noticed that it will become smaller as the frequency rises if $\frac{C}{K}$ is greater than $\frac{L}{R}$, that is if $CR > LK$, and this is emphatically true of cables in practice. Hence, though $e^{-\alpha x}$ grows less with rising frequency, the resistance q also becomes less ; so that the degradation of current is not so great as that implied in the mere increase of α with frequency.

In operating upon the equations (13) and (14) to obtain the equations (20) and (21), it may be well to point out that the quantity

$$\sqrt{(R^2+p^2L^2)(K^2+p^2C^2)}$$

may be written in the two ways :—

$$\sqrt{(KR \pm p^2LC)^2+p^2(LK \mp CR)^2}.$$

Prof. Fleming has chosen the top signs in order to consider the relation $LK-CR=0$.

But if the bottom signs are chosen we may contemplate the equality of KR and p^2LC, in which case not merely does one of the squares under the radical sign disappear, but the values of α and of β coincide. The quantity $\overline{KR-LCp^2}$ does in fact vanish when $K=0=L$, which more nearly coincides with fact than Prof. Fleming's critical condition. Perhaps it may be said that preference may be given to one or other of the two cases according as it is desired to transmit the voice as in telephony, or sound signals on the Morse system as in wireless telegraphy. In the latter case we have a great choice in the value of p which may be kept constant.

Mr. F. Jacob said the following corrections should be made to the data of the 1880 Anglo-American Cable. The length is 1852 nauts, the dielectric resistance of the cable as it lies in the sea is of the order of 20,000 megohms per naut, the figure 400 being at 75° F., and atmospheric pressure after 1 minute's electrification, but, on the other hand, the electrification or absorption of the dielectric masks all the effect of the high insulation value.

300 to 400 henrys per naut may be considered as absolutely beyond the realms of possibility, commercially or

otherwise. What is demanded is not of a higher order than one henry per naut.

The model is simply that of a wave motion with attenuation, and he failed to see how it helped the "practical man" to see the advantageous effect of adding inductance to an ordinary Line.

Dr. Fleming, in reply, said that the relation between the electrical constants of the cable which gave distortionless transmission was discovered and announced twenty years or more ago by Mr. Oliver Heaviside. All that he (Dr. Fleming) had attempted to do was to simplify and illustrate the proofs of this condition. The employment of complex quantities in the discussion of alternating current problems led to a considerable simplification of the analysis. As in practice we were only concerned with the maximum or root-mean-square values of the potential and current, the introduction of the time-variable into the equations, as in the ordinary method of handling the problem, led to complicated expressions, the physical meaning of which was not readily apparent to most students. All the usual algebraic expressions for the current and potential at any point in the cable could, however, be readily obtained from the vector equations he had given.

As regards the practical realization of Heaviside's distortionless condition for submarine cables, the figures given by him (Dr. Fleming) were merely intended to indicate the remoteness of its possibility.

The correction furnished by Mr. Jacob for the insulation resistance of the dielectric when the cable was immersed did not affect the general conclusions, as if the insulation resistance of the cable was in fact greater than 300–400 megohms per naut, the departure from the distortionless condition would be then still greater.

On the other hand, he quite recognized that the so-called "absorption" of the dielectric introduced a factor not taken into account in the mathematical theory. As regards the practical superior limit of inductance which might be given to a cable, no one could say what invention might not bring forth. He was glad to hear that an inductance of one henry per naut might improve matters sensibly, because he had devised a plan by which an approximation to this might be reached; but Mr. Jacob only confirmed the statement in his paper in expressing the opinion that an inductance of 300 to 400 henrys per naut was impracticable, at least at present and perhaps always.

The model was only intended as an assistance to those whose previous acquaintance with the subject was small, but it had other applications than to electrical problems, and might be used, for instance, to illustrate the decay of temperature waves in the earth's crust, penetration of current or magnetism into bodies, or in fact any class of oscillation covered by the equations under discussion.

Having regard to the fact that electricians of great practical experience in telephony had in the past advocated theories of the subject which were not capable of being reconciled with accepted scientific principles, he thought that an excuse might be made for going again over worn ground. There was still a great field open for advance in cable telephony, and improvements were only likely to be made by examining carefully how far theoretical remedies could be put into practice.

As regards the criticism that a knot is a measure of speed, the Author said that he had followed a custom adapted in most electrical manuals of calling a nautical or sea mile of 2029 yards a "knot," although he was aware that it was also employed as the name for a speed of one nautical mile per hour.

XXIX. *Chemical Dissociation and Electrical Conductivity.*

By A. E. GARRETT *and* R. S. WILLOWS, *B.A.*, *D.Sc.* (Communicated by Prof. J. J. THOMSON, *F.R.S.*)[*]

IN the Philosophical Magazine for July 1899 there appeared a paper by Beattie entitled "Leakage of Electricity from Plates." When various mixtures of substances were placed on the insulated plate of an air-condenser and heated, it was found that a leakage of electricity took place, this being, in several cases, apparently independent of the sign of the charge. The temperatures used were between 300° and 400°. Similar effects were obtained if the substances were placed on the earthed plate.

Shortly after the appearance of this paper one of us repeated the experiments and verified the results generally. The apparatus used was similar to Beattie's. It was found, however, in a good many cases, that the rate of leak depended in a great measure on the sign of the charge given to the insulated plate. In a later paper[†] Beattie found the same result. He showed that if positive electricity leaked from the plate when the substance was on it, then it was incapable of holding a negative charge when that substance was transferred to the earthed plate of the condenser.

The objects of the experiments described in the following paper were to discover the origin of the leak, or, in terms of the ionization theory, to find out the seat of the ionization, and also obtain information concerning the nature of the ions in some of the cases. As a good many of the new observations and facts have been anticipated in Beattie's second paper, only those are given which bear directly on these objects.

Here is a typical instance from Beattie's papers, and one which we have had under investigation. An air-condenser is formed of zinc plates; when iodine is placed in it no leak occurs beyond the small amount due to defective insulation of the supports, although the temperature is raised to 350°. All the iodine has long before this disappeared as vapour. Common salt alone in the condenser also produces no effect when heated to the same temperature; but if the salt is sprinkled with iodine and placed on the insulated plate, then a large leak of positive electricity takes place when the temperature is raised. If the mixture of salt and iodine is placed on the earthed plate, the sign of the discharge is changed, otherwise no difference is noticed. Bromine water can replace the iodine without much affecting the result.

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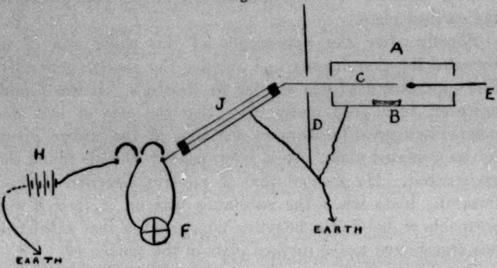
[*] Read June 24, 1904.

[†] Phil. Mag. April 1901.

Experience had shown that Beattie's form of apparatus was troublesome to use and unsuitable for quantitative experiments. That finally used is shown in fig. 1.

Fig. 1.



A is a metal cylinder about 12 cms. long and from 4 to 6 cms. in diameter. Caps of the same material partially close the ends. C is a wire placed along the axis of A, it passes through a hole in a metal plate D and then through ebonite plugs in a tube J; the latter is to screen it from external disturbances. D serves the same purpose for all parts of the apparatus on the left. During the early part of the experiments the measuring instrument was a gold-leaf electroscope. This was connected to C and charged, the case of the instrument, D and A all being earthed. Finally, two hundred small cells were obtained and an electrometer was used. One pole of the battery was connected to C by means of a paraffin switch, the other pole was connected to earth. Both pairs of quadrants were first connected together to C and to the insulated pole of the battery H. A, D, and J were permanently earthed. The quadrants were then disconnected, one pair was kept on the battery, the other pair to C. A leak of electricity from C is then shown by a constantly increasing deflexion of the electrometer-needle. E is a thermometer reading to 360°.

The insulation was tested when the apparatus was cold, and also when A was heated to 360°. The heating was done electrically by means of a coil of wire wound non-inductively on A and carefully insulated therefrom by a layer of asbestos paper. In series with the heating-coil was an adjustable rheostat which enabled the temperature to be varied within the limits desired, or kept constant within 1° for a considerable time. The substance to be experimented with was placed in a shallow metallic dish B.

Influence of the Nature of the Plate when Salt and Iodine are used.

When the plate B was zinc, Beattie's results were first verified. It was found, however, that the nature of the plate, and, in some cases, of the tube A had an influence. Up to a temperature of 360°, with salt and iodine placed on plate B, the following results with different metals were obtained. The electroscope was used in these experiments; the leaves were observed through a microscope having a scale in the eyepiece, and the time taken for the leaves to fall through the same five divisions was recorded. Cylinder A was made of brass.

Metal.	Charge on insulated electrode.	Time to leak through 5 divisions.		Temperature.
		min.	sec.	
Zinc ...	Negative	4	0	20°
		4	5	112
		3	55	175
		2	50	225
		1	45	242
			35	255
			30	259
			25	262

With a positive charge a slightly increased leak was obtained on heating; this was irregular.

Iodine vapour was visible at about 100°; its appearance did not coincide with an increased leak, sometimes, in fact, a charging up took place. This might easily be due to the projection of solid particles of iodine. The plate with its mixture could be used after standing for several days, when on heating, and without the addition of fresh iodine, the same behaviour was observed.

Metal.	Charge.	Time to leak through 5 divisions.	Temperature.
		min. sec.	°
Copper.	Negative	1 15	20
		1 20	185
		1 20	236
		1 15	256
		1 5	272
		1 0	286
		45	291
		45	293
		40	300
		27	302
		15	307

These are results of the same order as for zinc plates. Only a small leak was obtained with a positive charge. It is noticeable that the temperature at which the leakage begins is higher with copper than with zinc.

Metal.	Charge.	Time to leak through 5 divisions.	Temperature.
		min. sec.	°
Brass...	Negative	3 5	22
		2 25	225
		3 0	260
		2 20	279
		1 20	290
		1 20	296
		1 0	305

With a positive charge a small leak was obtained. The observations show that the effects are not so great with brass as with copper and zinc.

Metal.	Charge.	Result on heating.
Aluminium............	−ve	Practically no increased leak.
	+ve	Practically no increased leak.
Iron	−ve	No increased leak.
	+ve	No increased leak.
Tinned iron	−ve	Increased leak.
	+ve	Scarcely any increased leak.
Platinum	−ve	No leak.
	+ve	No leak.

From these results it is seen that the metal is a necessary part in the changes causing the conduction; the changes might therefore be looked for at the metallic surface.

From a zinc plate that had been used the adherent mixture was lightly brushed off; this caused no change in the electrical conductivity. If the surface was then cleaned, either with glass-paper or by washing well in distilled water, it was found that its activity had been removed, but it could be restored by fresh addition of salt and iodine.

The washing water was tested, after filtering, for free iodine and for iodine in combination. No free iodine was found, but a large amount was present in combination.

If a zinc plate that had been used was washed in dilute HCl instead of in water, a very large leak was obtained when it was replaced in the apparatus without drying and reheated. It was found then that a clean piece of zinc if dipped in HCl gave the same result, and this although it was previously heated to nearly 200° to drive off any adherent acid and moisture. The zinc plate was now replaced by one of platinum, on which was put some zinc chloride. When heated, this gave a very great leak with a negative charge on C, a smaller one with a positive charge. For example, at ordinary temperatures the time taken to leak through 5 divs. was 5 minutes, when heated to 300° the time required was 12 sec. only, the charge being negative.

The outer cylinder A had been made of brass to this point, but after heating in it zinc chloride it was found to be impossible to make it insulate properly at the higher temperatures. The electroscope also was not suitable for exact measurements. At this point, therefore, an iron cylinder was obtained, the insulated electrode C was formed from a platinum wire, and the dish B was also platinum. No increased leak was then noticed on reheating, no matter what substance had been used. The electroscope was replaced by an electrometer, and cells were used for charging the apparatus.

Experiments with Zinc Chloride.

A small piece of solid $ZnCl_2$ was placed on B, and the current between C and A measured at different temperatures. While an observation was being made the temperature was allowed to rise very slowly, the mean of the values at the beginning and the end of the reading being taken as the temperature of the experiment.

Greater regularity could be attained in this way than by trying to keep the temperature quite constant. The numbers given in the following table will show the magnitude of the results. In some cases a capacity was connected to the quadrants to reduce the rate of deflexion.

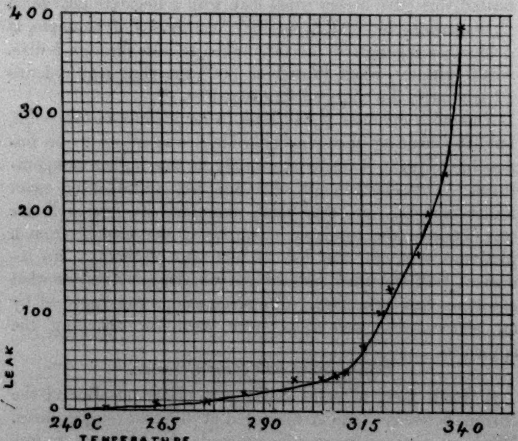
Leak with $ZnCl_2$ at different Temperatures.

Temp.	Leak.	Temp.	Leak.	Temp.	Leak.
125	2 in 30^s	275	17 in 10^s	319	200 in 10^s
157	2 ,,	284	32 ,,	322	250 ,,
208	1 ,,	297	60 ,,	329	160 in 5^s
223	0 ,,	303	65 ,,	332	200 ,,
248	8 ,,	307	70 ,,	336	240 ,,
263	32 ,,	310	80 ,,	340	390 ,,
		315	130 ,,		

C in this case was charged to —400 volts.

The numbers are represented graphically in fig. 2.

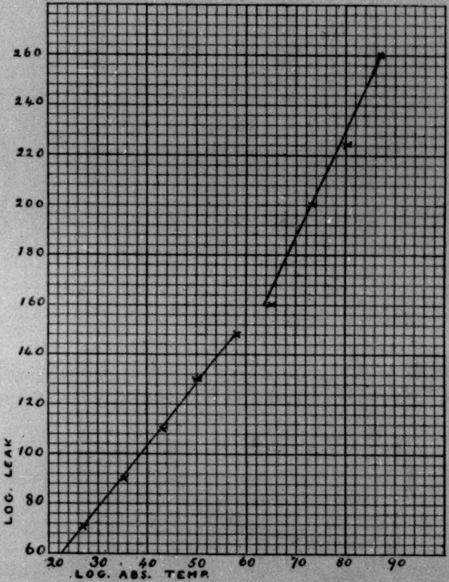
Fig. 2.



It is seen that the rate of increase with temperature is much quicker after passing 312°. This, as is seen below, is due to another source of ionization being tapped. In such a case it is hardly to be expected that the results can be expressed throughout their whole range by any simple law.

Fig. 3 gives the relation between the logarithms of the leak and the logarithms of the corresponding absolute temperatures. It shows, while the conduction is produced by the same mechanism, a straight-line law. Within these limits of temperature, therefore, the current I through the gas can be expressed by a formula of the type $I = aT^n$, a and n being constants, of which the former depends on the units employed. For the lower part of the curve $n = 24$, for the upper part $n = 41$.

Fig. 3.



When the direction of the field is reversed the current is greatly diminished, the proportion depending upon the temperature. The following is a typical set of readings:—

C charged to +200 volts.
Leak with $ZnCl_2$ at different Temperatures.

Leak.	Temp.	Leak.	Temp.
4	20	21	344
4	150	36	348
4	300	60	351
4	320	106	354
5	330	148	356
12	338		

Fig. 4.

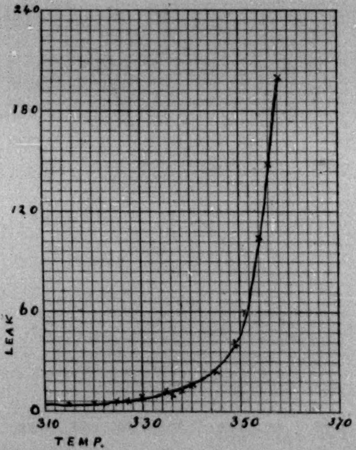
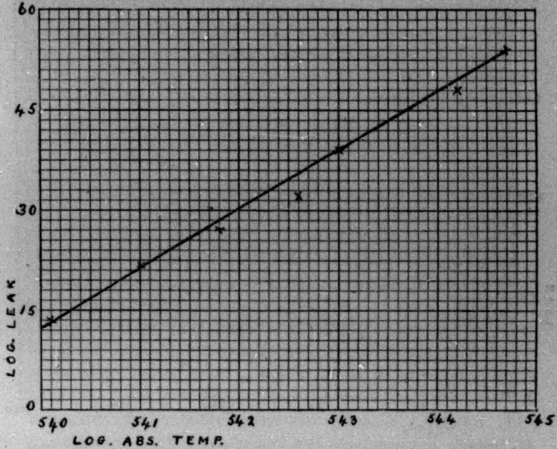


Fig. 4 is plotted from these readings. It shows that the conduction starts at a much higher temperature in this case than in the previous one; 246° in the one and about 322° in the other. It also throws light on the rapid increase in conduction shown in fig. 3 after temperature 320° is passed. At this temperature a new source of ionization appears which produces negative as well as positive ions.

Fig. 5 gives the relation between the logarithms of the leak and of the absolute temperature. It follows that the current is again expressed by a formula of the type $I=aT^n$. In this case $n=100$.

Fig. 5.

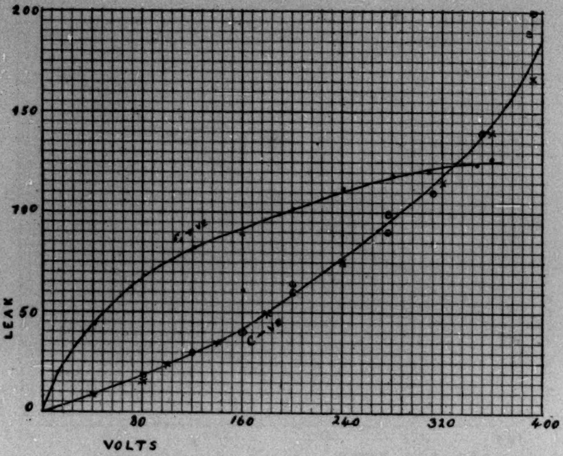


The alteration in current caused by an increase in voltage was next investigated. The temperature was kept constant within 0°·5. The table above is a typical set of readings; the temperature of the experiment was 345°.

C charged −ve.		C charged +ve and leak multiplied by 2.	
Volts.	Leak.	Volts.	Leak.
40	8	18	11
80	17	40	45
120	30	80	70
160	40	120	82
200	62	160	90
238	73	200	105
276	92	238	112
314	110	276	120
352	140	314	120
390	180	352	125
		390	132

The curves in fig. 6 are plotted from these numbers. They resemble the figures given by H. A. Wilson* for the case of the current between a hot platinum wire and a hot platinum tube outside it. When the current is carried by positive ions (C negative) no evidence of a saturation value is evident: this is most probably due to their small velocity, which renders it difficult to drive them out from the thin layer in which they are produced. The negative ions moving more quickly, are more easy to remove.

Fig. 6.



Another explanation may, however, be given. Prof. J. J. Thomson advances the opinion that the emission of negative corpuscles (or electrons) from heated substances is a general property of the atom†. An experiment with sodium vapour is given as an instance where the emission takes place at a temperature as low as 300°. If the negative ions are produced in this manner in the present instance, they would be more or less distributed throughout the space between the electrodes. In that case, it is known that a saturation-current is more easy to produce than it is when the ionization is confined to a single layer near one electrode.

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* H. A. Wilson, Phil. Trans. A, cxcvii. p. 415 (1901).

† 'Conduction through Gases,' p. 168.

It can be shown, on certain assumptions*, that the current should vary as the square of the potential-difference between the electrodes when the ions are produced in a narrow space near an electrode. When the current is carried by positive ions, it was found in our experiments that it is expressed with considerable accuracy by a formula of the type $I=av+bv^2$, where I=current, v=voltage, and a and b are constants. The points on the diagram marked with a cross were calculated from such a one. The other curve of fig. 6 is nearly parabolic.

If the conduction caused by heating a mixture of zinc, salt, and iodine be due to the formation of $ZnCl_2$, one way of identifying this would be to measure the rate of decay of the current produced by $ZnCl_2$ when kept for some time at a constant high temperature, and show that the law is the same as for the mixture.

The conduction obtained with $ZnCl_2$ alone was very irregular in amount; it had decreased by less than 10 per cent. when kept for 3½ hours at 360°. After standing for some hours in the apparatus, the current was largely increased on reheating, due probably to the absorption of moisture. The experiments are being repeated and extended with substances that have been carefully dried.

Experiments with a Mixture of Zinc, Salt, and Iodine.

About ½ gram of a mixture of zinc, salt, and iodine was placed on the earthed plate of the condenser and heated, the current due to 200 volts being noted at different temperatures. One set of readings is given below.

C charged to −200 volts.

Temp.	Leak per min.	Temp.	Leak.
20	3	337	28
95	Irregular. Iodine-vapour coming off.	342	52
		346	80
240	3	351·5	120
302	3	354	155
312	2	357	210
323	4	359	250
332	13		

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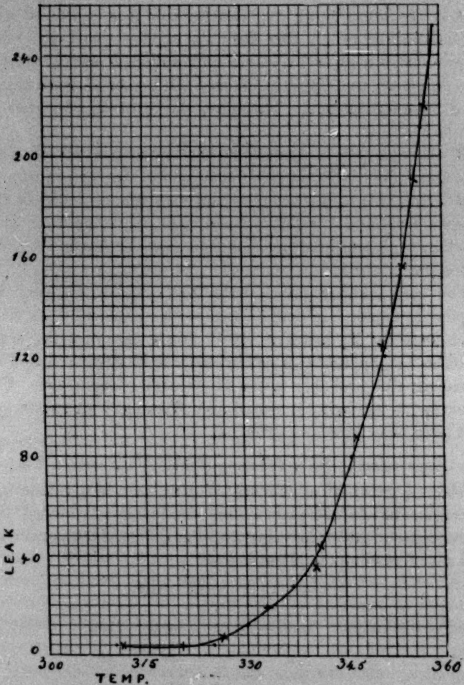
* 'Conduction through Gases,' p. 175.

The current apparently starts at 323°. The numbers are represented graphically in fig. 7. The relation between current and absolute temperature is again of the form $I = aT^n$ with $n = 60$.

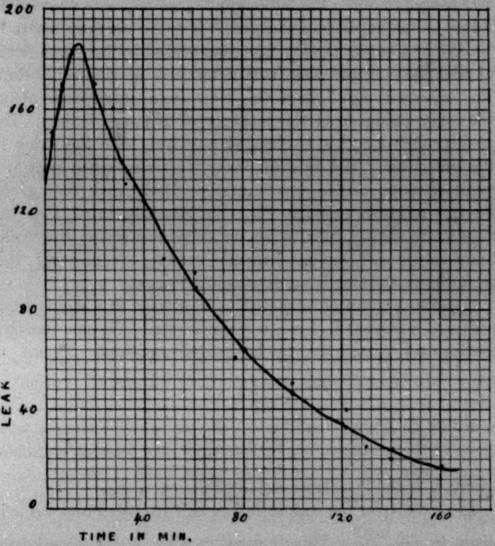
Fig. 7.



With the field reversed the leak was about 10 divisions per min. at 360°, altogether different from that due to $ZnCl_2$.

A fresh portion of the mixture was put in the apparatus and heated to 355° as quickly as possible. When the temperature had been constant for 5 min., observations of the current were begun and were kept up for about three hours. The result is shown in fig. 8.

Fig. 8.



When the current had thus been reduced to 8 per ½ min., no recovery was found to have taken place after standing for fifteen hours, neither did the addition of moisture cause any alteration. A further amount of iodine was added without effect on the leak when the inner electrode was negative, but caused a slightly increased leak when the field was in the opposite direction. After heating for some hours the leak after addition of iodine was, with a positive charge on C, equal to 10 per ½ min., with negative charge 4 per ½ min. The same weight of mixture freshly heated gave 10 per ½ min. with a positive charge. Prolonged heating had therefore no effect on the number of negative ions produced.

Experiments with Zinc Iodide.

Sodium iodide and zinc iodide are two compounds that may be formed when the above mixture is heated.

No increased current could be observed when the former was heated to 360°; the latter gave large currents above 320° when C was negatively charged. If these currents are multiplied by a constant, they are represented by the same curve as that showing the leak from a mixture of zinc-salt-iodine. The crosses in fig. 7 show the leak from zinc iodide alone, the other points from the mixture of zinc-salt-iodine. The positive currents, *i. e.* C charged positively, are the same in the two cases. The rate of decay with time when kept at a constant temperature agrees pretty closely with that given by the mixture (see fig. 8). We scarcely think that the maximum shown here is an experimental error, due to the temperature not having become steady. Neglecting this part, the remainder is an exponential curve.

Suppose a gram of zinc iodide is giving, at a steady temperature, a current I. Let a certain fraction λ become inactive during each minute due to chemical dissociation. Then

$$-\frac{dI}{dt}=\lambda I$$

and

$$I=I_0e^{-\lambda t},$$

where I_0 is the maximum value of the current.

The time, t, for the leak to fall to half its maximum value is given by $t=\frac{\log 2}{\lambda}$. This is equal to 42 min. for the mixture zinc-salt-iodine and 40 min. for zinc iodide. It thus becomes apparent that the conduction observed by Beattie when salt, sprinkled with iodine, is heated on a zinc plate is due to the formation of zinc iodide at the lower temperatures, and a subsequent breaking up as the temperature rises. The sodium chloride appears to have a catalytic action during the formation, as the direct heating of zinc and iodine together causes no leak.

It is to be noted that the expression $I=aT^n$ does not give a definite temperature at which the conduction begins; if, therefore, it accurately represents the current, a more sensitive apparatus would detect this at a lower temperature than in our experiments.

Experiments with Zinc Bromide.

As far as our experiments have gone at present, the results with zinc bromide are similar to those obtained with the iodide. The current, however, alters much more rapidly with the time.

In the preceding experiments, the zinc chloride was not specially dried. When heated, it is known that HCl is formed. In order to see whether this would account for the leak from the chloride, the apparatus was gradually heated and dry HCl gas admitted. An appreciable current was noticed at a temperature of 200°, and this did not depend on the direction of the field. When air was drawn over the chloride and bubbled through $AgNO_3$ a precipitate was at once formed; but if the nitrate was replaced by water and the bubbling took place for half an hour, no zinc could be found in the solution. When the bubbling took place through starch and KI solution, no coloration was noticed, so that no free chlorine was present. Next the air was pulled through an insulated test-tube packed with cotton-wool and coated on the outside with tinfoil. This was connected to the electrometer quadrants. When the foil was charged negatively, there was a further charging up when the air was drawn through; when charged positively, a greatly increased leak. There would therefore appear to be an excess of negative electricity drawn off from $ZnCl_2$ at this temperature (360°).

Some experiments were next carried out with the object of discovering the nature of the ions concerned in the conduction. For this end their velocity was determined at a temperature of 360°.

A method previously used by McClelland[*] to measure the velocities of ions from arcs was used. The apparatus is shown in fig. 9. The substance is heated at R, and air is drawn over it from left to right. The tube and electrode P are earthed and a field is put on at Q sufficiently large to remove all the ions reaching it; this number being proportional to the rate of deflexion of the electrometer-needle which is connected to Q. A field is now put on at P, and so some ions are removed, the reduced rate of deflexion of the electrometer connected with Q measuring this amount. Then, if ρ is the proportion removed by the field at P; A the difference of potential between P and the tube; b=radius of tube, a=radius of P; and t=time taken by the current of air to pass along P, the velocity of the ions u is given by*

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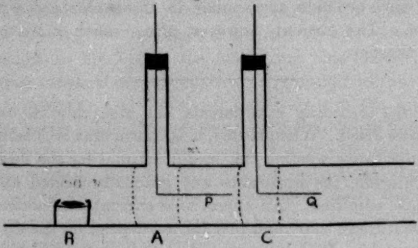
[*] Proc. Camb. Phil. Soc. x. p. 241 (1899).

Fig. 9.



$$u=\frac{\rho(b^2-a^2)\log_e(b/a)}{2At}.$$

To ensure insulation, the tubes carrying P and Q were made several inches long, and to facilitate the fixing of the electrodes the sections of the tube A, C, could be screwed off the other parts. The electrodes were made sometimes of platinum, sometimes of aluminium, the outer tube was iron.

The velocities for the positive ions under a potential gradient of one volt per centimetre were as follows:—

ions from $ZnCl_2$ =·0062,
ions from $ZnBr_2$=·0059,
ion from ZnI_2 =·0057;

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* 'Conduction in Gases,' J. J. Thomson, p. 48.

i. e., the velocities were the same in all cases within the limits of experimental error. The relative velocities are probaoly much more accurate than the absolute velocities. In order to be able to interpret definitely the results of the measurements, the field at Q should be sufficient to produce the saturation-current. Owing to the smallness of the velocity, we could not fulfil this condition as well as would be desirable. The results for the negative ions from zinc chloride were very irregular. The mean of a large number of results, which differed considerably from each other, gave a velocity =·02 cm. per sec. under one volt per centimetre.

During some of the experiments, we had occasion to draw air over the substances into a large bottle containing water. We usually, but not always, noticed in these circumstances an exceedingly dense fog, which was produced without expansion, and which hung in the bottle for half an hour before dispersion*. The bottle was several yards away from the apparatus. The air was drawn through a tube much narrower than that shown in fig. 9, and a strong field created between it and an electrode along the centre. We were unable to stop by this means the formation of fog. Bubbling through several inches depth of various liquids also failed to stop it, as did also a tight plug of cotton-wool when placed near the water-bottle. When the plug was placed near the apparatus, the fog was greatly reduced. The air could be allowed to remain in the connecting tubing for several minutes before admission to the water-bottle without greatly impairing its efficiency as a fog producer. The introduction of iodine vapour into the bottle, brought about by sucking air over boiling iodine, caused no fog.

These fog experiments seem to show that the ions are bodies of considerable size, while those on their velocity would, if trustworthy, show that they are similar whether they come from the chloride, iodide, or bromide, and would thus preclude the halogens themselves. Their source would thus be narrowed down to either the zinc or, perhaps, the oxygen of the air. The former of these views would correspond to the ejection of α rays from radioactive substances, the fact of the chemical dissociation of the compound, bromide, iodide, or chloride into its elements rendering this ejection possible. The latter of the two explanations would proceed on the lines marked by Prof. J. J. Thomson * to explain the power of oxygen to change a negative into a positive electrification, in the gas near an incandescent solid. Experiments to test these views are now in progress.

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* Similar effects have been noticed by Townsend, Wilson, and Meissner (see 'Conduction in Gases,' p. 337) for gases from electrolysis, &c.

The energy required to ionize a gram-molecule of zinc-chloride vapour may be calculated, if we take the curve in fig. 4 and assume that the negative ions are ejected from the $ZnCl_2$ vapour. This quantity, q, in calories, is given by the formula †

$$\frac{q}{2}\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right)=\log_e\frac{n_1^{\,2}}{n_2^{\,2}},$$

n_1 and n_2 being the small currents, in arbitrary units, at absolute temperatures θ_1 and θ_2, where $\theta_1-\theta_2$ is only a small interval of temperature. For a mean temperature of 335° C. the result is 165,000 calories. Taking the charges on the ions to be the same as in electrolysis, an estimate may be formed of the potential-difference, V volts, through which the charge must fall freely in order to gain energy enough to ionize a molecule. Since it requires 96,000 coulombs to electrolyse a gram-molecule of a substance, the energy spent $=96{,}000\times10^7$ V ergs, and this is the equivalent of 165,000 calories. Hence

$$96{,}000\times10^7\,\mathrm{V}=165{,}000\times4{\cdot}2\times10^7$$

and $$\mathrm{V}=7 \text{ volts},$$

in good agreement with the estimates of Townsend, H. A. Wilson, J. Stark, and Langevin.

Sir John Cass Technical Institute,
Jewry Street, E.C.

Discussion.

Mr. **Skinner** expressed his interest in the paper and pointed out that the chief leak did not begin until the iodine was completely vaporised. He asked if the ions were produced in the mixture or in the gas. Had the authors tried the effect of screening the mixture from the electroscope?

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* 'Conduction in Gases,' p. 180.

† H. A. Wilson, Phil. Trans. cxcvii. A. p. 429 (1901).

Dr. C. CHREE remarked on the fact that the authors employed formulæ of the type $I=at^n$, t representing absolute temperature, under circumstances in which n was a large positive number, in one case as big as 100. To fit such a formula I must be practically either infinitely small or infinitely large, except over a very limited range of temperature. He should like some further details as to the method of arriving at formulæ of a type so remarkable physically, and as to the range of temperature wherein they were supposed to hold.

Dr. WILLOWS, replying to Mr. Skinner, said that the effect was still produced when the mixture was covered with a metal plate, but they had not tried screening with mica. In reply to Dr. Chree, he said that their formula held over a range of 100° C., and that Dr. H. A. Wilson had arrived at a similar formula which applied over a range from 200° C. to 1000° C.

XXX. *The Magnetization of Iron in Bulk.* *By* W. M. THORNTON, *D.Sc.* (*Vict.*). (Communicated by Sir OLIVER LODGE, *F.R.S.*) *

§ 1. A method of testing the permeability of iron in bulk, with experiments on large rings.

§ 2. The influence of core-currents upon the growth of the magnetizing current. Comparison of coils with solid and laminated cores.

§ 3. Calculation of subsidence or growth of current in coil with large solid core of circular section. Evidence of sudden molecular movement as the iron approaches saturation.

THE permeability of specimens of iron has long been known to depend upon their size, shape, and previous history. The usual method of determining magnetic properties by tests on small samples cut or specially cast from a larger mass cannot be entirely relied on to give the same values that would be obtained if it were possible to apply tests to the material in bulk. It is especially necessary, in order to have uniform results, even from a batch of samples of the same metal, that they should have had the same annealing or have been subjected to the same process and time of cooling. If they could be conveniently compared, a 20 per cent. difference between the behaviour of a small specimen and of the actual magnet frame would be found to be not uncommon, and the permissible variation between castings of the same group is sometimes specified of this magnitude.

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* Read June 24, 1904.

One method of testing has been developed* free from most objections but not entirely free, because the amount of metal examined is a very small portion of the whole. It seems, in fact, that accuracy cannot be guaranteed unless the specimens are large enough to be considered as "in bulk," and are tested as a whole.

The two methods at present in use with the smaller specimens are the ballistic and the magnetometric. The latter is only suitable for long thin rods, the former for cores which, either by reason of the low inductance or high resistance of their windings, have small time-constants, so that the total inductive change following any alteration of the exciting current is practically instantaneous.

Some years ago the writer had occasion to measure the slow magnetic movements which occur on making or short-circuiting the exciting current of large dynamo-electric machines†. To do this a recording millivoltmeter was connected to an exploring coil surrounding the core. Any change in magnetization was then registered by the induced voltage in this secondary circuit, and the absolute accuracy of the result was made to depend only on the standardization of a voltage and the speed of the record. For let a voltage e be induced during time t in a coil with n turns. The magnetic movement in the core which has produced this is then

$$N=\int\frac{e}{n}\,.\,dt,$$

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* Drysdale, Journal Inst. Elect. Engrs. vol. xxxi. p. 283.

† *Vide* 'The Electrical Engineer,' vol. xxix. p. 523 ff.

And if $\bar{e}$ be the mean voltage during the total time T of change

$$N=\frac{\bar{e}T}{n},$$

corresponding, where a is the cross-sectional area of the core, to a change in flux density of

$$B=\frac{\bar{e}T}{na}.$$

The immediate object of these tests was to measure directly the leakage coefficients of large machines. Incidentally the permeabilities of various parts of the magnetic circuits were also determined. It was then suggested by Mr. C. C. Hawkins that since these were values for iron in bulk, the same method might be used for testing large rings specially wound for the purpose.

The lower limit to the size of ring depends only on the point at which the iron can be considered to be in bulk. So far as this alone is concerned a mass of anything over 100 lb. would probably be sufficient, in the shape of a yoke-ring of the usual proportions; and it is evident that to avoid expense the smallest size should be used, provided that the method of testing to be employed can be applied with accuracy and convenience.

In the experiments previously referred to the magnet core was of cast steel 15 inches diameter, the whole frame weighing several tons. This circuit had an inductance of about 50 henries and a resistance of 10 ohms, the time-constant therefore being 5 seconds.

The method of testing adopted in this case was to make the exciting circuit, allow the current to become steady, and then to short-circuit the windings, recording photographically the voltage induced in the exploring coil. It was found that the methods were uniformly accurate, and in applying the test to sample rings a time-constant of this magnitude was aimed at.

Two rings were kindly prepared by Messrs. W. H. Allen, Son & Co., one of cast iron the other of cast steel *, both being the yoke-rings of standard machines with the brackets

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* *Vide* Journal Inst. of Elect. Engrs. vol. xxxiii. p. 539.

and facings turned off. The former was wound with 1000, the latter with 777 turns of copper wire to carry 50 amperes for a few seconds without undue heating, having respectively 2·9 and 1·9 ohms resistance cold. In each case, however, the attempt to use the "make, short-circuit" method, recording photographically, had to be abandoned on account of the difficulty experienced in obtaining satisfactory records. The motion of the spot of light was so rapid that the photograph was too faint for accurate measurement. Another method had then to be found, and recourse was eventually had to the simple device of altering the current gradually with several brief pauses, recording as before the induced voltage in the secondary or exploring coil.

A vertical wooden pipe 10 inches square by 6 feet long was filled with dilute sulphuric acid and fitted with lead end-plates to which the voltage required to give the desired current was applied. A movable lead electrode was suspended from a pulley so that it could be readily raised or lowered. The movable and lower electrodes were connected through a standardized Weston ammeter to the primary winding on the ring. With the movable plate at the bottom the winding was short-circuited, when at the top it received the full exciting voltage.

The galvanometer used in the secondary circuit was by Pitkin, of the moving-coil type with the lower suspension made bifilar. It was just aperiodic. The recording drum, 5 inches diameter by 10 inches long, was contained in a box provided with an adjustable shutter, and was at first driven by a clock within the drum. It made one revolution a minute and ran for ten minutes, the speed slowly decreasing from the start.

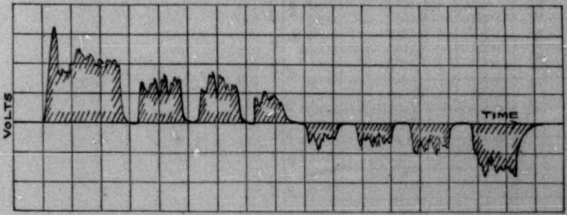
To avoid this drop of speed a small spindle was screwed into the end of the axle of the drum, and, passing through the end of the box, was fitted with a pulley so that by using a gramophone-clock as motor a suitable constant speed could be obtained for the record.

In applying the magnetizing current the pauses, which were about every ten amperes, were always long enough for the spot of light to remain for a moment at rest so that the curves as recorded were as follows (fig. 1).

The upper areas represent the changes of magnetic flux with increasing currents, the lower with decreasing. The area A of each curve was measured with a planimeter.

FIG. 1.—Typical Record for Half a Hysteresis Loop.

(Each area denotes the change of B for an observed change of H.)



Knowing, then, the voltage *e* for one centimetre deflexion and the seconds *s* the periphery of the drum took to move one centimetre, the area multiplied by the product *es* gave the total lines threading the exploring coil. The corresponding change in flux density

$$B=\frac{Aes\times 10^8}{na}=kA \text{ say};$$

A having in succession the values recorded. The flux density at any current is then *k* times the sum of the areas up to that point. The value of the magnetizing current at each pause was noted by the observer. The areas can be readily checked by allowing the current to change from zero to the maximum and back to zero without pause, when it is found that the agreement between the areas so found and the separate steps is within 1 per cent.

The difference between the areas above and below the zero line represents the residual magnetism.

The results of the tests on the two rings are given in Tables I. and II. and in fig. 2.

The observations are reduced in a manner similar to the "step by step" method of ballistic testing. The labour could be greatly reduced by the use of a quantometer [*], if that most

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[*] *Vide* 'The Electrician,' vol. l. p. 383.

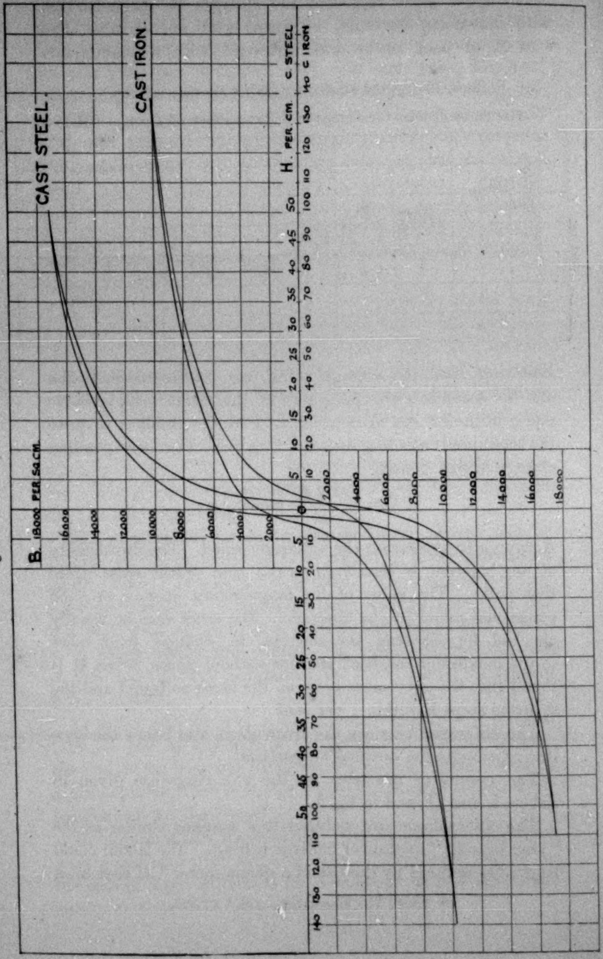

Fig. 2.—Magnetization Curves of Large Rings.

TABLE I.—Cast Steel Ring.

H.	B inc.	μ.	B dec.	μ.	w ergs per c.c.
0	−5000	$-\infty$	5000	$+\infty$	
·628	−4000	−6370	6500	10340	
1·25	0	0	7000	5510	
2·51	+5000	1990	8000	3180	1200
5·0	7900	1580	9600	1920	2490
7·54	9500	1260	11000	1460	3300
10·05	11000	1095	12000	1194	4300
12·57	12000	955	13000	1034	4900
17·6	13600	774	14200	807·5	5900
23·6	14500	614	15000	635	6630
25·1	15000	598	15300	610	7000
37·8	16300	432	16500	436	7800
50·4	17100	340	17150	341	8500
62·8	17300	275	17350	276	
75·2	17600	233	17600	233	
109·3	18100	165·5	18100	165·5	
125·7	18500	147·3	18500	147·3	
201	19600	97·6	19600	97·6	

Volume of Cast Steel Ring *14520* cub. centim.

TABLE II.—Cast Iron Ring.

H.	B increasing.	μ.	B decreasing.	μ.	w ergs per c.c.
0	−2900	∞	2900	∞	
2·5	−1800	720	3000	1353	
3·78	− 950	250	3400	900	
4·4	0	0	4000	796	
6·28	1000	159	4250	678	380
12·67	3000	239	5000	398	2800
25·14	5150	203	6100	243	5200
37·7	6500	172·5	7000	185·7	7700
50·28	7200	143	7550	150	8900
62·8	7800	124	8100	129	10,200
73·4	8300	113	8500	116	11,100
88·0	8750	100	9000	102·2	12,200
100·5	9150	91	9300	92·5	13,000
125·7	9800	78	9950	79	15,000
138·0	10100	73·5	10200	74	15,250
163·2	10600	65	10650	65·25	
188·5	11000	58·6	11000	58·5	
251·4	11900	47·3	11850	47·25	

Volume of Cast Iron Ring *32580* cub. centim.

useful instrument were generally available. Dr. Beattie having kindly lent his, the tests were repeated, obtaining agreement within 2 per cent. The method of testing was now as originally designed. The ring windings were connected to the generator through reversing and spring-break change-over switches *. The operations were then :—

1. Make the circuit and allow the current to reach a steady state.
2. Short-circuit the windings and allow the current to fall to zero.
3. Reverse the coil-connexions and repeat the process.

This was done several times for each current, the quantometer readings being eventually reduced as in ordinary ballistic testing. The instrument is so easily worked, and its readings are so constant when there is a strong field in its air-gap, that if it were better known it would no doubt be used for magnetic testing on a large scale in preference to any other.

For measuring large changes of magnetization, these methods, the only two which are not ballistic, represent extremes. In the first with a recording voltmeter the best results are obtained when the current is varied as slowly and uniformly as possible; in the second with the quantometer, when the change is as rapid as the inductance of the winding and the tailing-off effect of the core currents will permit. The latter do not affect the accuracy of either method.

§ 2. The rings exhibit in a very marked manner the effects of eddy currents and of change of permeability in the core on the growth of the magnetizing current. Curves I_1, I_2, fig. 3, were taken, using an oscillograph as a dead-beat ammeter, and recording the deflexions photographically, as previously described, but with the drum making one revolution a second. They are remarkable on account of the pause in the rate of growth observed in each, and for the difference between the two curves. I_1 corresponds to a molecular change from $+B_r$, the residual magnetism, to $+B_m$, the maximum induction-density ; I_2 from $+B_r$ to $-B_m$. The magnetic

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* 'The Electrician,' vol. lii. p. 479, "On a master switch for Central Stations."

movement is therefore much greater in the second case, and the current is accordingly more retarded.

In order to study the reaction of the core currents on the magnetizing circuit, it is necessary to compare the curves of

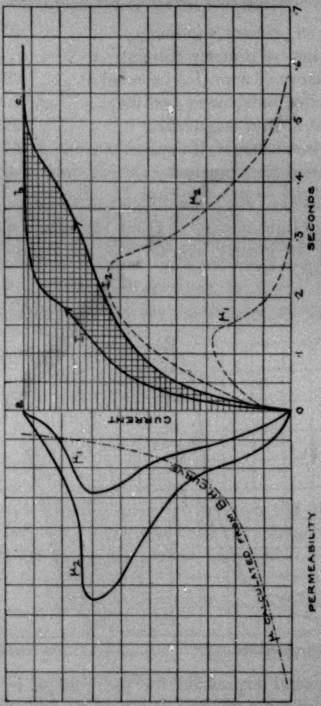


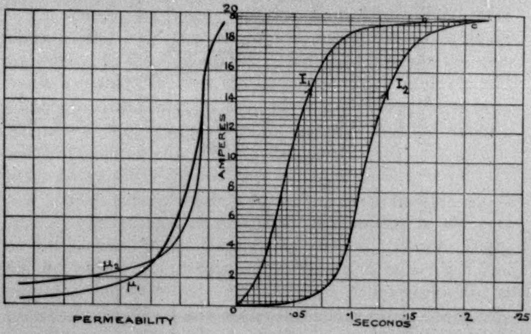
FIG. 3.—Rise of Current Magnetizing Cast Steel Ring.

fig. 3 with those obtained when the core is laminated. For this purpose a 20-kilowatt transformer was used, having about the same weight of iron as the rings. The high-tension winding was excited from storage-cells, and on closing the switch the current rose as shown in fig. 4. Here the curve I_1 indicates as before the rise from zero after the core has been already magnetized in the same direction; I_2 when the magnetization is being reversed. The following are the details of construction affecting the curve:—

Turns on primary winding	97
Resistance of primary winding . . .	·75 ohm.
Reluctance of core (l/A calculated) . .	$\cdot 15/\mu$.
Calculated inductance $\mu=100$ . . .	L=·31 henry.
Weight of iron magnetized	510 lb.
Maximum value of H applied to core .	67
Calculated time constant	·41 second.

The curves of rise were difficult to obtain photographically, the spot of light moving at about 140 cms. a second. It was necessary to greatly overdevelop the extra-rapid bromide paper used.

FIG. 4.—Rise of Current in Coil with Laminated Core.



The general expression for the voltage in an inductive circuit to which a steady voltage e is applied is

$$e=ri+\frac{dN}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here N is the total magnetic change and equals Li. The inductance L is, in all cases where iron is used, a variable. Hence

$$e=ri+\mathrm{L}\frac{di}{dt}+i\frac{d\mathrm{L}}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and since L is proportional to μ we may write

$$e=ri+k\mu\frac{di}{dt}+ki\frac{d\mu}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$k\mu\frac{di}{dt}+i\left(r+k\frac{d\mu}{dt}\right)=e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In curve I_2, fig. 4, the current pauses for some time after switching on, $d\mu/dt$ must therefore be exceedingly large since both i and di/dt are small. If desired it can be found, for all the other quantities in (4) are determinable.

In curve I_1 there is scarcely any initial pause, which is to be expected, since the core starts well on the way to saturation.

It is of some practical importance to notice that the horizontally shaded areas *abo*, *aco* represent the magnetism which the current induces whilst rising.

Equation (1) may be written

$$d\mathrm{N}=(e-ri)dt;$$

so that

$$\frac{\mathrm{N}}{r}=\int(i_0-i)dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where i_0 is the final steady current.

The value (i_0-i) is the intercept between the curve and the horizontal tangent. The right-hand side therefore is the area A between the curve, the vertical axis, and this tangent. Thus

$$\mathrm{N}=k'\mathrm{A}\times10^8=k\mathrm{A} \text{ say}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

k' being the volt seconds per sq. cm. of the record, known from the scale of the instrument. This is independent of whether the core is solid or laminated.

If A_1, A_2 are the respective areas *abo*, *aco*, c the cross-sectional area of the core, and t the turns on the winding, then the change of flux density corresponding to an area is

$$\mathrm{B}=\frac{\mathrm{N}}{tc}=\frac{k\mathrm{A}}{tc}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Denoting by B_1, B_2, the flux densities corresponding to A_1, A_2, the following relations are evident :—

$$\frac{B_1+B_2}{2}=B_m, \text{ the maximum induction,}$$

$$\frac{B_2-B_1}{2}=B_r, \text{ the residual magnetism.}$$

Thus
$$\frac{B_1+B_2}{B_2-B_1}=\frac{B_m}{B_r}. \quad \ldots \ldots \quad (8)$$

Also
$$A_1+A_2=B_m \cdot \frac{2tc}{k} \quad \ldots \ldots \quad (9)$$

$$A_2-A_1=B_r \frac{2tc}{k}. \quad \ldots \ldots \quad (10)$$

The sum of the horizontally shaded areas therefore represents the maximum induction density reached, their difference, that is the vertically shaded area forming the loop *bco*, the residual magnetism. It is clear that if there were no retentivity the curves would be identical.

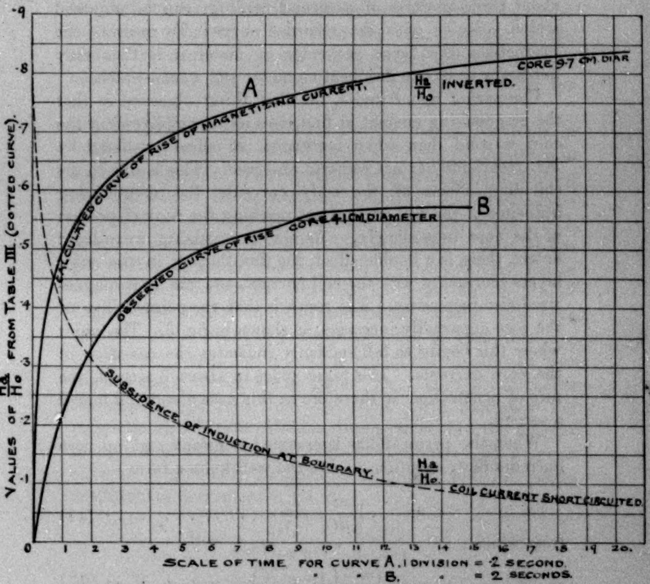
These relations can be illustrated from figs. 3 & 4 as follows :—

	Cast Steel Ring.	Transformer.
Area A_2 (by planimeter)	6 sq. in.	10·75 sq. in.
„ A_1	3·1 „	4·67 „
(A_2+A_1)	9·1 „	15·42 „
(A_2-A_1)	2·9 „	6·08 „
Ratio B_m/B_r	3·1	2·5
Sectional area C	75·1 sq. cm.	420 sq. cm.
Turns on winding	777	97
Resistance in circuit ...	1·9 ohms.	5 ohms.
Scale of record	10 amp.=1 inch. ·125 sec.=1 inch.	4 amp.=1 inch. ·05 sec.=1 inch.
k	$2·3\times10^8$ per sq. in.	$1·0\times10^8$ per sq. in.
B_m (from eqn. 9)	17,900 per sq. cm.	18,900 per sq. cm.
B_r („ „ 10)	5,770 „	7,560 „
H max.	225 „	67

Comparing the values of B_m and B_r obtained in this way with those of § 1 for the ring, it is seen that there is a fairly close agreement for the maximum induction, but about 15 per cent. difference in the residual. Of the two methods, the latter is the more likely to be correct in determining the residual flux.

These relations suggest a very simple method of measuring the induction through the windings of large electromagnets, such as the exciting circuits of large dynamo-electric machines, which may be in use but has not come to the writer's notice. It is to record the rise of current when the magnetization is suddenly switched on, and to measure the area corresponding to *abo* in fig. 3. The curve will in general shape resemble curve A in fig. 5; and since most large exciting currents take from 5 to 60 seconds to reach their full value, the speed at which the record must be made is not excessive. Proceeding in this way, the magnetic leakage between the

Fig. 5.—Calculated and Observed Coil Currents.



windings and the armature can be measured by ordinary test-room instruments.

A still further simplification suggests itself in the case of machines where the time constant is above five seconds. Instead of *recording* the current rise, the times, taken by a stop-watch, for the current to reach given values in turn are plotted. The first rush cannot be measured, but after about three-quarters of a second readings can be obtained which coincide with the recorded curve. To measure the magnetism of a large generator or motor it is then only necessary to have a deadbeat ammeter and a stop-watch.

Comparing figs. 3 and 4, the first thing observed is that the magnetizing current at first rises more rapidly when the core is solid than when laminated, an effect predicted by Mr. Heaviside but not hitherto observed. This is due to the shielding effects of the eddy currents, the magnetizing influence of the coil current not reaching the central portion of the core immediately. As the magnetizing wave-front enters, there is a double effect, the flux density in the outer layers increasing with the coil current, and the area magnetized also increasing. The result is that the permeability of the core apparently increases, as shown in fig. 3. The point where this begins to fall suddenly indicates the cessation of the core currents. This takes place in about ·3 second, the rate of penetration is therefore in this case about 3·3 inches a second.

When the permeability is assumed constant and no core currents flow, equation 2 takes the well-known form

$$\mathrm{L}\frac{di}{dt}+ri=e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

from which

$$i=\frac{e}{r}\left(1-\epsilon^{-\frac{rt}{\mathrm{L}}}\right). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If this curve be drawn with a mean value of L suitable to the circuit, the corresponding curve with core currents allowed to flow appears with a superposed wave. The solution of (4) is impossible unless μ can be expressed as a function of time; and even if this were known, there would be a further complication in expressing the magnetic force as a function of the radius. The only course open is to find experimentally the curve of rise, and from it to deduce the changes of μ. It is seen from figs. 3 and 4 that the relation of μ to t does not lend itself to simple treatment.

The effect of the core currents may be summarized thus. When the core is of such dimensions or shape that the magnetizing wave from the circumference reaches the centre in a time equal to or less than the time-constant of the circuit, the rise is wavy; when it is greater the core currents merely prolong the time of growth without distorting the curve of rise.

§ 3. The distribution and reaction of induced currents in solid cores present the most difficult and complex problems in applied magnetism. The regulation of alternators, the heating of magnet cores due to armature reaction, the sluggishness of response exhibited by machines with solid cores which depend upon series excitation, are examples. Most of the practical cases are beyond the reach of exact analysis; but in order to see whether Mr. Heaviside's solution for the rise of magnetizing current in a coil surrounding a solid core gave any clue to the observed pause in the rate of rise when the core is very large, the following numerical illustration was worked out. The curves he gave * for a core 1 cm. radius are smooth and free from distortion. He expresses the magnetic distribution within a circular solid core, the magnetizing coil of which has just been short-circuited, by a series of terms of gradually diminishing amplitude and increasing coefficient of decrement †, viz.:

$$\mathrm{H}_r = \frac{2\mathrm{H}_0}{na(1+s^2n^2)} \cdot \frac{\mathrm{J}_0(nr)}{\mathrm{J}_1(na)} \cdot e^{\mathrm{D}t} ;$$

where H_r is the magnetic intensity at a radius r, H_0 the initial value at the boundary. The radius of the core is a; n the successive roots of the equation

$$\mathrm{J}_0(na) = sn\mathrm{J}_1(na),$$

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* Electrical Papers, vol. i. p. 398.

† *Loc. cit.* p. 397.

J_0 and J_1 being the usual Bessel functions. The coefficient $s=\frac{L\rho}{2\pi\mu a R}$, in which L is the inductance and R the resistance of the coil winding; so that

$$s=\frac{2A\rho T^2}{laR};$$

where A is the cross-sectional area of the core, l its mean length, ρ its specific resistance, μ its permeability, and T the turns on the winding.

D the damping coefficient $=\frac{-\rho n^2}{4\pi\mu}$.

In the case of the cast steel ring $s=\cdot6055$, assuming as a first approximation that the core is circular.

For $a=4\cdot87$ cms. $l=193$ cms.

$\rho=12{,}000$. $T=777$.

$r=1\cdot9\times10^9$.

Thus

$$\frac{H_a}{H_0}=\frac{\cdot2486}{1+\cdot3675n^2}\cdot e^{Dt}.$$

The n's are the roots of $\frac{J_0(z)}{J_1(z)}=\cdot1243z$.

This can be most readily solved by inspection and trial from tables of J_0 and J_1. The roots so found are given in the following Table, which also gives the working of the different terms. It will be seen that the sum of the first twenty terms of the amplitude is only ·9 instead of nearly unity. The reason for this is that in the expression for s the conductivity enters as an unknown figure. The best values for cast steel containing silicon give an average of 12,000 for ρ, but the composition of the steel has so great an influence that any numerical estimate is uncertain. If the value chosen were too large the roots would also be too large, and the amplitudes therefore diminished. The ratio H_0/H_a is plotted in fig. 5; and since the values of H_a and H_0 are both proportional to the coil-current, this ratio shows the theoretical

TABLE III.

					$t=$	·2 sec.		·5 sec.		1·0 sec.		2 sec.		3 sec.		4 sec.	
$z=na$.	n.	n^2.	$(1+\cdot3675n^2)$.	A.	D.	ϵ^{Dt}.	$A\epsilon^{Dt}$.	ϵ^{Dt}.	$A\epsilon^{Dt}$.	ϵ^{Dt}.	$A\epsilon^{Dt}$.	ϵ^{Dt}.	$A\epsilon^{Dt}$.	ϵ^{Dt}.	$A\epsilon^{Dt}$.	ϵ^{Dt}.	$A\epsilon^{Dt}$.
2·13	·437	·1915	1·07	·2325	−·333	1·068	·218	1·18	·197	1·39	·1673	1·94	·12	2·717	·085	3·79	·061
4·94	1·014	1·025	1·377	·1805	−1·78	1·42	·127	2·43	·074	5·94	·0304	34·9	·005	206	·0008		
7·85	1·613	2·61	1·96	·1270	−4·53	2·45	·0517	96	·0134	89	·0014						
10·83	2·225	4·96	2·82	·0885	−8·62	5·6	·0156	74	·00119								
13·86	2·845	8·12	3·98	·0625	−14·1	16·75	·0037										
16·9	3·47	12·06	5·44	·0457	−20·95												
Ratio H_a/H_o =				·7367	..	..	·4162	..	·2856	..	·1991	..	·125	..	·0858	..	·061

The sum of the amplitudes of the first 15 terms = ·87, of first 20 terms = ·9.

change of the latter with time when the permeability is constant. The decay of current in a short-circuited coil is in this case the same as the growth of current when first switching on, the curve as drawn being merely inverted.

The curve B of fig. 5 is the rise of current in a coil surrounding a solid core 41 cms. diameter. The small dip observed beyond the bend must be attributed to a rapid molecular movement. The change in the curvature is sudden, and is always obtained even with repeated magnetizations in the same direction. The above analysis shows that no such dip would be obtained if the permeability were constant. It can then only be attributed to a sudden increase in the number of molecular magnets brought into alignment. The observed effect resembles the phenomenon of crystallization in a supersaturated solution, starting at the circumference. It is in fact the direct observation, only possible when dealing with large masses of iron, of the sudden slip noticed in the magnets of Ewing's hysteresis model when the field approaches saturation.

The growth and subsidence of magnetic induction in rectangular cores cannot well be calculated except on the assumption that the core currents follow rectangular paths. This is approximately true when the thickness is small compared with the length of the section, but not when it is comparable to it.

Photographs have been taken by Professor Hele-Shaw of the flow of liquid between the minor and major semiaxes of rectangles. These show clearly the transition from a circular path at the centre to an approximately rectangular path at the outer boundary. At intermediate points the curves are too complex for the action of the core currents to be calculated where the thickness is not small compared with the length, on account of the difficulty of finding an expression for the instantaneous magnetic force in terms of the distance from the centre.

Discussion.

Prof. S. P. Thompson expressed his interest in the paper, and referring to the curves showing the rise of the magnetizing current, said he thought the explanation given by the author was not the only possible one. He was not inclined to think that the effect was due to internal eddy currents in the core.

Mr. W. H. **Pretty** said that the question of magnetization tests of iron and steel in bulk was a very important one. The curves shown by the author suggest the possibility of detecting cavities in steel castings, as such castings would doubtless give curves differing from those representing a truly solid block of the same material cast at the same time and subjected to the same after-treatment. It would be interesting to have a series of tests on the same block of steel in which each process in the manufacture is represented, *i. e.* as cast, after annealing, and after the surface-scale or skin has been removed to various depths; also after the block has been cut up into slabs or laminæ and put together again. Referring to the dip in the curves of rise of magnetization observed with large cores, Mr. Pretty said it was not unlikely due to internal mechanical stress and the accompanying strain, and that as it became possible to obtain steel more and more homogeneous such internal molecular changes would become more apparent.

Dr. **Thornton**, in reply to Mr. Pretty, said that he was now experimenting upon the effect of cutting up a large core into laminæ and then building up the core again.

XXXI. *Exhibition of Admiral Fleuriais's Gyroscopic Horizon.* *By* M. E. J. GHEURY *.

IF, in the vertical plane passing through a celestial body and the eye of an observer, a pendulum is made to oscillate in such a way that the extreme positions reached by its bob in swinging are about on the horizontal passing through the eye of the observer, and if the latter observes the altitude of the body with regard to the centre of the bob at three extreme consecutive positions, he can calculate the altitude with regard to the true horizon, by taking the mean of his observations and applying a correction depending upon the rate of decrease of the amplitude of the oscillations of the pendulum.

This method, however, cannot be used at sea, as the displacement of the support of the pendulum, caused by the motion of the vessel, would introduce forced vibrations which would vitiate the observations. The only way to overcome this difficulty is to use a pendulum with a time of oscillation so great as not to be influenced by the motion of the vessel. This would lead to a pendulum of prohibitive length.

In Admiral Fleuriais's Gyroscopic Horizon, recently improved by Messrs. Ponthus and Therrode, of Paris, the essential feature is a gyrostat, revolving, in a vacuum, with a speed of about a hundred revolutions per second, in an aluminium case fixed on a sextant. During its motion of precession the successive positions of the axis in a given plane are nearer and nearer the vertical, and are, therefore, similar to the successive positions of a pendulum, with the difference that the pendulum is inverted, and that its time of complete oscillation is equal to that taken for a cycle of precession, that is, in the actual instrument, about two minutes.

The gyrostat may therefore be assimilated to a pendulum of very long time of oscillation and of very small size, and can therefore be used for taking altitudes at sea as explained above.

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* June 10, 1904.

To enable these observations to be made with ease, the cylindrical case in which the gyrostat revolves has two windows facing each other. Through these the reflected light of the sun or, at night, of a small glow-lamp is sent through the case and received in the observing telescope. In this beam of light, fixed on the top of the gyrostat, is a lens having at its focus a grating of horizontal transparent lines on an opaque background.

At every revolution of the gyrostat the optical axis of this arrangement is in the direction of the beam passing through the case, and an image of the grating is seen in the telescope. As the gyrostat revolves rapidly the successive images form a perfectly continuous grating of luminous lines on a dark background, quite steady and well-defined. As precession goes on, this grating is shifting slowly upwards and downwards across the field of vision. The image of the observed body being brought down upon that of the grating in the usual way, its apparent displacement with regard to the lines of the grating is observed and its position noted when its motion is reversing three times in succession—this corresponding to extreme positions of the pendulum—and the corresponding time noted.

From these data the distance of the image of the body from the central line of the grating when the axis of rotation is truly vertical is calculated and reduced in angular measures by multiplying it by an instrumental factor.

An instrumental error due to the fact that the central line of the grating is not exactly the trace of the true horizontal plane through the eye, even when the axis of the gyrostat is exactly vertical, is applied to this angle, together with a correction for the influence of the rotation of the earth upon the precession of the gyrostat; the corrected angle is then added or subtracted—as the case may be—from the reading on the limb of the sextant, corrected for the usual instrumental errors. This altitude is then corrected as usual for refraction, parallax, and semi-diameter, to obtain the true altitude.

The corrections and calculations are easily and rapidly made by tables and curves. The gyrostat is set in rotation by ten or twelve strokes of a portable vacuum-pump, two more strokes of which give a vacuum of 70 to 72 cms. of mercury, and the instrument is then available for observations during about half an hour. The total weight of sextant and gyrostat is 4½ lbs.; this is not so great as to be an objection, the time required for an observation being but a couple of minutes.

The gyroscopic horizon has been submitted, at the French Hydrographic Service, to tests extending over two years, which gave evidence that the instrument is likely to be of great use at sea, and, when skilfully used, yields remarkable results.

Fig. 1.

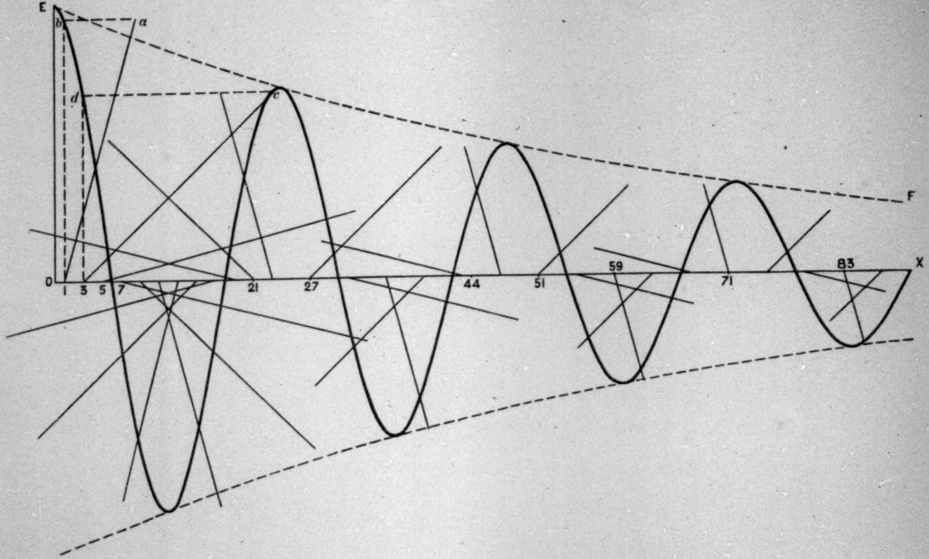
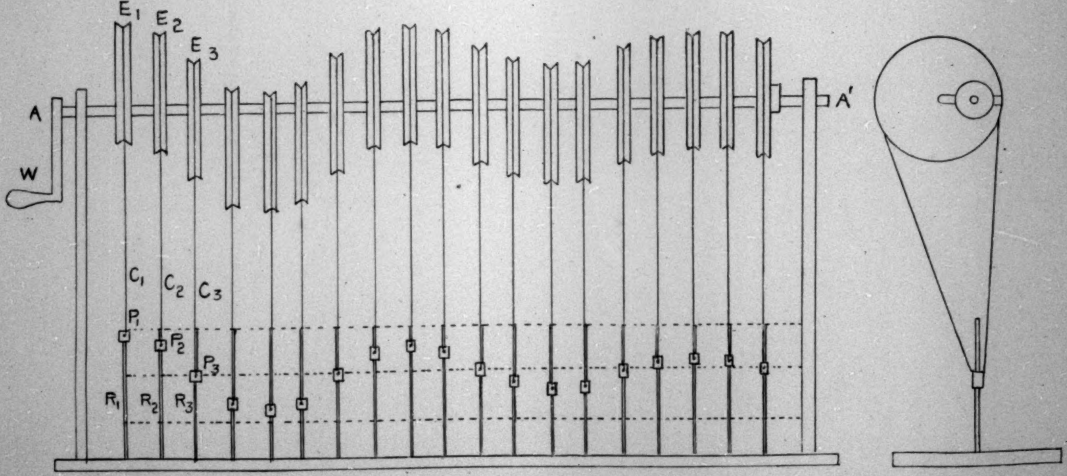

E
b
a
d
c
F
O
1
3
5
7
21
27
44
51
59
71
83
X

FIG. 2.


E₁
E₂
E₃
A
W
C₁
C₂
C₃
P₁
P₂
P₃
R₁
R₂
R₃
A′

Elevation.

Section.

XXXII. *An Interference Apparatus for the Calibration of Extensometers.* By JOHN MORROW, *M.Sc. (Vict.), M.Eng. (Liverpool), and* ERNEST L. WATKIN, *M.A. (Cantab.), Lecturers in University College, Bristol* *.

Introduction.

BEING engaged in a research which requires the use of extensometers of considerable sensitiveness, the authors have found some difficulty in determining the constants of the instruments with sufficient accuracy; they also considered it advisable to find out how far the extension readings were reliable.

For these purposes it was decided to calibrate the instruments by interference methods; and accordingly, after some preliminary trials, the apparatus described in this paper was designed. The following points were kept in view as being essential:—(1) That the calibrator should be as far as possible self-contained and easily made ready for use; (2) that any displacement accidental or otherwise should affect both the interference apparatus and that being calibrated simultaneously and equally; (3) that it should be unaffected by external disturbances; and (4) that there should be no error possible in the determination of its own displacement.

In testing strain-meters by comparison with reading-microscopes or comparators there is great difficulty, owing to the unavoidable backlash of the screw, in ascertaining whether the instrument is itself free from a similar defect. The apparatus here described is not open to this objection.

Description of the Apparatus.

In figures 1 & 2, *ff* is a rigid cast-iron frame into the top of which is screwed a steel tube *a*. The lower portion forms guides *gg*, between which slides a gun-metal sleeve *c*. The steel tube *b* is attached to *c* by a set-screw, and the whole can be raised or lowered by the fine-threaded screw *s* actuated by the levers *l l′*.

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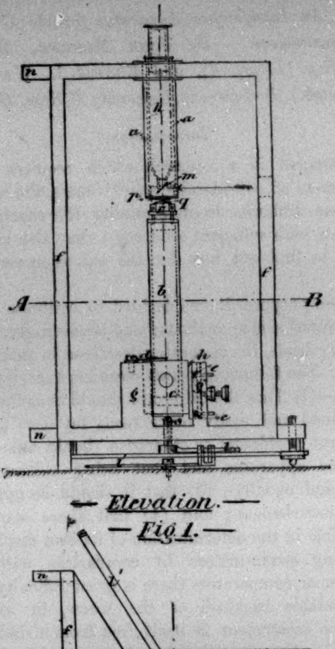
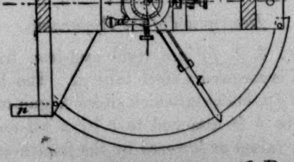
* Read October 28, 1904.


n
h
a
a
m
p
q
f
f
A
B
b
d
h
e
g
c
e
n
k
l

Elevation.

Fig. 1.


n
l'
f
d
c
b
h
g
e
f
l
n

Sectional Plan A-B.

Fig. 2.

The extensometer to be tested is attached by means of its gauge screws to *a* and *b* as to an ordinary test-piece. The displacement to be measured is therefore that of *b* relatively to *a*, and this is effected as follows :—

To the lower end of *a*, and at right angles to its axis, is attached a thin piece of optically plane glass *p*, whilst *b* carries at its upper extremity nearly in contact with *p* a lens *q* of black glass and small curvature. This lens is supported by the levelling table *r*, by means of which its centre of curvature is placed accurately on the axis of the instrument.

Light of known wave-length passing through a hole in *a* is directed to this optical system by a piece of plane glass *m*, set at an angle of 45°; and the interference-rings thus produced are viewed through the microscope *k* which slides in the upper tube.

When the screw *s* is turned *b* is raised or lowered through a small distance, and a ring appears or disappears for every half wave-length of light.

In the eyepiece of the microscope are three cross-wires, one at right angles to the other two, both extremities of the diameter of a ring being thus under observation. When in perfect adjustment, the centre of the rings remains midway between the two cross-wires.

The displacement is produced by turning the lever *l′* which is pivoted independently of the remainder of the apparatus, so that vertical pressure on it is not transmitted to *l* and *b*. The sleeve *c* is kept in contact with two worked surfaces on *g* by the piece *h* pressed against it by the springs *e e*, and at the same time its contact with the point of *s* is ensured by the springs *d d*.

To attach an extensometer the piece *b* can be lowered in its sleeve by releasing the set-screw ; the ends of the instrument can then be passed between *p* and *q*. Where this is not possible, the springs *d* are turned outwards so that the tube *b* can be removed altogether.

When the extensometer is one to be used in a horizontal position, the calibrator can be laid on its side resting on the legs *n n*.

The sodium light is brought to a focus at p by a condensing lens. It is obtained by placing a piece of platinum foil in the flame of a Bunsen burner, and allowing a solution of common salt to trickle very slowly down the foil.

Other methods which have been tried do not give a sufficiently intense light for a long period. This one, when properly arranged, burns continuously with little or no attention.

Each new ring corresponds to a distance of $29{\cdot}46 \times 10^{-6}$ centimetres.

Test of an Extensometer.

The points to be examined in testing an extensometer are—the value of the magnification constant; the action of the instrument at starting and reversing (in order to discover whether there is any 'lost motion'); and lastly—within what limits the readings obtained are proportional to the displacement and what correction is to be applied when this is not the case.

The following tests will serve as examples :—

The extensometer was of the differential mirror type arranged so that a displacement of 3×10^{-6} centimetres could easily be detected.

In the first place simultaneous readings of the extensometer and calibrator were taken for successive displacements in the same direction.

The test commenced with the extremities of an outside diameter of a ring at the cross-wires of the microscope. A displacement exactly equivalent to ten rings was then produced and the movement of the extensometer scale noted. The process being continued, the readings in Table I. were obtained.

The mean difference of 14·43 corresponds to a displacement of $29{\cdot}46 \times 10^{-5}$ cms. The magnification of the extensometer is therefore 3012.

On another occasion, the instrument was arranged to give a difference of 13·75 units for a displacement of ten rings, and was then tested for accuracy in reversing. Readings were taken for a displacement of five rings followed by a return

TABLE I.

Calibrator No. of Rings.	Extensometer.	
	Reading.	Differences in $\frac{1}{10}$ inch.
0	50·0	14·5
10	64·5	14·6
20	79·1	14·3
30	93·4	14·4
40	107·8	14·4
50	122·2	14·4
60	136·6	14·4
70	151·0	
Mean difference for 10 Rings ……		14·43

to the original position. Three such series are given in Table II.

TABLE II.

Calibrator No. of Rings.	Extensometer.					
	Readings	Diff.	Readings	Diff.	Readings	Diff.
0	163·3	6·8	119·9	7·0	120·0	6·8
5	170·1	6·9	112·9	6·9	113·2	6·9
0	163·2		119·8		120·1	

It will be seen that the instrument proved correct within the limits of accuracy of the observations.

The authors feel that the use of a calibrator of this kind will facilitate accurate work in elasticity and lead to a greater uniformity in the determination of coefficients.

University College, Bristol.
June, 1904.

DISCUSSION.

Prof. S. P. **THOMPSON** said he was pleased to see that optical devices were being applied practically for measuring small lengths and for testing extensometers. It was a step in the direction of realizing Maxwell's idea regarding the utilization of the wave-length of light as a standard of length. He would like to know the "order" of the rings which were used in the experiments, and also if there was any difficulty due to the interference of the light emitted by the two sodium lines. It would be interesting to know how the Authors had measured fractions of a ring and what fraction of a ring it was possible to measure with accuracy.

Dr. R. T. **GLAZEBROOK** said the method employed was similar to that used by Fizeau: a method carried to perfection by Benoit in the measurement of the coefficient of expansion of small objects. The principle had been applied by the Authors to practical mechanical problems.

Mr. **MORROW**, in reply to Prof. Thompson, said the rings used were near to the centre of the system. In fact the two surfaces could be brought into contact and the experiments commenced from that position. Fractions of a ring were estimated by eye, but there would be no difficulty in employing an eyepiece with a micrometer-scale. It was easy to estimate to one-tenth of a ring with the apparatus shown, but no mechanical device was accurate to anything like the same extent.

XXXIII. *Investigation of the Variations of Magnetic Hysteresis with Frequency.* *By* THOMAS R. LYLE, *M.A., Professor of Natural Philosophy in the University of Melbourne* *.

[Plate XV.]

IN the following paper are given some results obtained by my wave-tracer, of which a description has been published in the 'Philosophical Magazine' for November 1903.

The work described, which is of a preliminary nature, was in great part performed more than a year ago, but a severe illness has prevented me until now both from preparing it for publication and from continuing as intended the same work in a more accurate way. It is hoped that the experimental results given below will be of sufficient interest to merit publication now.

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* Read November 11, 1904.

1. The experiments were made on two rings of laminated annealed iron, in one of which the radial breadth of the iron was considerable relative to its mean radius. These rings were magnetized by alternating currents of different strengths and periods; both the magnetizing-current wave and the magnetic-flux wave were quantitatively determined by the wave-tracer, using the galvanometer method described in the paper already quoted, and the wave-forms so obtained were subjected to harmonic analysis.

The experiments were divided into series in which the period and wave-form of the magnetizing current were kept as nearly constant as possible throughout any one series, while its strength was varied. The analytic expressions for the associated current and flux waves for a few series are given in tabular form, and some of their more interesting amplitude and phase relations are shown by means of curves.

From the analytic expressions for each pair of associated waves the total iron loss (I, say) per cubic centimetre per cycle was calculated, and it was found, when the magnetizing current is approximately sinusoidal, that I is given with considerable accuracy by the formula

$$\mathrm{I} = (\cdot 00186 + \cdot 000026\, n)\, \mathfrak{B}^{1\cdot 67}$$

for ring I.,

and by

$$\mathrm{I} = (\cdot 001684 + \cdot 0000272\, n)\, \mathfrak{B}^{1\cdot 5}$$

for ring II.,

where n is the number of periods per second and $\mathfrak{B}$ (called the effective induction) is $\sqrt{2}$ times the root of mean square of $d\mathrm{B}/dt$, for all values of n, and for all values of the induction between 1000 and 12,000.

When from the total iron loss I per cm.^{3} per cycle the sum of the statical hysteresis (U, say) previously obtained by Ewing and Klaasen's method, and E the value that theory assigns to eddy-current loss, was subtracted, a considerable quantity (I—U—E) remained, which increased both when the frequency and when the flux-density increased. This quantity, called by Fleming the kinetic hysteresis, has been obtained for each experiment, and is given in the tables that are to follow, and curves are also given which show how it varies with the frequency and flux-density. That such a source of loss exists when iron is subjected to alternating magnetizing forces has already been shown by Steinmetz and by Siemens by other means.

An interesting case of transformation and what is called reflexion of energy is drawn attention to and discussed. If the E.M.F. impressed on the magnetizing circuit on the iron ring be sinusoidal, the flux-wave produced contains third, fifth, &c. harmonics. These higher flux-harmonics induce currents in the magnetizing circuit which are dissipated as heat in it. Thus we have a transformation of electric energy due to alternating currents of frequency *n* to energy of currents whose frequencies are $3n$, $5n$, &c., which is reflected back into the magnetizing circuit. This the author believes is a hitherto unnoticed source of transformer loss.

2. Fig. 1 (Pl. XV.) shows the arrangement of the apparatus. The magnetizing current was obtained from a four-pole rotary converter T supplied with direct current from storage-cells. By means of rheostats placed both in the armature and field-circuits, the speed could be varied and adjusted. One end of the spindle of the commutator C was directly connected to the spindle of this converter, and on the other end was a screw thread which worked a tangent wheel. On this wheel an ebonite stud is fixed which momentarily breaks the circuit of an electric chronograph once every revolution, thus recording the time of every 200 periods.

The magnetizing current, drawn from the slip-rings of the rotary converter, passed through a regulating resistance, a Kelvin balance, the primary of the air-circuit transformer M, and the primary coil on the iron ring B.

One end of the secondary of M is joined to one end of that of B, and the junction connected to one of the fixed brushes of the commutator. The other ends of the secondaries are connected, as shown in fig. 1, to the three-way key K, from the lever of which connexion is made to the other fixed brush. By this means either secondary can be joined to the commutator, and thence from the movable brushes of the latter through a reversing key and a high resistance to the galvanometer.

A Clark's cell with a megohm is arranged so that it may be used at any time for the purpose of determining the reducing factor of the galvanometer. The deflexions of the latter were so nearly proportional to the currents producing them over the part of the scale used that no calibration was necessary.

3. The details of the rings, called Ring I. and Ring II. respectively, are as follows :—

	RING I.	RING II.
No. of laminæ	18	31
Internal diameter (r_1)	7·6 cm.	15·223
External diameter (r_2)	11·58 cm.	16·477
Mean thickness (x)	·0475 cm.	·0443
Section of magnetic circuit (a)	1·701 cm.^{2}	·861
Length of magnetic circuit (l)	30·12 cm.	49·79
Specific resistance of iron at 12° C. (ρ)	12590	13600
No. of primary turns (n)	164	406
No. of secondary turns (n_2)	5 or 10	10

The laminæ were well annealed, the oxide removed, and they were insulated from each other by oiled paper. The mean thickness was determined from weight, area, and specific gravity. The values obtained for the specific resistance seemed high, so the determination was checked.

4. The method by which the current and flux waves were determined was practically the same as that described in the paper already quoted, except that readings of the galvanometer which give the ordinates for a definite phase were only taken for every six degrees on the divided circle which carries the movable brushes of the commutator. This gave 15 ordinates per half wave, and from these, without plotting, the first, third, and fifth harmonics were easily obtained by an arithmetical method of analysis when the seventh and higher harmonics could be neglected. Whether or not the latter assumption was legitimate would appear during the analysis, and in all cases in which the amplitude of the

seventh harmonic was greater than one per cent. of that of the first, it was determined, and though its values are not given in the tables that are to follow, its effect was allowed for.

5. In order to explain the procedure by which the results arrived at were deduced, the galvanometer readings for a pair of associated waves will be given, and the treatment to which these readings were subjected will be indicated.

The following table gives four times the galvanometer deflexions for a pair of associated current and flux waves, with the corresponding divided-circle readings. The latter, when doubled, give the corresponding phase-angles, as one complete revolution of the divided circle corresponds to two full waves, the rotary converter and the commutator both being four-pole. The readings for the current-wave are represented by γ, and those for the flux-wave by β. The other necessary details of the experiment are also given.

Experiment with Ring I.

Resistance in γ circuit $= r_1 = 5030\ \omega$.
" in β circuit $= r_2 = 15030\ \omega$.
Period by chronograph $= \mathrm{T} = {\cdot}0307$ sec.
Kelvin-balance reading of current C at start $= {\cdot}1873$ amp.
" " " at finish $= {\cdot}1868$ amp.
Reducing factor of galvanometer $= \lambda = 1{\cdot}287 \times 10^{-8}$ amp.
M of air-circuit transformer $= 0{\cdot}00061$ henry.
Secondary turns on ring $= n_2 = 5$.

Divided Circle.	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
4γ ……	103	338	583	795	1003	1160	1270	1323	1314	1239	1069	872	638	395	147
4β ……	−636	−561	−450	−296	−82	183	449	661	774	804	795	785	760	729	690

Each of these numbers is the sum of four separate readings of γ or of β, two of these being at corresponding points on the positive and negative halves of each wave, and the other two being a similar pair got after switching the galvanometer.

Analysing the above, we obtain

$$4\gamma = 1293 \sin \omega t - 41 \sin 3(\omega t - 7) + 5 \sin 5(\omega t - 23{\cdot}4)$$

$$4\beta = 868 \sin(\omega t - 51{\cdot}76) + 120 \sin 3(\omega t - 63{\cdot}1) + 23 \sin 5(\omega t - 69{\cdot}2).$$

6. The factors (c and f say) to be applied to the different values of γ and of β in order to obtain the corresponding values of current (C) and of flux (F) are (see Phil. Mag. *loc. cit.*)

$$c=\frac{\lambda r_1 T}{4M}, \quad f=\frac{\lambda r_2 T}{4n_2}, \quad . \quad . \quad . \quad . \quad (1)$$

which become

$$c=\cdot 00008145, \qquad f=29\cdot 7$$

for the experiment being discussed when the figures given in § 5 are substituted.

These factors can also be obtained from the Kelvin-balance reading C of the magnetizing current; for if γ be the R.M.S. of the γ readings

$$C=c\bar{\gamma}$$

and

$$\bar{\gamma}^2=\tfrac{1}{2}\{\gamma_1^2+\gamma_3^2+\gamma_5^2+\ldots\}$$

where $\gamma_1, \gamma_3, \gamma_5$, &c. are the amplitudes of the first, third, fifth, &c. harmonics of γ, and these have already been obtained by analysis (§ 5).

The factor f can then be deduced from c, as equations (1) above give us the relation

$$f=\frac{r_2}{r_1}\frac{M}{n_2}c.$$

By this method we obtain for the experiment in hand

$$c=\cdot 00008178, \qquad f=29\cdot 8,$$

which agree satisfactorily with the values obtained above by the other method.

The latter method of obtaining c and f is the one usually adopted.

The factors (h and b say) to be applied to γ and β in order to obtain H and B, where H is the M.M.F. round the ring divided by its mean circumference l, and B is the total flux F divided by the iron cross-section a, are

$$h=\frac{4\pi n_1 c}{l}, \qquad b=\frac{f}{a}.$$

Substituting the values of n_1, l, and a given in the details of ring I. in § 3, and those of c and f just obtained, we find that

$$h = \cdot 0056, \qquad b = 17\cdot 52\,;$$

so that

$$\mathrm{H} = \cdot 0056\gamma, \quad \mathrm{B} = 17\cdot 52\beta.$$

Hence the final expressions for the pair of associated waves investigated are

$$\mathrm{H} = 1\cdot 81[\sin \omega t - \cdot 0317 \sin 3(\omega t - 7) + \cdot 004 \sin 5(\omega t - 23\cdot 4)]$$

$$\mathrm{B} = 3802[\sin(\omega t - 51\cdot 76) + \cdot 1382 \sin 3(\omega t - 63\cdot 1) + \cdot 0265 \sin 5(\omega t - 69\cdot 2)].$$

7. It is easy to show that as the secondary (galvanometer) current is inappreciable, the energy (D say) dissipated per cycle in the iron of the ring is given by

$$\mathrm{D} = n_1 \int_0^{\mathrm{T}} \mathrm{C} \frac{d\mathrm{F}}{dt}\, dt,$$

which becomes (see § 6)

$$\mathrm{D} = n_1 cf \int_0^{\mathrm{T}} \gamma d\beta.$$

Hence when the galvanometer readings for β are plotted as ordinates against the corresponding ones for γ as abscissæ for a complete period a closed curve is obtained, whose area when multiplied by $n_1 cf$ gives D the total energy dissipated per cycle in the iron of the ring.

Dividing D by the volume $l\alpha$ of iron we get for quotient the space average throughout the ring of the iron loss per cubic centimetre per cycle. In the sequel this quantity will be represented by I.

I can also be obtained from the harmonic expressions for H and B in § 6.

For as

$$\mathrm{H}l = 4\pi n_1 \mathrm{C}, \qquad \mathrm{B}\alpha = \mathrm{F},$$

$$\mathrm{D} = \frac{l\alpha}{4\pi} \int_0^{\mathrm{T}} \mathrm{H} d\mathrm{B}\,;$$

hence

$$\mathrm{I} = \frac{1}{4\pi} \int \mathrm{H} d\mathrm{B},$$

and it is easy to show that if

$$\mathrm{H} = \mathrm{H}_1[\sin \omega t + h_3 \sin 3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5) + \&c.]$$

$$\mathrm{B} = \mathrm{B}_1[\sin(\omega t - \theta_1) + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5) + \&c.],$$

then

$$\frac{1}{4\pi} \int \mathrm{H} d\mathrm{B} = \frac{\mathrm{H}_1 \mathrm{B}_1}{4} \left\{ \sin \theta_1 + 3h_3 b_3 \sin 3(\theta_3 - \phi_3) + 5h_5 b_5 \sin 5(\theta_5 - \phi_5) + \&c. \right\}.$$

Applying this formula to the expressions for H and B in § 6 we find, for the experiment being discussed, that

$$I=1346.$$

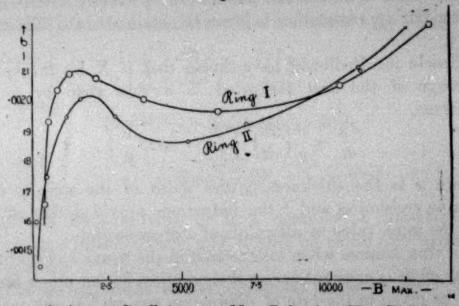
8. For the two rings used what has been called the static hysteresis (U, say) was determined for various induction densities by Ewing and Klaasen's method. It was found that the Steinmetz coefficients σ, where

$$U=\sigma B_{\text{Max.}}^{1\cdot 6},$$

for them were not constant but varied with the induction, and that the variation was quite as great for the narrow ring as for the broad one.

In fig. 2 the Steinmetz coefficients σ for these two rings are plotted against maximum induction (B_0 say). It will be seen that σ tends to be very small in weak fields, possibly vanishing with B_0, that it increases rapidly to a maximum, then diminishes to a minimum, and then steadily increases. It is possible that the maximum σ point, which is very marked, may give the induction at which some definite physical change due to magnetization begins to take place in the iron, perhaps that at which magnetization begins to produce lengthening.

Fig. 2.



Steinmetz Coefficient σ v. Max. Induction ($U=\sigma B^{1\cdot 6}$).

In the case of magnetization by means of alternating currents, it is fair to assume that the static hysteresis will depend on the maximum value of B, (B_0 say), during the alternation. B_0 is obtained from the maximum β reading by multiplying the latter by b (see § 6), and

$$U=\sigma B_0^{1\cdot6},$$

in which for σ we take from the curve in fig. 2 for the ring used the value corresponding to B_0.

For the experiment being discussed,

$$\text{Max. } \beta=\beta_0=201 \text{ (see § 5)}$$

$$B_0=17\cdot52\beta_0=3522 \text{ (see § 6),}$$

for which induction for ring I.

$$\sigma=\cdot002012,$$

hence

$$U=952.$$

9. No formula is available by means of which the eddy-current loss in an annular lamina can be exactly determined, but a fair approximation to it can be obtained in the following way.

Searle and Bedford* have shown that if X be the space average of the heat dissipated in a thin strip by eddy-currents,

$$\frac{dX}{dt}=\frac{x^2}{\rho}\left(\frac{db}{dt}\right)^2\left\{\frac{1}{12}-\cdot0525\frac{x}{y}\right\},$$

where x is the thickness, y the width of the strip, ρ the specific resistance, and b the induction, provided that db/dt has the same value at all points of a cross section.

In ring laminæ when magnetized in the usual way neither b nor db/dt is constant across their section, but an upper limit will be given to the rate of dissipation per cm.^{3} at any point where the induction is b by the equation

$$\frac{dX}{dt}=\frac{x^2}{12\rho}\left(\frac{db}{dt}\right)^2,$$

and the rate of dissipation of energy by eddy-currents in the whole ring will be

$$=\frac{x^2}{12\rho}\times\text{vol. of ring}\times\text{space average of }\left(\frac{db}{dt}\right)^2\text{ throughout the ring.}$$

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* Phil. Trans. vol. cxcviii.

Now it can be shown that this average will not differ much from $(dB/dt)^2$ where B has the meaning already assigned to it, namely, the average induction across any section. The latter statement is roughly indicated by the fact that though the amplitude of b and hence of db/bt may vary considerably from the inner to the outer radius of the ring, being greatest at the inner radius as the amplitude of the magnetizing force (which varies inversely as the distance from the centre of the ring) is greatest there, still, since the inner circumference is less than the outer one, there will, in making up the space average of $(db/dt)^2$, be a smaller relative volume of the iron at the high induction than at the low induction. Hence, finally, if E be the average eddy-current loss throughout the ring per cm.^{3} per cycle, we have, approximately, that

$$E = \frac{x^2}{12\rho}\int_0^T \left(\frac{dB}{dt}\right)^2 dt.$$

If

$$B = B_1[\sin \omega t + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5) + \&c.]$$

then

$$\int_0^T \left(\frac{dB}{dt}\right)^2 dt = \frac{2\pi^2}{T}\mathfrak{B}^2,$$

where

$$\mathfrak{B}^2 = B_1^2\{1 + 9b_3^2 + 25b_5^2 + \&c.\}$$

and

$$E = \frac{\pi^2 x^2}{6\rho T}\mathfrak{B}^2.$$

It will be seen that $\mathfrak{B}$ is a quantity of considerable importance in this theory, and it will be called the *effective induction.*

It is the amplitude of the sinusoidal induction-wave that would by its variation generate in any circuit looped on it the same virtual E.M.F. as would be produced in the same circuit by the actual induction B; or otherwise stated,

$$\mathfrak{B} = \sqrt{2}\ \text{R.M.S.}\left(\frac{dB}{dt}\right).$$

When we write the equation for E in the form

$$E = e\frac{\mathfrak{B}^2}{T},$$

e will be a constant for a particular ring and we find, using the values for x and ρ given in § 3, that,

$$\text{for Ring I. } e=2{\cdot}95 \times 10^{-7},$$

$$\text{and for Ring II. } e=2{\cdot}373\times 10^{-7}.$$

In the experiment with Ring I. that is being discussed,

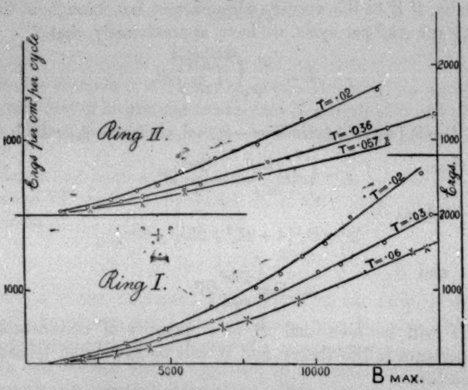
$$\mathfrak{B}^2=17{\cdot}2\times10^6$$

$$T={\cdot}0307$$

hence $$E=165.$$

10. In every case with either ring it was found that the

Fig. 3.



Kinetic Hysteresis v. Max. Induction.

sum of the statical hysteresis (U) and the theoretical eddy-current loss (E) was less than the total iron loss (I) as determined by the wave-tracer. The difference I−(U+E), which Fleming has called the kinetic hysteresis, was determined for each experiment, and is given in the tables that are to follow; and its values for different induction densities and frequencies for both rings are plotted in fig. 3 against B_0 the maximum induction in each case. The results expressed by these curves seem to thoroughly verify the observations of

Steinmetz and of Siemens both as to the existence of such a quantity as kinetic hysteresis and as to the general character of its variation with induction density and period. As this question has been much disputed the verification, by a new method, of the results obtained by previous investigators is not without significance.

11. In Table I. (p. 382) are given the analytical results deduced as indicated in the preceding paragraphs from a series of experiments with Ring I., in which the period was approximately ·019 sec. The wave-forms of the magnetizing currents in experiments 1 to 13 were approximately sinusoidal, and the E.M.F. impressed on the circuit was practically of the sine form in all experiments with the exception of No. 15, Table I.

In fig. 4 (p. 383) the more important characteristics of the induction-waves given in Table I. Nos. 1 to 13, *i. e.* produced by q.p. sine currents, are plotted against the amplitudes B_1 of the first harmonic of these waves. These curves are typical of any series of induction-waves of constant period produced by currents of similar wave-forms. No such regularity, however, would be obtained if the wave-forms of the magnetizing currents were allowed to vary, as will be seen by marking on fig. 4 the points for the characteristics of the induction-wave of Experiment 15, in which the H wave was greatly distorted (made saddle-shaped) by artificially distorting the applied E.M.F. wave.

The characteristics μ_0 and θ, which are the connecting links between the H wave and the B wave it produces, both fall to small values for small values of B_1, θ probably vanishing with B_1. Fig. 4 shows clearly how both rise to maxima and then diminish as B_1 increases.

The curve for b_3 the ratio of the third (B_3) to the first harmonic B_1 of B is striking, apparently issuing from the origin (see series 2, Table II. p. 384, & fig. 5, p. 387), it rises quickly between $B_1=0$ and $B_1=1000$ (q.p.), from which it continues for larger values of B_1 as a straight line. Hence for all values of B_1 greater than 1000 the amplitude B_3 of the third harmonic of B is of the form

$$\alpha B_1+\beta B_1^2.$$

TABLE I.—Ring I.

$T = 2\pi/\omega = \cdot 019$ (q.p.).

$\mu_0 = B_1/H_1$.

$\mathfrak{B} = \sqrt{2}\,.\,\text{R.M.S.}\left(\frac{dB}{dt}\right)$.

Type forms $\begin{cases} H = \dfrac{\text{M.M.F.}}{\text{length}} = H_1[\sin \omega t + h_3 \sin 3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5) + \ldots]. \\ B = \dfrac{\text{Flux}}{\text{area}} = B_1[\sin(\omega t - \theta) + b_3 \sin 3(\omega t - \theta - \psi_3) + b_5 \sin 5(\omega t - \theta - \psi_5) + \ldots]. \end{cases}$

$I = \dfrac{\text{Iron loss per cycle in ring}}{\text{vol. of iron in ring}}$. $U = \dfrac{\text{Statical hysteresis per cycle in ring}}{\text{vol. of iron in ring}}$ $E = \left\{ \begin{array}{l} \text{Theoretical eddy-current} \\ \text{loss per cm.}^3 \text{ per cycle.} \end{array} \right.$

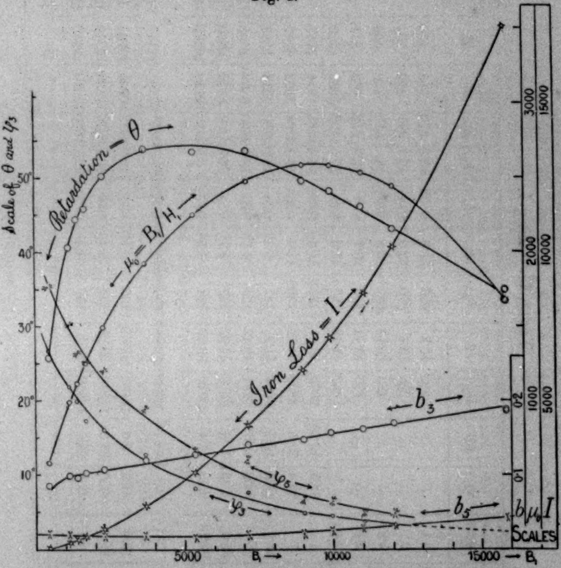
No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	I.	U.	E.	I-U-E.
1 ...	·01877	0·773	·0463	−0·12	...	...	448	25·75	·085	26·76	·021	35·25	580	450	465	37·2	31	3·4	2·8
2 ...	·01887	1·156	·042	1·9	...	...	1147	40·65	·099	21·89	·020	30·14	992	1124	1204	216	158	22·6	35·4
3 ...	·01849	1·261	·033	4·17	...	...	1407	44·55	·096	19·84	·016	26·32	1116	1370	1468	311	219	34·4	57·6
4 ...	·01894	1·35	·037	2·67	...	...	1689	45·85	·1025	17·32	·017	25·15	1251	1643	1772	409	293	49	67
5 ...	·01860	1·536	·034	1·14	...	...	2291	50·34	·108	16·05	·016	24	1492	2199	2414	680	463	92·5	124·5
6 ...	·0190	1·93	·025	0·0	...	...	3708	53·93	·120	12·73	·017	18·85	1921	3482	3956	1452	936	243	273
7 ...	·01875	2·377	·012	17·4	...	...	5360	53·54	·128	8·10	·015	13	2255	4884	5753	2573	1588	521	464
8 ...	·01892	2·892	0	...	...	...	7160	53·66	·140	7·60	·019	12	2476	6466	7800	4170	2457	948	765
9 ...	·01878	3·497	·020	57	·005	21·0	9040	49·61	·148	4·82	·021	6·76	2585	8032	9934	6027	3443	1550	1034
10 ...	·01905	3·897	·014	41·4	·007	14·5	10050	48·18	·157	4·56	·025	6·5	2580	8855	11170	7260	4115	1935	1210
11 ...	·01868	4·347	·017	50·36	·008	16·0	11010	46·06	·162	3·84	·030	4·92	2534	9604	12350	8614	4762	2408	1444
12 ...	·01835	4·943	·022	42·85	·008	8·8	12070	43·10	·170	3·28	·033	4·90	2442	10440	13690	10170	5540	3015	1615
13 ...	·01840	5·67	·040	36·53	·014	7·4	13020	40·35	·175	2·65	·034	3·30	2296	11250	14500	11820	6380	3560	1880
End of approximately sinusoidal magnetizing waves.																			
14 ...	·01869	9·42	·1635	8·72	−·056	29·78	15860	34·94	·186	4·77	·043	9·18	1677	13670	18542	17500	9530	5427	2543
15 ...	·01844	2·886	−·218	−·45	·042	26·09	6488	40·84	·134	−2·9	·022	−1·5	2248	5780	7037	3451	2059	792	600

Within the limits of experimental error b_5 may be taken as a straight line, so that B_5 is also of the form

$$\gamma B_1 + \delta B_1^2.$$

The curves for ψ_3 and ψ_5 show the gradual change of position as regards phase of the third and fifth harmonics with respect to B_1 the first harmonic of B as B_1 increases.

Fig. 4.



Both ψ_3 and ψ_5 seem to attain small limiting values for very high values of B_1, which means that all the harmonics of B pass through their zero values in the same direction at nearly the same instant when B_1 is very large.

In Tables II. and III. are given the analytical results for two other series of experiments with Ring I. for periods ·03 sec. and ·058 sec. approximately, and in Tables IV., V., and VI. are given the results of three series of experiments with Ring II.

TABLE II.—Ring I.

No.	T.	H.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	I.	U.	E.	I-U-E.
1...	·0302	·4	·054	5·1	...	...	148	13·9	·053	37·8	·0153	45·3	370	145	150	3·2			
2...	·0299	·577	·048	2	...	...	258	19·4	·070	32·0	·014	40·1	447	259	264	12·2			
3...	·03	·783	·047	4·2	...	...	483	25·11	·081	26·8	·019	38	617	479	500	39·5	35·3	2·5	1·7
4...	·0303	·955	·043	2·12	...	...	776	32·41	·103	23·1	·024	32·7	813	768	818	98·4	83·2	6·5	8·7
5...	·0305	1·202	·038	2·46	...	...	1432	41·18	·113	18·5	·024	27·5	1192	1388	1520	282	224	22·3	36
6...	·0303	1·366	·043	4	·005	12·25	2025	47	·1253	16·4	·0284	24·4	1481	1944	2182	505	381	46	78
7...	·0307	1·807	·031	7	·004	23·5	3802	51·76	·1382	11·4	·0265	17·4	2104	3522	4147	1344	952	165	227
8...	·0297	2·114	·021	13·9	·008	25·3	5050	51·7	·147	9·43	·028	14·9	2389	4613	5568	2078	1445	308	325
9...	·0295	2·708	·014	25	·008	11·7	7284	49·13	·155	6·65	·030	10·2	2690	6462	8105	3692	2455	657	580
10...	·0296	3·366	·018	35·6	·009	11·8	9318	46·07	·167	4·77	·035	6·9	2769	8142	10550	5590	3580	1110	900
11...	·0295	4·53	·044	29·1	·012	7·4	11720	39·89	·181	2·34	·042	3·9	2587	10120	13580	8310	5234	1844	1232
End of approximately sinusoidal magnetizing waves.																			
12...	·0295	4·737	·115	13·9	·036	3·0	12400	43·88	·163	6·17	·034	12·0	2618	10900	13964	9308	6026	1950	1332
13...	·0295	6·243	·148	11·6	−·046	34·3	14210	39·28	·178	5·46	·038	10·7	2276	12420	16340	12115	7817	2665	1633
14...	·0301	9·83	·1895	6·5	−·058	26·5	16310	31·1	·196	4·4	·052	8·2	1660	13970	19440	15630	9930	3704	2000

Table III.—Ring I.

No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	I.	U.	E.	I-U-E.
1 ...	·0574	·774	·038	1·6	...	...	483	23·62	·089	24·08	·021	34·9	624	477	500	36·8	...	...	
2 ...	·057	·971	·04	0·6	...	...	847	31·34	·123	19·2	·026	28·7	873	825	911	105	94	4·3	6·7
3 ...	·0572	1·211	·032	7·4	...	...	1625	39·8	·131	16·5	·031	22·6	1342	1550	1763	311	267	16	28
4 ...	·0576	1·348	·045	7·5	·016	8	2102	42·5	·143	14·2	·038	27·7	1560	1965	2322	470	386	27	57
5 ...	·0578	1·653	·026	9·9	·008	16·3	3398	46·0	·155	11·2	·042	15·5	2055	3118	3814	997	790	74	133
6 ...	·0582	1·94	·024	15·1	·009	15·7	4765	46·35	·165	8·7	·043	12·4	2456	4275	5413	1645	1284	148	213
End of approximately sinusoidal magnetizing waves.																			
7 ...	·0593	2·583	·074	14·6	·027	17·9	7685	48·46	·170	7·6	·043	12·0	2975	6800	8785	3545	2668	384	493
8 ...	·0585	2·91	·082	14·5	·026	2·8	8690	46·84	·174	7·1	·042	11·4	2987	7686	9975	4337	3255	501	581
9 ...	·0602	3·737	·108	13·9	·036	0·5	10780	43·17	·179	5·7	·038	10·7	2885	9440	12450	6255	4620	760	875
10 ...	0590	6·41	·293	15·6	·061	10·7	14710	37 71	·123	8·6	·018	19·9	2295	13330	15750	11730	8982	1241	1507
11 ...	·0643	9·328	·189	3·45	·074	22·6	16165	28·93	·203	4·0	·053	7·0	1733	13800	19500	13090	9800	1748	1542

Table IV.—Ring II.

No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	I.	U.	E.	I–U–E.
1 ...	·0210	·582	·034	−1·4	...	...	291	21·2	073	25·9	·017	35·9	500	291	300	15			
2 ...	·0216	·894	·036	1·8	...	...	712	32·4	·082	22·6	·015	34·4	796	703	736	85	69	6	10
3 ...	·0204	1·161	·029	6·2	...	...	1357	41·45	·089	19·2	·016	26	1169	1316	1410	260	193	23	44
4 ...	·0205	1·46	·0306	7·2	...	...	2325	47·2	·101	16	·016	24·5	1592	2230	2430	621	453	68	100
5 ...	·0212	1·737	·03	6·8	...	...	3225	48·97	·116	15·1	·022	22·5	1857	3047	3435	1055	724	132	199
6 ...	·0209	2·046	·032	17·5	...	...	4173	49·45	·118	12·5	·023	19·3	2040	3916	4450	1604	1059	225	320
7 ...	·0204	2·276	·036	21·4	·008	15	5011	48·75	·1195	11·5	·020	19·25	2202	4657	5350	2110	1383	333	394
8 ...	·0204	2·647	·042	18·6	·008	18·7	5910	47·45	·127	10·6	·022	17·2	2233	5443	6356	2824	1775	470	579
9 ...	·0208	3·18	·052	19·8	·013	13·8	7300	44·6	·136	8·8	·029	15	2295	6638	7956	3948	2477	722	749
10 ...	·0206	4·234	·068	19·1	·015	5·3	9074	38·78	·151	7·32	·031	12·7	2143	8094	10050	5707	3575	1163	969
11 ...	·0203	5·65	·076	16·6	·016	0	10900	32·51	·155	6·09	·035	9·19	1930	9612	12165	7760	4763	1730	1267
12 ...	·01994	10·57	·076	11·5	−·0256	25·8	14260	21·4	·184	2·8	·052	4·4	1350	12250	16700	12710	7716	3318	1676

TABLE V.—Ring II.

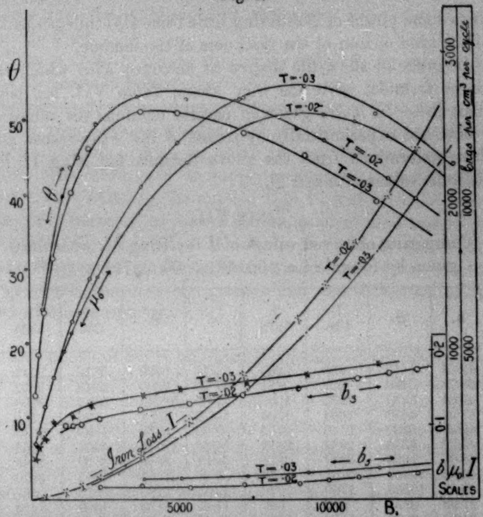
No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	𝔅.	I.	U.	E.	I–U–E.
1 ...	·0352	0·881	·051	4·6	...	...	802	34·17	·10	22·6	·025	30·8	938	788	836	97	78	4·7	14·3
2 ...	·0358	1·445	·047	7	·005	5·0	2610	45·82	·131	14·7	·038	19·8	1861	2448	2850	670	521	54	95
3 ...	·0360	1·734	·048	11·6	·005	12·5	3636	46·36	·140	13·0	·032	18·8	2097	3372	3990	1120	839	105	176
4 ...	·0359	1·997	·050	16·2	·011	13	4503	45·32	·146	11·1	·032	17·0	2255	4128	4970	1555	1147	163	245
5 ...	·0360	2·573	·065	17·6	·017	10·2	6110	42·07	·149	9·2	·032	14·6	2375	5560	6770	2513	1841	302	370
6 ...	·0358	2·843	·073	19·4	·019	6·8	6770	40·43	·15	8·7	·032	13·3	2381	6133	7500	2948	2172	373	403
7 ...	·0359	4·425	·088	16·7	·023	2·1	9570	32·22	·158	5·6	·037	8·5	2163	8420	10740	5247	3780	763	704
8 ...	·0350	9·97	·210	3·23	− ·07	25	14030	24·65	·148	8·5	·034	14·6	1407	12540	15550	10900	8100	1641	1160
9 ...	·0359	14·1	·171	2·57	− ·068	17·1	16180	20·96	·176	6·33	·053	11·5	1147	14120	18860	14140	10470	2354	1316

TABLE VI.—Ring II.

No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	𝔅.	I.	U.	E.	I–U–E.
1 ...	·0572	1·13	·0363	0·9	...	...	1525	38·7	·133	15·4	·0355	22·1	1350	1437	1670	265	224	10·5	30·5
2 ...	·0567	1·392	·0268	11·4	·01	1·9	2525	41·54	·143	12·7	·0374	17·7	1814	2330	2795	573	486	32·5	54·5
3 ...	·0573	1·959	·0581	15·3	·02	3·3	4557	41·2	·163	9·6	·0437	14·6	2326	4138	5170	1400	1151	111	138
4 ...	·0572	2·56	·065	13·65	·015	1·87	6206	38·88	·171	8·1	·0465	11·6	2425	5544	7120	2350	1832	210	308
5 ...	·0568	4·185	·090	13·0	− ·023	29·1	9340	30·93	·173	5·4	·049	7·7	2232	8168	10770	4550	3543	484	523
6 ...	·0565	10·38	·1583	1·38	− ·062	11·8	14370	22	·176	6·1	·05	10·65	1384	12490	16710	10230	8067	1174	990

In fig. 5 some of the characteristics of the induction-waves produced in Ring I. for two different speeds are plotted against B_1 for the purpose of showing the general effect of change of frequency on these characteristics.

Fig. 5.



12. For the two rings experimented with I find that I, the total iron loss per cm.^{3} per cycle, is given very approximately in terms of $\mathfrak{B}$, the effective induction, and n the number of periods per second, when $\mathfrak{B}$ lies between 1000 and 12,000, by a simple formula analogous to that of Steinmetz for statical hysteresis.

Thus for Ring I.,

$$I = (\cdot 00186 + \cdot 000026\, n)\, \mathfrak{B}^{1\cdot 57},$$

and for Ring II.,

$$I = (\cdot 001684 + \cdot 0000272\, n)\, \mathfrak{B}^{1\cdot 57}.$$

That the same power of the effective induction should appear in the formula for each ring is striking, and suggests the probability that for any given frequency when the wave-form of H is approximately sinusoidal the total iron loss per cm.^{3} per cycle is proportional to

$$\mathfrak{B}^{1·57},$$

or to some power of $\mathfrak{B}$ differing little from 1·57 independently of the kind of iron or the thickness of the laminæ.

In order to show the degree of accuracy with which the above formulæ give the iron losses, Table VII. has been compiled. In it are given in parallel columns the values of I obtained experimentally by means of the wave-tracer and those determined from the above formula for Ring II. for different values of n and $\mathfrak{B}$.

TABLE VII.

Comparing observed values of I for Ring II. with those given by formula $I=(·001684+·00002717\ n)\ \mathfrak{B}^{1·57}$.

n.	$\mathfrak{B}$.	I. Obs.	I. Calc.	n.	$\mathfrak{B}$.	I. Obs.	I. Calc.
49·07	1410	260	265	27·77	6770	2513	2517
48·71	2430	621	621	27·90	7500	2948	2962
47·24	3435	1055	1056	27·86	10740	5247	5200
47·91	4450	1604	1596	34·03	3061	793	775
49·02	5350	2110	2153	34·22	3780	1080	1083
49·02	6356	2824	2819	34·66	4770	1569	1568
48·03	7956	3948	3975	34·91	5818	2139	2145
48·54	10050	5707	5760	33·49	7536	3165	3170
49·21	12165	7760	7710	17·62	2795	573	557
27·90	2854	670	649	17·45	5170	1400	1459
27·76	3990	1120	1098	17·48	7120	2350	2409
27·81	4970	1550	1551	17·60	10770	4550	4625

13. In the tables giving the analytical results of the different experiments, it will be seen that as H_1 gets large the wave-form of H differs more and more from the sine form, and that in the slower speeds the range over which it was possible to obtain magnetizing currents of approximately sine wave-form was less. This was due to the reaction of the iron in the ring and may be explained as follows:—

If we assume that the applied E.M.F. was sinusoidal [it was nearly so], a current would flow which would produce a flux wave that contained 1st, 3rd, 5th, &c. harmonics. The

3rd, 5th, &c. harmonics produced would induce in the magnetizing circuit currents of periods $3n$, $5n$, &c., which would appear as 3rd, 5th, &c., harmonics in the magnetizing current, and would be greater as the impedances of the primary circuit for currents of these different periods were less, if we assume that the amplitudes of the upper harmonics of the flux are fixed.

Now in order to obtain the larger magnetizing currents it was necessary to reduce the impedance, and hence in these cases the higher harmonics produced as above by the reaction of the ring became greater and the magnetizing current more distorted. Also in order to get the slower speeds fewer cells were used to supply current to the rotary converter, thus reducing the applied E.M.F., and in order to get the same currents as before the impedances used had to be proportionally reduced. Hence, in the slower speeds the upper harmonics of the current due to the reaction of the ring would be relatively greater and the distortion of its wave-form more marked.

This phenomenon is an interesting case of transformation and what may be called reflexion of energy. Thus the iron receives energy from the first harmonic (frequency n) of the exciting current, some of which it transforms to vibratory energy of frequencies $3n$, $5n$, &c., and sends back energy to the primary circuit by means of currents of these higher frequencies, where in general this reflected energy will be dissipated as heat.

This reflexion of energy occurred in nearly all the experiments recorded in this paper, and the amount in each case for each harmonic is obtained when calculating the iron loss by the analytical method given in § 7.

Thus the energy D received per cycle by the iron and dissipated as heat in it is given by

$$\begin{aligned} \mathrm{D} &= n_1 \int_0^{\mathrm{T}} \mathrm{C} \frac{d\mathrm{F}}{dt} dt \\ &= n_1 \int_0^{\mathrm{T}} \left\{ \mathrm{C}_1 \frac{d\mathrm{F}_1}{dt} + \mathrm{C}_3 \frac{d\mathrm{F}_3}{dt} + \mathrm{C}_5 \frac{d\mathrm{F}_5}{dt} + \&\mathrm{c.} \right\} dt \\ &= \mathrm{D}_1 + \mathrm{D}_3 + \mathrm{D}_5 + \&\mathrm{c.}\ (\text{say}), \end{aligned}$$

where C_1, C_3, C_5, F_1, F_3, F_5 are the 1st, 3rd, and 5th harmonics of current and flux respectively.

Now if, for instance,

$$D_3 \text{ or } n_1 \int_0^T C_3 \frac{dF_3}{dt} dt$$

is negative, the iron by means of the third harmonics is sending back per cycle to the magnetizing circuit energy to the amount $-D_3$, and similarly if D_5 &c. are negative.

We have (see § 7)

$$\frac{D}{v} = I = \frac{H_1 B_1}{4} \{\sin \theta_1 + 3h_3 b_3 \sin 3(\theta_3 - \phi_3) + 5h_5 b_5 \sin 5(\theta_5 - \phi_5) + \&c.\}$$

$$= I_1 + I_3 + I_5 + \&c. \text{ (say)},$$

where v is the volume of iron in the ring:

hence
$$D_1 = I_1 v = v \frac{H_1 B_1}{4} \sin \theta_1$$

$$D_3 = I_3 v = v \frac{H_1 B_1}{4} \times 3h_3 b_3 \sin 3(\theta_3 - \phi_3)$$

$$D_5 = I_5 v = v \frac{H_1 B_1}{4} \times 5h_5 b_5 \sin 5(\theta_5 - \phi_5).$$

For example, in calculating I by means of the above formula for experiment 14 (Table I.) in which

$$H = 9{\cdot}42\{\sin \omega t - {\cdot}1635 \sin 3(\omega t - 8{\cdot}72)$$
$$- {\cdot}0565 \sin 5(\omega t - 29{\cdot}78) + {\cdot}0159 \sin 7(\omega t - 15{\cdot}86)\}$$
$$B = 15860\{\sin (\omega t - 34{\cdot}94) + {\cdot}1865 \sin 3(\omega t - 39{\cdot}71)$$
$$+ {\cdot}0434 \sin 5(\omega t - 44{\cdot}12) + {\cdot}0117 \sin 7(\omega t - 47{\cdot}9)\}$$

we find that

$$I = 37350\{{\cdot}5727 - {\cdot}0914 - {\cdot}0116 - {\cdot}0009\}$$
$$= 21390 - 3414 - 434 - 34$$
$$= 17508,$$

or
$$I_1 = 21390, \quad I_3 = -3414,$$
$$I_5 = -434, \quad I_7 = -34,$$

which shows that per cm.^{3} of iron per cycle 21390 ergs entered the ring by means of the first harmonics, and of this there was sent back or reflected to the primary circuit 3414 ergs by means of the third, 434 ergs by means of the fifth, and 34 ergs by means of the seventh harmonic, while the remainder, $=I=17508$ ergs, was dissipated as heat in the iron of the ring.

These reflected current harmonics will obviously, if of sufficient magnitude, greatly modify the wave-form of C, and they will also by their reaction reduce the amplitudes and change the phases of the corresponding harmonics of F from which they arise, thus modifying the wave-form of F (the flux).

Now I find that when the upper harmonics of the flux are damped as above, μ_0 or B_1/H_1 becomes greater, and the iron loss I for the same value of $\mathfrak{B}$, or even of B_1, becomes greater. To make up for this greater iron loss as well as to supply the energy that is reflected, more energy must enter the iron by means of the first harmonic of C, and this is effected by a considerable increase in θ_1, the angle of lag of F_1 behind C_1.

If the applied E.M.F. (supposed sinusoidal) and the circuit be so controlled that while F_1 or αB_1 is kept constant the impedance is regularly diminished, the reflected energy due to any of the upper harmonics (C_3, F_3, say) will not go on increasing as the impedance diminishes, but will at first increase to a maximum and then diminish, in the limit vanishing when the impedance becomes zero. This would follow if we assumed that on account of some property of the iron there is always associated with a given primary sinusoidal flux oscillation F_1 or αB_1 of frequency n a series of magneto-motive forces M_3, M_5, &c., of frequencies $3n$, $5n$, &c. given in amplitude and fixed in phase relative to F_1.

F_3 would then, for a given impedance, be the flux produced by the vector resultant of M_3 and $4\pi n_1 C_3$, where C_3 is the reflected current produced in the circuit by variation of F_3.

In the simple case in which the circuit outside the ring is non-inductive, and when magnetic lag is neglected, the relations between M_3, C_3, F_3, and the reflected energy can be shown by means of a vector right-angled triangle of which the hypotenuse is proportional to M_3 and is fixed. One of its sides is proportional to C_3 and the other to F_3, while the area of the triangle is proportional to the reflected energy. The locus of the right-angled vertex is thus a semicircle, and as C_3 increases from zero until the triangle becomes isosceles, the reflected energy increases from zero to a maximum, after which any further increase of C_3 causes a diminution in the reflected energy, which finally vanishes with F_3.

If we took account of magnetic lag and of the inductance of the circuit the vector diagram would be more complicated, but the general result would be similar.

The following results of three experiments with Ring I. will illustrate the preceding.

The ring was rewound with two layers of 207 turns each, one or both of which could be used, thus enabling me to obtain for the same values of F_1 two series of values, one double the other, for the reaction E.M.F.'s due to variation of the upper harmonics of F.

EXPERIMENT 1.

$T = {\cdot}0295$ sec. Magnetizing current sent through 207 turns. Full pressure of secondary battery on to rotary converter, and current in magnetizing circuit cut down by iron-less inductances to keep the reflected currents due to higher harmonics of F as small as possible.

$$\begin{aligned} H = {} & 4{\cdot}53\{\sin \omega t - {\cdot}0443 \sin 3(\omega t - 29{\cdot}11) \\ & \qquad + {\cdot}0124 \sin 5(\omega t - 7{\cdot}41)\} \\ B = {} & 11720\{\sin(\omega t - 39{\cdot}89) + {\cdot}1814 \sin 3(\omega t - 42{\cdot}23) \\ & \qquad + {\cdot}042 \sin 5(\omega t - 43{\cdot}8)\}, \end{aligned}$$

$\mu_0 = B_1/H_1 = 2587.$ $\mathfrak{B} = 13580.$ Max. $B = 10120.$

$I_1 = 8541,$ $I_3 = -203,$ $I_5 = -1,$

$I = I_1 + I_2 + I_3 = 8310,$

$I/\mathfrak{B}^{1{\cdot}57} = {\cdot}0027.$

EXPERIMENT 2.

$T = {\cdot}02957$ sec. $n_1 = 414$ turns.

E.M.F. the same as in Exp. 1, but current cut down by non-inductive resistances.

$$\begin{aligned} H = {} & 4{\cdot}4\{\sin \omega t - {\cdot}1758 \sin 3(\omega t - 22{\cdot}56) \\ & \qquad + {\cdot}039 \sin 5(\omega t - 1488)\} \end{aligned}$$

$$\begin{aligned}\mathrm{B}=11850\{&\sin(\omega t-48{\cdot}04)+{\cdot}1246\sin 3(\omega t-56{\cdot}82)\\&+{\cdot}0164\sin 5(\omega t-66{\cdot}73)\}.\end{aligned}$$

$\mu_0=2693.$ $\mathfrak{B}=12700.$ Max. $\mathrm{B}=10840.$
$\mathrm{I}_1=9692.$ $\mathrm{I}_3=-830.$ $\mathrm{I}_5=-42.$
$\mathrm{I}=8820.$ $\mathrm{I}/\mathfrak{B}^{1{\cdot}57}={\cdot}00318.$

Experiment 3.

$\mathrm{T}={\cdot}0294$ sec. $n_1=414$ turns.

Impressed E.M.F. reduced and non-inductive resistances proportionally reduced.

$$\begin{aligned}\mathrm{H}=4{\cdot}23\{&\sin\omega t-{\cdot}25\sin 3(\omega t-30{\cdot}25)\\&+{\cdot}0572\sin 5(\omega t-32{\cdot}24)\}\end{aligned}$$

$$\begin{aligned}\mathrm{B}=11560\{&\sin(\omega t-50{\cdot}77)+{\cdot}0718\sin 3(\omega t-62{\cdot}19)\\&+{\cdot}0096\sin 5(\omega t-82{\cdot}7)\}.\end{aligned}$$

$\mu_0=2734.$ $\mathfrak{B}=11840.$ Max. $\mathrm{B}=10900.$
$\mathrm{I}_1=9465.$ $\mathrm{I}_3=-654.$ $\mathrm{I}_5=-32.$
$\mathrm{I}=8780.$ $\mathrm{I}/\mathfrak{B}^{1{\cdot}57}={\cdot}00353.$

Though the applied E.M.F.'s in these experiments were only approximately sinusoidal, they illustrate the main features of the phenomenon being discussed. Thus we have with diminution of the impedance of the circuit while B_1 is kept approximately constant, the increase in the reflected upper harmonics of C and the diminution of the corresponding harmonics of F, the increase of μ_0 and of I and the increase and subsequent decrease of the reflected energies I_3 and I_5.

Attention is drawn to the great variation in the ratio of I to $\mathfrak{B}^{1{\cdot}57}$ with change of wave-form of H, and therefore of B, thus showing that the formula given in § 12 for I can only apply to a series in which the wave-form of H is nearly sinusoidal.

In order to fully investigate the phenomenon drawn attention to in this paragraph, it would be necessary to determine the impressed E.M.F. wave as well as that of H and of B. This is being done, and will form the subject of a future communication.

Finally, the matter dealt with in this section seems to me to have an important practical significance in the case of transformers. In calculating their losses all that has hitherto been counted as iron loss is the energy that is dissipated as heat in the iron. But now we see that it is possible for the iron to send back a considerable amount of energy to the primary circuit, where it also is dissipated as heat. This reflected energy should be counted against the iron.

Thus Experiment 2 above may, for the sake of illustration, be supposed to refer to a transformer on open secondary. The true dissipation of energy associated with its operation per cm.^{3} of iron per cycle is I_1 or 9692 ergs, of which $I=8820$ ergs, hitherto called the total iron loss, is dissipated as heat in the iron, and $-(I_3+I_5)=872$ ergs is reflected by the iron to the primary circuit, where it also is dissipated as heat.

I desire to thank Mr. K. S. Cross and Mr. E. Machin for valuable assistance in carrying out the experiments discussed in this paper.

DISCUSSION.

Dr. J. A. **FLEMING** said that he had hardly yet given Professor Lyle's paper the study it deserved, but several of the matters dealt with seemed interesting. As regarded the law of iron core loss for transformers, the formula given by Prof. Lyle, representing it as the product of a factor and the maximum induction density raised to the power of 1·57, agreed fairly well with practice. On the other hand, it must be remembered that the so-called law of static hysteresis, representing it as proportional to the induction density raised to the power of 1·6, was a purely empirical law and only agreed with facts for a certain range of induction density varying from about 2000 to 9000.

In connexion with this matter, he desired to refer to a paper brought before the Physical Society in 1899 by himself and Messrs. Ashton and Tomlinson, "On the Magnetic Hysteresis of Cobalt," in which it was shown that over about the same range the same 1·6 law of hysteresis held good. In this paper on Cobalt it was also pointed out that in many ways it would be preferable to represent the hysteresis loss as a function of the magnetization rather than the induction density.

As far as regards practical construction of transformers, the exact enunciation of the law of so-called static hysteresis was not so important as the knowledge of the manner in which the total iron loss varied with the induction density. Iron for transformer manufacturing was now made and sold to a specification defining the loss in fractions of a watt per pound for a standard frequency and induction density, and in the predetermination of transformer performances it was therefore necessary to be able to determine from the specified magnetic value of the iron the loss to be expected in the iron at other frequencies and induction densities. In this, account must be taken of the eddy-current loss as well as any variation in the hysteresis, due to frequency. It followed from this that experiments made with many forms of hysteresis meters were of little practical use to the manufacturer. He himself (Dr. Fleming) had contended now for eleven years, since the publication of a paper read before the Institution of Electrical Engineers in 1893, that the only satisfactory method of measuring iron losses for transformer purposes was by means of a properly constructed wattmeter, and he believed that others of experience were now of the same opinion. Moreover, experience shows that tests made on small specimens of iron are not much guide to the results obtained with the iron in bulk. In Germany it was now becoming customary to test transformer-iron in large sheets by an arrangement in connexion with which a wattmeter was used. In this way the figure determined was the actual loss for the whole sheet, and not merely that of a small slip cut from it.

The result obtained by Prof. Lyle, that energy is reflected back to the primary circuit by the iron, was scientifically interesting, but it did not appear to him that the paper left the matter in a sufficiently definite form to become a guide in practice, as, unless the true iron loss could be determined from the figures of observation obtained by the wattmeter or some equivalent method without much trouble, it would not be of assistance to transformer manufacturers.

When using the wattmeter, it is always possible to separate out from the observed reading that amount which is due purely to the resistance of the copper circuit. Then also it had to be remembered that in all wattmeter readings the wattmeter measured also the loss in itself; but if this were taken into account, the deficit would always represent very closely the true loss in the iron resulting from eddy-current loss and what he had called kinetic hysteresis.

Mr. A. **Campbell** drew attention to the work of an earlier investigator on the same subject. Max Wien, in a paper on "Magnetization by Alternating Currents" (Wied. *Ann.* (13) Aug. 1898), gives results of an investigation on the change in flux-density and hysteresis loss due to variation in frequency of the magnetizing alternating current. His chief method of measurement was based on Maxwell's method of comparing two self-inductances. One of the inductive coils so compared was the winding of the iron ring under test, the source of voltage being alternating (128 to 520∼per sec.), and the indicating instrument a tuned optical telephone or vibration galvanometer. The bridge measurements (starting with direct current) gave the "effective" resistance and self-inductance of the ring winding, and from these (with the dimensions &c.) were deduced the values of B and the hysteresis loss due to the fundamental component of the alternating flux, allowance for eddy-currents having been made. A number of conclusions are drawn from the results; amongst others, that increase of frequency decreases B for a given value of H, and increases the hysteresis loss for a given value of B. This appears to indicate that with quickly alternating fields B lags behind H. It is interesting to notice that for one of the rings tested the "kinetic hysteresis" found for $n = 128$ is of somewhat the same order of magnitude as that given by the rings of Prof. Lyle at lower frequencies.

Dr. D. K. **Morris**, in a letter to the Secretary, said he had been much interested in read Prof. Lyle's paper and in noting the way in which he had overcome the considerable experimental difficulties which beset the determination of the variation of magnetic hysteresis with frequency. Much experimental work on this subject had been lately carried out (chiefly by Mr. R. P. Hulse, B.Sc.) in his laboratory at Birmingham; and it had always appeared that with good transformer iron, the increase of hysteresis with frequency was *less* than that found by Prof. Lyle. These experiments were made with ring-magnets, built up of stampings properly insulated from one another, and having a small difference only between the internal and external radii.

But another method of experimenting was the following:—If the no-load loss of a transformer be measured at different frequencies, the voltage being so adjusted as to give constant flux-density, then by dividing each measurement by the frequency and plotting the quotients against the frequency, a straight line is obtained, from whose inclination the eddy-current constant can be obtained on the assumption that hysteresis does not vary with the frequency. Prof. Lyle in § 9 quotes $\frac{\pi^2}{6}=1{\cdot}654$ as this constant*. Different methods of arriving at it, as variously given, show that it certainly lies between 1·6 and 2·0. Measurements on transformers by the above method give figures between 4 to 5, as far as Dr. Morris's experience goes. This meant that that portion of the loss which does not simply increase proportionally with the frequency is rather more than double that which is ordinarily explained by eddy currents. It would be interesting to know whether anyone present has made similar measurements. Such measurements had been made for a maximum flux-density ranging between B=2500 and 5000 lines per sq. cm., and a comparison with the last two columns of the tables on pp. 384–386 showed that Prof. Lyle's results were very closely in agreement with this. Dr. Morris had, however, usually attributed the excess eddy-current loss to current flowing across from plate to plate. Such currents certainly did not occur in Prof. Lyle's experiments. Sometimes the discrepancy between observed and calculated eddy currents was due in part to the existence of higher harmonics in the flux-wave for which the relatively large allowance had not been made. But if it was not due to either of these causes, it certainly would seem to indicate "kinetic hysteresis"—perhaps largely "magnetic viscosity," arising from some kind of molecular eddy current flowing possibly in paths whose specific resistance locally was lower than the measured average value.

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* That figure, namely, which when multiplied by the (thickness of stamping)^2 × (induction density)^2 × (frequency)^2 and divided by the specific resistance of the stampings will give the eddy-current loss.

In § 1 Prof. Lyle asserted that "if the E.M.F. wave impressed on the magnetizing winding is sinusoidal, the flux wave will contain 3rd harmonics etc." It was quite true that the magnetizing *current* contained such harmonics, but unless the resistance of the magnetizing winding was exceptionally high, these harmonics would surely not appear appreciably either in the back voltage or the corresponding flux. The practical significance of the "reflected energy" referred to on p. 393 was therefore very slight.

It seemed to him also that Prof. Lyle's increase of hysteresis might be partly caused by the existence of irregular or unsinusoidal magnetization, which might not be fully allowed for by the method of p. 389. Partly, also, it might be due to imperfect allowance for eddy currents, since these would probably be considerably greater than those calculated owing to the difference in flux-density at the inside and outside of the ring. In this connexion it might be noted (see fig. 3) that Ring I, in which this difference of flux-density is large, gave proportionally greater "kinetic hysteresis" than Ring II.

Mr. T. H. BLAKESLEY said that 10 or 12 years ago he read a paper before the Society describing experiments which pointed to the existence of kinetic hysteresis. The present paper substantiated what he said then.

XXXIV. *On the Practical Determination of the Mean Spherical Candle-Power of Incandescent and Arc Lamps. By* G. B. DYKE, *B.Sc., Salomons Scholar of the Institution of Electrical Engineers.* (Communicated by Prof. FLEMING, F.R.S.) *

[Plate XVI.]

THE following experiments were undertaken at the suggestion of Dr. J. A. Fleming, F.R.S., as a piece of work during the tenure of a Salomons Scholarship, and were carried out at the Pender Laboratory, University College, London, under his direction.

The principal object of the research was to investigate the relation between the mean spherical candle-power of various types of incandescent and arc lamps, and the candle-power measured in a horizontal direction.

It is obvious that some improved method of expressing the efficiency of electric lamps is needed, having regard to the fact that a nominal 16-candle power incandescent lamp may give from 12 to 20-candle power in different directions in a horizontal plane, corresponding to a variation in so-called efficiency of from 5 to 3 watts per candle.

Such a method has been pointed out by Dr. Fleming †, consisting in stating the whole flux of light in lumens per watt, which gives a figure of about 3·5 for glow-lamps.

The expression of the efficiency in this manner involves the determination of the mean spherical candle-power; and the method described in this paper affords accurate results with no greater expenditure of time than that required for one, or at most two, measurements by existing methods.

Apparatus and Method.

The experiments were carried out with the following objects in view:—

(i.) To obtain a series of curves showing the variations of candle-power in incandescent lamps, in a horizontal plane.

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* Read November 11, 1904.

† "The Photometry of Electric Lamps," by Dr. J. A. Fleming, F.R.S. Journal Instit. Electric Engineers, vol. xxxii. p. 165.

(ii.) To obtain reduction factors by which the mean horizontal candle-power may be calculated from the maximum horizontal candle-power, for different types of filament.

(iii.) To obtain reduction factors for obtaining the mean spherical candle-power from the maximum or mean horizontal candle-power.

Objects (i.) and (ii.) were met by designing a carriage, to run on the existing photometric bench in the Pender Electrical Laboratory, provided with a scale of degrees and means for rotating the lamp.

The carriage may be briefly described as follows :—

A strong cast-iron frame A (Pl. XVI. fig. 1) carries four flanged wheels B and a cord grip C. In the centre of the frame a bearing for a vertical shaft D is arranged, supported at E by a collar-bearing. This shaft is hollow for more than half its length, and carries a second sliding tube, which can be fixed in any position by the nut F. To the upper end of this is screwed the lamp-holder G.

On the lower portion of the shaft is fixed a bevel-wheel H and a pair of slip-rings J arranged in a horizontal plane on an ebonite base. On these rings press a pair of gauze brushes K K fixed in spring-holders and mounted on insulating supports.

A scale of degrees is arranged at L (shown dotted in plan), the pointer being fixed to the collar-bearing E.

The vertical shaft is driven by a motor M carried on the same carriage. The motor is mounted in a gun-metal ring N by three screws, which, together with the slots at O O, allow of a small movement in any direction.

A pinion on the motor-shaft engages with the bevel-wheel H, the reduction ratio being 4 to 1. The motor voltage is adjusted until the lamp rotates about four times per second.

Two pairs of flexible leads are brought from suitable plugs, supplying current to motor and lamp respectively, and also a pair for measuring the voltage on the lamp-terminals. All the electrical measurements were accurately made on a potentiometer.

The slip-rings are connected by flexible leads passed through the hollow shaft to the terminals of the lamp-holder. A cord grip P relieves the motor and brush supports from the drag of the flexible cord.

The carriage is pulled along the photometric bench by an endless cord passing over pulleys at the ends of the bench and gripped by a cord grip C. The position of the lamp on the bench is read by means of the wire Q.

The advantage of this rotator is that it is self-contained, and hence can be arranged so as to enable a fixed photometer-screen to be used, making it much more convenient to work with.

The method employed was to fix the lamp in the holder with the filament so placed that the axis of the block on which the filament was formed coincided with the axis of the photometric bench.

This was called the zero position.

The voltage on the lamp terminals was kept at that required to give the marked candle-power in the zero position.

Readings of candle-power were then taken at frequent intervals round the lamp, special attention being paid to those positions in which the candle-power was changing rapidly.

Finally the lamp was rotated, and a series of readings of the mean horizontal candle-power taken.

The value thus obtained was checked against that given as the mean ordinate of the curve plotted from the step-by-step readings; the results in almost every case agreeing to within three-quarters of 1 per cent.

To obtain rapidly the relation between mean spherical and mean horizontal candle-power, a modification of the method devised by Mr. C. P. Matthews was employed.

This method consists in placing several pairs of mirrors round the circumference of a semicircle, and so arranging them that light emitted from a source placed at the centre of the system shall be incident on a white screen, also situated at the centre, at the same angle with the vertical as that which it made on emission from the source. Then, if the source is symmetrical about the diameter of the semicircle, and if Lambert's cosine law holds for the screen used, it can be shown that the intensity of illumination on the screen is proportional to the mean spherical candle-power.

A proof of this is given in Mr. Matthew's paper*; but an interesting method is to deduce it from the well-known Rousseau diagram, as follows:—

Let n = number of pairs of mirrors used.

I_θ = intensity of illumination in a direction making an angle θ with the horizontal.

$I_{max.}$ = maximum intensity = OB.

M.S.C.P. = mean spherical candle-power.

and M.H.C.P. = mean horizontal candle-power.

Let OG1 (Pl. XVI. fig. 2) be the polar intensity curve of a source symmetrical about the vertical AB.

Let CDEF be the Rousseau diagram obtained in the usual way from the polar curve.

Divide the semi-circumference A 1 B into $(n+1)$ parts at the points 6, 5, 4, ... 7.

Project these points horizontally on to CF, and through the projections draw ordinates to the Rousseau curve, shown dotted in the figure.

Bisect the arcs A6, 65, &c., and project the points of bisection in a similar manner, obtaining the full-line ordinates of the curve.

Then, if n is sufficiently large, the whole area CDEF may be considered to be made up of small strips, e. g., $a\,b\,c\,d$, cut off by the full-line ordinates of the curve.

Hence the area CDEF $= \Sigma$ (area $a\,b\,c\,d$)

$$= \Sigma\, \{ad \times \tfrac{1}{2}\,(a\,b + c\,d)\}.$$

Now, with a sufficient degree of approximation,

$$\tfrac{1}{2}\,(a\,b + c\,d) = I_\theta,$$

and

$$a\,d = \frac{\pi}{n+1} \cos\theta\, I_{max.}$$

Hence the area CDEF $= \dfrac{\pi}{n+1} I_{max.}\, \Sigma\, (I_\theta \cos\theta)$.

But M.S.C.P. $= \dfrac{\text{area CDEF}}{\text{CF}}$

$$= \frac{\text{area CDEF}}{2 \times I_{max.}}.$$

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* "An Integrating Photometer for Glow-Lamps and Sources of like Intensity," by C. P. Matthews. Trans. Amer. Inst. Elec. Engin. vol. xix. Nov. 1902, p. 1467.

Hence $\text{M.S.C.P.} = \frac{\pi}{2(n+1)} \Sigma (I_\theta \cos \theta)$.

Now if D = total distance in feet by any one path from source to screen,

and K = reflexion coefficient of a pair of mirrors,

and if the cosine relation holds,

then the intensity of illumination of the screen in candle feet $= \frac{K}{D^2} \Sigma (I_\theta \cos \theta)$.

Hence $\text{M.S.C.P.} = \frac{K}{D^2} \frac{\pi}{2(n+1)} \times$ illumination of screen in C.F.

The following description briefly outlines the construction of the apparatus designed to obtain the ratio $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$.

A wrought-iron band, 4 in. × $\frac{1}{4}$ in., A (fig. 3), stiffened with a small L section, is bent into a semicircle 6 feet in diameter, and fixed by wood supports with its diameter vertical.

Eleven bosses, B, fitted to the outside of the semicircle, carry the mirror supports C; these supports slide radially through the bosses and may be fixed in any position by set screws.

Each mirror support carries a pair of brass plates to which mirrors are clipped, each plate being independently adjustable.

The arrangement is shown in Pl. XVI. fig. 3. The distance from centre to centre of the mirrors of each pair is $8\frac{3}{4}$ inches.

The lamp to be tested is mounted on the rotator at D, at the centre of the semicircle and in the plane of one set of mirrors.

The photometer, of the Lummer-Brodhun pattern, is placed at E, also at the centre, but in the plane of the other set of mirrors.

Both lamp and photometer are so mounted as to allow of their being set accurately to the central position.

The photometer has a 1-inch disk, and the case is cut away at F, so as to permit light to reach the whole surface of the disk from any angle in the vertical plane containing the disk.

The object being to find the relation between the mean spherical and mean horizontal candle-powers, the side G of the photometer must be illuminated with an intensity proportional to the mean horizontal candle-power.

This is done by arranging the pair of mirrors H to slide in a horizontal plane along a railway J formed of an inverted T section.

A matchwood screen K prevents any light reaching the photometer except by way of the mirrors.

This screen is provided with windows opposite the mirrors fitted with slides, so that each pair of mirrors may, if desired, be used separately: this arrangement is of great service during the setting of the mirrors.

The position of the sliding pair is read on a scale arranged on the box-shaped screen protecting them from stray light. This scale is graduated to read off directly the distance from lamp to photometer.

The above system of screens is of course only suited for use in a dark room, but, with a simple arrangement of curtains, no difficulty should be found in using the apparatus in broad daylight.

An approximate idea of the dimensions of the apparatus may be formed from the scales included in fig. 1 and fig. 3.

It will be seen that the formula given above for the determination of the mean spherical candle-power depends for its validity upon the truth of the cosine relation, and also on the identity of the reflexion coefficients of the various pairs of mirrors.

Taking first the cosine relation, this was investigated in the following manner :—

The mirrors on one side of the upper quadrant of the apparatus having been removed, a lamp-holder was devised to slip on the rods of the mirror supports and to be clamped with a set screw in such a position that a lamp placed in the holder in a radial direction pointed directly head-on to the photometer disk.

A 32 C.P. lamp was used for this purpose.

A 16 C.P. lamp, which had been previously calibrated, was then placed in the rotator and, all the window-slides having been closed, the 32 C.P. lamp was placed on each of the mirror supports in turn and successive balances obtained, and the corresponding scale-readings taken.

From these readings, and a knowledge of the reflexion coefficient of the sliding pair of mirrors, the illumination of the photometer screen was deduced and the curves of fig. 4 obtained; the full-line curve taken from the experimental results, and the dotted curve showing the values which would have been obtained had the cosine relation held.

From these two curves fig. 5 follows immediately, and shows the percentage variation from the cosine relation for the particular Lummer-Brodhun screen under test.

The reflexion coefficients of the mirrors were obtained by the usual method on an ordinary photometric bench, each pair of mirrors being tested together.

To compensate for both these disturbing influences, each pair of mirrors was placed at such a distance from the photometer that the illumination on the screen due to each was the same as it would have been, had the cosine relation been rigidly followed and the reflexion coefficients been all the same as the coefficient for No. 1 pair of mirrors (the mirror pairs being numbered as in fig. 2).

The following table shows the results of these investigations and adjustments.

No. of mirror pair.	Illumination less than cosine. per cent.	Reflexion coefficient. per cent.	Illumination less than No. 1. per cent.	Distance lessened. per cent.	Total distance lamp to photometer. inches.	Distance mirror to photometer inches.
1	0·0	68·5	0·0	0·0	73·75	32·50
2	0·1	68·5	0·1	0·1	73·68	32·47
3	0·9	71·4	−2·0	−1·0	74·49	32·87
4	2·1	68·9	1·7	0·9	73·10	32·18
5	3·6	69·1	3·0	1·5	72·66	31·96
6	6·1	69·5	5·1	2·6	71·84	31·55
7	6·1	66·8	7·8	3·9	70·88	31·06
8	3·6	67·3	4·8	2·4	71·99	31·62
9	2·1	69·2	1·4	0·7	73·24	32·25
10	0·9	69·2	0·2	0·1	73·68	32·47
11	0·1	65·1	3·5	1·8	72·44	31·85
12	0·0	70·5		(sliding	pair)	

These adjustments having been made it will be sufficient to insert in the formula

$$K = \cdot 685 \text{ and } D = \frac{73\cdot 75}{12} = 6\cdot 15 \text{ feet.}$$

The formula given above was

$$\text{M.S.C.P.} = \frac{D^2}{K} \cdot \frac{\pi}{2(n+1)} \times \text{illumination of screen in C.F.}$$

But the illumination of the screen

$$= k\frac{\text{M.H.C.P.}}{d^2},$$

where d = total distance in feet from source to screen by way of the sliding pair, and k = reflexion coefficient of sliding pair of mirrors.

Hence $$\text{M.S.C.P.} = \frac{k}{K}\left(\frac{D}{d}\right)^2 \frac{\pi}{2(n+1)} \cdot \text{M.H.C.P.},$$

or, putting in the values of the constants:

$$\text{M.S.C.P.} = \frac{70\cdot 5}{68\cdot 5} \times \left(\frac{6\cdot 15}{d}\right)^2 \times \frac{\pi}{2(11+1)} \times \text{M.H.C.P.}$$

$$= \frac{5\cdot 09}{d^2} \times \text{M.H.C.P.}$$

$$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}} = \frac{5\cdot 09}{d^2} \text{ where } d \text{ is in feet.}$$

Hence a scale could be easily constructed from which the ratio $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$ might be read off directly.

Before passing on to the results of the tests, some interesting figures relating to the absorption of ground glass might be quoted.

In designing the apparatus described above, attempts were made, at the suggestion of Dr. Fleming, to diminish the dimensions of the integrating photometer by fixing ground-glass screens in the windows between the mirrors in order to absorb a portion of the light and so reduce the illumination on the photometer-screen, which with an instrument of small radius would otherwise be uncomfortably brilliant, especially when dealing with arc lamps.

The validity of this device must of course depend upon the constancy of the absorption coefficient of the screen when illuminated by lights of very different spectral composition, and the following experiments were made to investigate this point.

The screen used was a sheet of glass of an average thickness of ·08 inch, ground by sand-blast on both sides.

This screen was set up in a vertical position, 25 inches from the photometer-screen and at right angles to the photometric bench. The source of light under test was set up behind the screen at a fixed distance from it, a photometric balance being obtained by means of a lamp, on the other side of the photometer, whose distance could be adjusted.

The sources used were a 50 C.P. incandescent Focus lamp, a 100 watt Nernst lamp, and a hand-fed continuous-current arc.

However, the results obtained with the Arc lamp can scarcely be compared with the others, as the distance of the lamp from the screen was much greater in this case than in the others, due to the intensity of the light. In the case of the Focus and Nernst lamps the distance from lamp to screen was 25 inches.

Readings were obtained with and without the screen in position, the absorbing power being taken as the ratio of the illumination on the photometer without the screen to the illumination with the screen.

Observations were made on the Nernst lamp from 20 per cent. below its marked volts until the automatic heater came into action, and on the Focus lamp from 2 per cent. above to 10 per cent. below its marked volts.

The results of these experiments are shown in the accompanying curve (Pl. XVI. fig. 6), in which absorbing power is plotted against watts per candle.

The curve shows the general result that the absorption increases with the watts per candle, whence it follows that the red rays suffer a greater absorption than the blue.

It was at once seen from these results that any such screen could not possibly be used for sources of different spectral character, and hence the somewhat bulky apparatus of fig. 3 was necessitated.

RESULTS.

Incandescent Lamps.

In all, nine types of incandescent lamps were tested; the total number of lamps under observation being twenty-five.

The horizontal candle-power was investigated as indicated above, the average number of positions photometered being 26 for 16 C.P. lamps and 84 for 32 C.P. lamps; the greater number in the case of the 32 C.P. lamps being necessitated by the greater distance between lamp and photometer, and the consequent diminished sharpness in the gradations of illumination of the photometer as the lamp was turned round by hand.

The following is a table of the different types of lamps investigated:—

1. Ediswan Flat-Loop Filament. 50–16–B 1·12.
2. Ediswan Double-Loop Filament. 100–16–A ·64.
3. Edison Double-Loop Anchored Filament. 110–V–16 C.P.–E.
4. Ediswan Triple-Loop Doubly Anchored Filament. 110–16–A.
5. Ediswan Triple-Loop Doubly Anchored Filament. 210–16–B.
6. Royal Ediswan Double-Loop Filament. 110–16–A ·58.
7. Ediswan Double-Loop Filament. 110–8–A ·29.
8. Ediswan Quadruple-Loop Doubly Supported Filament. 110–32–B.
9. Premier Double-Loop Filament. 32–110–D. C.

The curves in fig. 7 represent graphically the result of the observations, the ordinates reading candle-power and the abscissæ degrees of angle in a horizontal plane round the lamp; the zero angle, as before mentioned, corresponding to that position of the lamp in which the axis of the loops of the filament is parallel with the axis of the photometric bench.

The full-line curves show the variation in candle-power in a horizontal plane, and the line of strokes gives the M.H.C.P.

These curves show in a very evident manner the very large changes in candle-power which may occur in any lamp when photometered in different positions.

The sharp peaks and depressions which occur in some of the lamps are particularly noticeable; and with a view to investigating the appearance of the filament when observed in these directions, photographs of the lamp when incandescent were taken in a few of these positions.

These show that, as would be expected, the sudden depressions in the curves are due to parts of the filament becoming hidden, either behind another part of the filament, or, in some of the multiloop lamps, behind the anchoring support.

The high peaks, on the other hand, are, as Mr. Russell * pointed out some little time ago, due to the back of the bulb acting as a mirror, the peaks of the curve corresponding to the positions in which part of the filament happens to coincide with the principal focus of the part of the bulb directly behind it. The photographs, two of which are reproduced in Pl. XVI. fig. 8, show this very well indeed. Whilst all of them show a large number of images of the filament in the glass, the sharpness of these images proves that they are at approximately the same distance from the camera as the filament; but in the cases where a peak occurs, and only in such cases, the photograph shows a more or less pear-shaped and blurred bright region; indicating the presence of an image which is not at the same distance.

With regard to the relations between the mean spherical, mean horizontal, and usually measured candle-power, the results of the observations on the types of incandescent lamps tested are shown in the following tables, in which the following abbreviations have been used:—

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* "Mean Horizontal and Mean Spherical Candle-power," by Alex. Russell, M.A., Journ. Inst. Elect. Eng. vol. xxxii. p. 635.

CP. = mean of C.P.'s in positions in which axis of loops of filament is parallel to photometric bench.

M.H.C.P. = mean horizontal candle-power.

M.S.C.P. = mean spherical candle-power.

Lamp No.	C.P.	M.H.C.P.	M.S.C.P.	Reduction Factors.		
				$\frac{\text{M.H.C.P.}}{\text{C.P.}}$	$\frac{\text{M.S.C.P.}}{\text{C.P.}}$	$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$
1	16·6	16·1	12·5	·97	·75	·78
2	16·6	16·0	12·8	·97	·77	·80
3	16·4	16·2	13·0	·99	·79	·80
4	15·9	14·8	12·5	·93	·79	·85
5_A	16·0	14·9	11·7	·94	·73	·78
6_A	16·0	15·1	12·2	·94	·76	·81
7_A	8·0	7·5	6·2	·93	·77	·83
8_A	30·5	29·3	24·5	·96	·80	·84
9_A	31·8	29·9	24·2	·94	·76	·81

It is obvious, however, that these reduction factors are of very little use unless they remain sensibly constant for different lamps of the same type of filament.

The following table shows the reduction factors for seven lamps of type 5 :—

Lamp No.	Reduction Factors.		
	$\frac{\text{M.H.C.P.}}{\text{C.P.}}$	$\frac{\text{M.S.C.P.}}{\text{C.P.}}$	$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$
5_A	·94	·73	·78
5_B	·92	·71	·78
5_D	·92	·72	·78
5_E	·96	·74	·78
5_F	·93	·72	·78
5_G	·95	·74	·78
5_H	1·00	·79	·79

This table shows clearly that only one reduction factor is of practical utility, namely, the ratio $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$.

This one factor, however, is remarkably constant and appears to be very little altered by a distorted filament; thus, although in 5_H the filament was so much bent as to almost touch the bulb in one place, the factor is only altered by 1 per cent.

The unsatisfactory nature of the present method of photometering incandescent lamps is seen in the second reduction factor $\frac{\text{M.S.C.P.}}{\text{C.P.}}$.

This shows that the mean intensity of the light emitted is only about 75 per cent. of the nominal or marked intensity, and also that the mean and marked intensities bear no definite relation to one another, even in the case of lamps of the same type of filament.

A more satisfactory method would be to determine the mean horizontal candle-power of each lamp by using a rotator similar to that described in the paper, and then to obtain the M.S.C.P., employing a factor previously determined for the particular type of filament by means of an integrating photometer as explained.

The marking on lamps thus photometered would then be proportional to the amount of light emitted, and hence would be a figure definitely determining their utility as a means of illumination.

In this manner the suggestion made by Dr. Fleming in his paper on the "Photometry of Electric Lamps"* could be practically carried out with but little more trouble than the present unscientific method of marking incandescent lamps.

Arc Lamps.

The Arc lamp being symmetrical about the vertical will not require revolution in the integrator, but the readings will give directly the ratio of the mean spherical to the horizontal candle-power.

Preliminary experiments only have been made with arc lamps, but these have been sufficient to show that arc lamps may be dealt with in the integrator with but little more trouble than incandescent lamps.

In conclusion, the author wishes to express his indebtedness to Dr. J. A. Fleming, F.R.S., and to Mr. W. C. Clinton, B.Sc., for their constant advice and assistance during the progress of the work.

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* Journ. Inst. Elec. Eng. vol. xxxii. p. 165.

Discussion.

Dr. J. A. **Fleming** said that he was glad to have an opportunity of congratulating Mr. Dyke on the completion of a useful piece of work. The paper was not a long one but it represented a good deal of work, both constructional and observational, and owing to the necessity for possessing the exclusive use of a secondary battery during the observations, that work had chiefly to be done at night. He regretted that it had not been possible to bring down the integrating photometer to exhibit it to the Meeting, but it was a heavy and somewhat bulky object, and the journey might have deranged the setting of the mirrors. The great feature of this form of integrating photometer, due to Mr. C. P. Matthews, was that it was so easy to compensate for variations from the cosine law and for irregularities of reflecting power by altering the distances of the pairs of mirrors from the centre. The combination of such a form of integrating photometer with a rotating lamp-holder afforded the means of obtaining a mean spherical candle-power with the greatest ease when once certain constants had been obtained.

The chief result in Mr. Dyke's paper, viz.: the constancy of the ratio between mean spherical candle-power and mean horizontal candle-power for different types of glow-lamp, was certainly a very useful and important practical result. The figures and the curves given by Mr. Dyke showed what great precautions must be taken in drawing inferences from single observations in a horizontal direction made on an incandescent lamp. Although in some lamp factories, spinning lamp-holders had been employed to get a mean horizontal candle-power, yet this figure was not of much scientific value unless it served, as Mr. Dyke had shown it might do, as a stepping-stone to obtain the mean spherical candle-power, and therefore the total flux of light from the lamp.

The instrument described was also interesting, as it showed how conveniently it could be employed for investigating the law of variation of brightness of various substances when illuminated at different angles. Mr. Dyke's figures showed that the screen of the Lummer-Brodhun photometer obeyed very closely the cosine law up to about 60 degrees, and, as had been mentioned, a compensation for deviation beyond that angle or at any angle could be made by setting in the double mirrors.

In conclusion, Dr. Fleming expressed the hope that Mr. Dyke might find an opportunity for continuing the work with other types of lamps, as it was a result of very considerable practical value to know that the mean spherical candle-power of an incandescent lamp could be obtained by multiplying the mean horizontal candle-power of the lamp placed with its axis vertical by the factor 0·78 very approximately the same for all classes of incandescent lamps.

Mr. C. C. PATTERSON said that a set of experiments similar to those described by Mr. Dyke were carried out at the National Physical Laboratory in the early part of the year. A large number of lamps with different types of filament had to be compared for their percentage drop in candle-power on life test. In order to compare them satisfactorily, it was necessary to obtain the exact watts per candle at which they were burning. Owing to the different types of filament employed the watts per M.H.C.P. would not have been a measure of the intrinsic brightness of the carbon filament, and it was found necessary in the case of one lamp of each type to measure the mean spherical candle-power, using the factor so obtained for reducing the M.H.C.P. of the other lamps of the same type.

The method adopted was to measure the candle-power of each lamp in nine vertical planes, taking points in each plane at angular distances of 10 degrees. For deducing the M.S.C.P. from the mean distribution diagram so obtained, the method given by Mr. Russell in the paper referred to by Mr. Dyke was used. The surface of the sphere was divided respectively into 4, 6, and 10 zones of equal area, and in each case the average of the mean illuminations for the zones was taken as the M.S.C.P.

The following values were thus obtained:—

Mean Candle-power of

	4 zones.	6 zones.	10 zones.
Lamp 1	14·47	14·3	14·29
„ 2	...	5·16	5·14
„ 3	12·26	12·16	12·12

The close agreement between the mean of 6 zones and 10 zones showed that it was only necessary to take readings at 6 points in a vertical plane, rotating the lamp at the same time, in order to obtain an accurate measurement of the M.S.C.P. This method was adopted in the remainder of the lamps to be tested, and the following results, corresponding to the values given by Mr. Dyke, were obtained for the ratio $\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$

Lamp 1	0·84	Lamp 5	0·88
„ 2	0·84	„ 6	0·88
„ 3	0·85	„ 7	0·90
„ 4	0·87		

The reason why these ratios were rather higher than those given in the paper was due to the fact that the method adopted did not take into account the light cut off from the lamp by its socket.

Mr. W. **Duddell** pointed out that if the peaks in the curves of horizontal intensity were due to reflection from the bulb, one would expect that their position would vary with the distance between the lamp and screen. He asked the Author if he had noticed any such variation. The integrating photometer described in the paper was very cumbersome, and he would like to know in what respects it was superior to Blondel's lumenmeter.

Dr. C. **Chree** remarked that it would be interesting mathematically to express the illumination in a series of spherical harmonics.

Dr. R. T. **Glazebrook** expressed his interest in the fact that some of the results of the Author had been confirmed by Mr. Patterson at the National Physical Laboratory. He had recently been in America, and found that the integrating photometer was in use for measurements upon arc-lamps, but that incandescent lamps were measured with a rotator, no particular regard being paid to the M.S.C.P. With regard to the observations of Dr. Chree, he said that Mr. Patterson could give him sufficient data to carry out the calculations.

The AUTHOR, in reply to Mr. Duddell, said that although he had not varied the distance between the lamp and the screen, he had taken photographs of the "blaze" of the lamp at different distances, and found that it was always the same.

XXXV. *The Measurement of Small Differences of Phase.*

By W. E. SUMPNER, *D.Sc.**

WHILE investigating recently the behaviour of wattmeters, transformers, and similar apparatus, the writer has found it necessary to measure small phase-differences occurring in the working of alternating-current plant. The methods hitherto available for such measurements are not at all satisfactory when the angles to be determined are small. Phase-meters have been constructed for commercial purposes, but the angular deflexion of the pointer of these instruments is, as a rule, smaller than the phase-difference to be measured, so that when the latter is as small, or smaller, than one degree, such instruments, even if perfectly accurate, are quite unsuited for the purpose. All other known methods necessitate the simultaneous reading of three instruments. The best-known method needs a wattmeter, a voltmeter, and an ammeter. Among wattmeters we may include all instruments of the double current, or double voltage, type, whether dynamometers, current-balances, or electrometers. In all these cases the value of cos θ, where θ is the angle of phase to be determined, is measured by dividing the wattmeter reading by the product of the readings of the other two instruments. The percentage error made in measuring cos θ is thus greater than the corresponding errors made in reading the separate instruments. The method only gives fair results when the angle θ is large, and instruments of suitable range are available. When θ is small the method is hopeless, for since cos 1° is ·99985, and cos 0°·5 is ·99996, it is clear that no such method involving the measurement of three deflexions can be anything like accurate enough for determining values of θ less than one degree.

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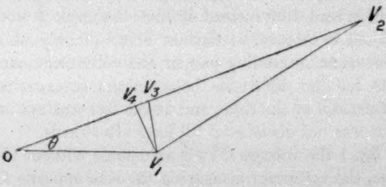
* Read November 25, 1904.

All other known methods are based upon the three-voltmeter method of measuring power factor, or phase-difference. This method has been much criticised at different times, but its limitations were all fully pointed out in the original paper (see Proc. Roy. Soc. vol. xlix. March 1891) in which Professor Ayrton and the present writer first drew attention to it. It is a method for measuring power, or power factor, which cannot compare in ease or simplicity with the wattmeter method, when a suitable wattmeter is available, and the accuracy of the instrument is not in doubt. But wattmeters and other alternating-current meters, owing to the absence of iron-cored magnetic circuits, are not nearly so sensitive as, and do not possess anything like the range of, the corresponding direct-current instruments. A wattmeter of the right range is not always available, and in such cases the three-voltmeter method, or some modification of it, has often proved a convenient substitute.

The errors arising in practice in measuring phase by the three-voltmeter method are serious for low-power factor circuits, that is for values of θ approaching 90 degrees, but are not so important when high-power factors have to be measured. The perfection of the method in theory, and its limitations in practice, are exactly comparable with the determination of the angles of a triangle from measurements of its three sides. If these sides are measured accurately the angles can be correctly calculated in all cases, but for given percentage errors made in estimating the sides, the resulting error made in calculating the angle will largely depend on the shape of the triangle. Thus, in fig. 1, the angle θ will be determined much less accurately from measurements of the sides of the triangle OV_1V_2, where OV_1 and OV_2 are very different in magnitude, than from equally accurate measurements of the sides of OV_1V_3, in which OV_1 and OV_3 are supposed to be nearly of equal length. The measurement will be most accurate, for given percentage errors in the sides, if the length of the perpendicular V_1V_4 on the line OV_2 can be measured, since then the value of sin θ will be known as accurately as the ratio of V_1V_4 to OV_1 can be determined. If the sides of the figure represent voltages in different phases, the measurement of the ratio in question can in this case be determined as accurately as two voltmeter readings can be read and compared.

Fig. 1.



If we denote by v_1 and v_2 the lengths of the sides of the triangle forming the angle θ, and if the third side of the triangle opposite θ is represented by v, it can easily be shown that

$$\cos\theta = 1 - \tfrac{1}{2}\phi^2, \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (1)$$

where

$$\phi^2 = \frac{v^2 - d^2}{v_1 v_2}, \text{ and } d = v_1 - v_2.$$

This is true whatever the value of θ, but for small values θ becomes identical with ϕ, and we have in circular measure

$$\theta = \sqrt{\frac{(v-d)(v+d)}{v_1 v_2}} = \frac{v}{\sqrt{v_1 v_2}} \sqrt{1 - \frac{d^2}{v^2}} \quad . \;\; . \;\; . \quad (2)$$

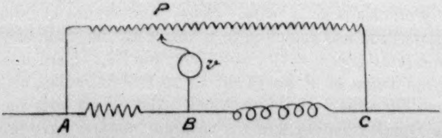
Some years ago the writer tested the phase-difference between the primary and secondary voltages of a small equal-ratio transformer by joining a low-reading hot-wire voltmeter to two terminals, one on each coil, and by connecting the other two terminals with a wire. Some numbers taken from an old note-book and referring to a test with the secondary on open circuit give $v_1 = 72$; $v_2 = 73{\cdot}8$; $v = 1{\cdot}9$. Here $d = 1{\cdot}8$, and the value of θ works out to be 0·0084 radian, or 0·48 degree, for which $\cos\theta$ is 0·99996. But recent tests on the same transformer by a more accurate method have proved that this estimate of θ is no less than 24 times too great, that its real value is only 0·00037 radian, or 0·021 degree, and that consequently the value of cos θ only differs from unity by 7 parts in a hundred million.

The explanation of the error made in the earlier test is simply the difficulty of determining the difference d with sufficient accuracy when it is about the same magnitude as v and small compared with v_1 or v_2. If, assuming v_1 correct, v_2 had been read 73·9 instead of 73·8, the angle θ would have worked out to be zero, so that an error of only about one-tenth per cent. in reading one of the voltmeters completely accounts for the difference between the two results. This was not noticed at the time, and as the test was not repeated, the error was not discovered till long afterwards.

If in fig. 1 the voltage OV_2 is adjustable without alteration of phase, the voltmeter measuring the side opposite θ can be made to give a minimum reading by altering OV_2. The minimum reading will be V_1V_4 perpendicular to OV_4, and we can determine sin $\theta = v/v_1$ as accurately as we can read the voltmeters.

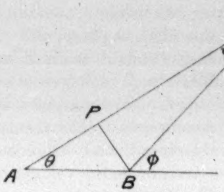
Thus the phase relation of the voltages of two coupled alternators can be determined by a two-voltmeter method in this way, by simply adjusting the excitation of one of them, this one being preferably run on open circuit. But in any case, by shunting the larger of the two voltages forming the angle θ with a non-inductive resistance, it is possible, by tapping this resistance at various points, to get a minimum reading of the voltmeter v, the ratio of which to the unshunted voltage will give the value of sin θ. This method is particularly suited to determine small angles of phase-difference, since voltmeters having different ranges may be used, and sin θ or θ can be measured as accurately as the two readings can be determined. The method is not always applicable, since the conductors absorbing the voltages to be compared

Fig. 2.



may be in series. In such cases the following bridge method can be used. Suppose that in fig. 2 A B represents a non-inductive resistance, and B C an inductive resistance in series with it; the phase-difference ϕ between the current flowing through B C, and the voltage across B C, can be determined by connecting the points A and C to the terminals of a suitable non-inductive resistance A P C, and finding upon it a point P such that when this point is electrically connected with B through a voltmeter the reading v of this instrument is a minimum. The vector figure representing the voltages is shown in fig. 3. In this figure B P

Fig. 3.



represents v and is perpendicular to A C. If v_1 and v_2 are the voltages A B and B C respectively, the phase-angles θ and ψ by which these voltages differ from the voltage A C are such that

$$\sin\theta=\frac{v}{v_1},\quad \sin\psi=\frac{v}{v_2},\quad \phi=\theta+\psi.$$

We also have $v_1 v_2 \sin\phi = v v_3$ where v_3 is the voltage A C.

The former equations illustrate a method of determining ϕ by observing v, v_1, and v_2, while the latter equation indicates a method of determining $\sin\phi$ from these measurements together with an observation of v_3. It is to be noted that the conductor A P C need not necessarily be noninductive. It should be of the same inductive character all through, and such that it is possible to adjust the point P connected with the voltmeter. The point P will then move along the vector A C (see fig. 3) as the adjustment is varied. Moreover there is no disadvantage in choosing a small noninductive resistance

A B such that the voltage v_1 is small compared with v_2, provided always that suitable voltmeters are available for making the requisite measurements. One of the disadvantages of the three-voltmeter method for the measurement of power factor, is that the noninductive resistance A B should absorb about the same voltage as the conductor B C, and hence a greater amount of power than the load under test. If the bridge method just described is used, the load B C may absorb several hundred volts, while v_1 and v may each be less than a volt. Where A B is negligible compared with B C, it of course follows that ψ is small compared with θ, and that θ and ϕ can in general be considered the same. Now in order to determine one of the angles by the above method, all that is really needed is to have two voltages, one like A B in phase with the current, and the other in phase with either B C or A C. One of these voltages should be fixed in magnitude, and the other adjustable. Such voltages can be obtained with all requisite accuracy for most purposes by the aid of current and voltage transformers. The former must have its primary in the main circuit, and its secondary closed through a noninductive resistance, the voltage on which will be in phase with A B The latter must be a constant potential transformer with its primary across A C or B C, and its secondary closed through a noninductive resistance, the voltage on which will be in phase with that on the primary. The secondary voltage need not be greater than corresponds with the length A P in the figure. The two secondary circuits must be connected together at one point to determine a common potential represented by the point A in fig. 3. Such an arrangement makes the method adaptable to loads absorbing high potentials, or taking heavy currents. The phase-differences introduced by such transformers, if of good design, can in practice be reduced to about a tenth of a degree, and hence, for values of θ such as are usual, no appreciable error will be introduced into the measurement by them.

The writer has tested this two-voltmeter method in all the various forms above indicated, and found it satisfactory. It is not so simple or so accurate as the wattmeter method when an instrument of the right range is available, and when the value of θ to be determined is large. But its great adaptability to loads of all kinds makes it a useful laboratory method. Under ordinary circumstances, a hot-wire voltmeter reading up to 1 volt can be used for taking the two measurements v and v_1, while the voltage v_2 on the load can be taken by the instrument necessarily used on the circuit. It is then only necessary to find two noninductive resistances, one such as to absorb about one volt when traversed by the load current, and the other suitable for the voltage v_2 or v_3, and having a portion of it (taking about a volt) of such a structure that a voltmeter can be connected to it at various points.

It is, however, when the phase-difference to be determined is small that the method is most useful and most accurate, while the wattmeter method altogether fails. The vector diagram, shown in fig. 3, applies whatever the frequency or wave form of the current, and even when the current is unidirectional but varying in strength. In a test in which a hand-regulated direct-current arc was put in series with a noninductive resistance, it was found, with $v_1=70$ volts and $v_2=40$ volts, that the minimum value of v was less than 0·1volt. A lower voltage could not be measured with any certainty with the hot-wire voltmeter used. It follows from this test, that the power factor of the direct-current arc cannot differ from unity by more than 5 parts in a million, and the phase-difference ϕ between the voltage and the current works out to be about 0·003 radian. A similar test, made on the small equal-ratio transformer already referred to, with the primary subjected to over 80 volts, gave as minimum voltage v, a value estimated to be only 0·03 volt. This would correspond with a phase-difference of only 0·00037 radian, or 0·021 degree, and a value of cos ϕ differing from unity by only seven parts in a hundred million.

Voltmeters for measuring minute alternate voltages are not procurable. For small phase-angles v will be small compared with either v_1 or v_2. If the apparatus to be tested is such that the application of high voltages to it would either do it injury or alter its working conditions, the method becomes impracticable when the angles to be measured are small.

In order to overcome the difficulty arising from the absence of sensitive alternate-current voltmeters, the writer has adopted the drastic method of commutation. By such means it is possible to make use of sensitive direct-current instruments for the measurement of alternating voltages. Suppose the alternator used for the tests is made to drive a synchronous motor constructed with at least four poles and having on its rotating spindle a commutator with as many metallic sectors as there are current reversals in the time of a revolution. Let half of the sectors (the first, third, &c.) be in metallic connexion with each other, and the other sectors be insulated. A pair of brushes suitably spaced on the commutator will then be in metallic connexion during half the period, and insulated from each other for the other half. But the instant at which the brushes are connected will not be the same as that at which the current reversal takes place, except for special positions of the brushes. If the brushes are adjusted to this position, and the alternating voltage to be measured has its terminals connected to the brushes with a direct-current voltmeter in circuit, this instrument will measure half the mean value of the ordinate of the positive portion of the alternating wave of potential, while the true value of the alternating voltage will be the square root of mean square of this ordinate. The sectors are supposed to be all equal to each other, and the width of the gap between successive sectors should be small compared with the circumferential width of the sectors, though for wave-forms approximately sinuous the gap width can be an appreciable fraction of the sector width without causing any appreciable effect on the readings. Theory shows that if the gap width is a small fraction ϵ of the sector width, the fractional error of the reading is represented by $\pi^2\epsilon^2/8$ for sine currents. It follows that if ϵ is one per cent. the error is only 1/80 per cent., while ϵ has to be 9 per cent. before the error amounts to one per cent. The sensitiveness of the arrangement might have been doubled if the commutator had been constructed so as to reverse the voltage at every half period by connecting alternate sectors to two slip-rings. But further sensitiveness was not required, and hence only the simplest form of cmomutator was tried.

For each wave-form there is a definite value f for the ratio of the square root of mean square of the ordinates to the arithmetical mean. For sine waves this ratio is

$$f = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11.$$

For such waves the reading of the direct-current voltmeter would have to be multiplied by 2·22 to get the reading of an alternate-current instrument arranged so as to measure the voltage directly. For other wave-forms the multiplier would be $2f$, where f may be called the form factor of the alternating voltage. By using a commutator in this way, the most delicate Thomson reflecting-galvanometer can be utilized for measuring alternating voltages of the same magnitude as the smallest direct-current voltages to which such an instrument responds.

The method would in practice prove very tedious owing to the need for carefully adjusting the brushes before taking a measurement, since the adjustment required will depend on the phase of the voltage under test. But if the commutator be rotated nearly, but not quite, synchronously with the speed correspondiog with the frequency, the galvanometer or voltmeter will show "beats," the interval between which lengthens as the speed of synchronism is approached. Under such conditions, the phase of the brush contact will automatically adjust itself whatever the phase of the voltage under test; and the maximum reading of the direct-current instrument will measure half the arithmetical mean of the ordinates of the curve representing the alternating voltage.

Any method of driving in which the speed can be delicately adjusted can be used for running such a commutator. A shunt motor with adjustable resistance either in series or in the shunt circuit would of course do. But for alternating currents of usual frequencies, a particularly convenient piece of apparatus is available in the induction motor. The speed of the rotor of such a motor, when running light, differs from the synchronous speed by a minute amount called the "slip," the amount of which in many cases is less than one-tenth per cent. For currents of 50 cycles per second such a value of the slip would correspond with beats at intervals of 20 seconds, or with a 10 second interval between the maximum positive and the maximum negative readings of the direct-current instrument.

A commutator of very simple construction, as above described, when attached to the spindle of an induction motor, and used with a direct-current instrument, forms in practice a satisfactory means of measuring minute alternate-current voltages. The writer has found it very serviceable in connexion with the measurement of alternating magnetic fields. The reliability of the apparatus was thoroughly tested in various ways, as, for instance, by putting in parallel (1) a standard voltmeter suitable for either direct or alternate voltages, and (2) a direct-current voltmeter in series with the commutator. The ratio of the two readings in volts of the two instruments was found to be exactly 2 on direct-current circuits with the commutator running; but for the alternating voltages produced by a small rotary transformer run from the direct current side, the ratio was found to be 2·28 when the instruments were shunted to a portion of a non-inductive resistance connected with two slip-rings of the rotary. These ratios were the same whatever instruments were used, and for different resistances put in series with the direct-current instrument, provided due allowance was made for these resistances when interpreting in volts the reading of the latter instrument. But although the ratio of the reading of the alternate-current voltmeter to the maximum reading of the direct-current instrument was always found to be 2·28 when only noninductive resistances were included in the circuit connecting two slip-rings of the rotary, this number was found to vary with the inductiveness of the load current taken from these slip-rings. Thus the parallel arrangement of instrument just described, when shunted to a suitable noninductive resistance traversed by the current passing through an iron-cored choking-coil consisting of one of the coils of an old hedgehog transformer, yielded a ratio of 2·36 for the readings of the two voltmeters when the current was 10·6 amperes, and the frequency 43 cycles per second. A similar experiment on a hand-regulated alternating-current arc for a current of 11 amperes of the same frequency, yielded a ratio of 2·55. In each case about a dozen different observations were taken, and the individual values of the ratio found in each set of tests agreed with each other as accurately as it was possible to read the instruments observed. Since the ratio between the two readings is $2f$ where f is the form factor of the voltage, or the ratio of the square root of mean square of the voltage to its arithmetical mean value, it follows that the form factor of the slip-ring voltage is 1·14, that of the current through the choking-coil is 1·18, and that of the alternating-current arc is 1·275, the theoretical value for sine-currents being 1·11. The wave-forms of the alternating current through an arc, or through a large choking-coil with a strongly magnetized iron core, are quite exceptional. For ordinary currents produced by the rotary, it was found sufficiently accurate to take a constant of 2·3 for direct-current voltmeters used with the commutator.

It now became possible to measure the voltages represented by the lines in figs. 1 and 3, even when some of these voltages were small compared with a millivolt. Moreover, as the angles to be measured are determined from the ratios of voltages, and not from their actual values, it was possible, by measuring each voltage by the commutator method, to eliminate the constant of the instrument, assuming only that the form factor is the same for all the voltages measured.

Thus the inductiveness of a pair of lead plates in slightly acidulated water was tested by the method shown in fig. 3. A current of 8·2 amperes was passed through a noninductive resistance AB in series with the lead plates BC. The voltage BC on the plates was 18·7 ; that of AB was $0·6\kappa$, while BP the minimum voltage v was $0·004\kappa$, where κ is the constant previously referred to. It follows that $\sin\theta$ is 0·0067 and $\cos\theta$ is 0·999977. A number of similar measurements were made with very satisfactory results.

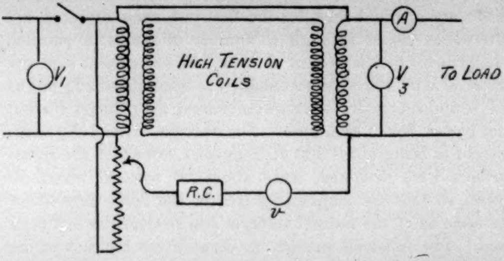
A small current-transformer for oscillograph purposes was tested for phase-difference between primary and secondary currents. This transformer was designed to reduce currents in the ratio of 20 to 1, and for use on a 6000 volt circuit, for a primary current not exceeding 10 amperes, and to supply a secondary current to any closed circuit not absorbing more than 2 volts. Noninductive resistances were put in series with the primary and secondary windings, and were electrically connected so as to form a point represented by O in fig. 1.

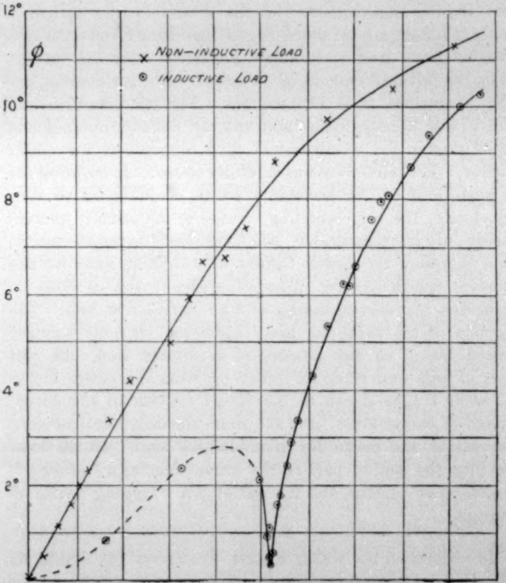
The former for a current of 8 amperes absorbed 0·66 volt, represented by OV_1. The secondary resistance could be tapped at various points, and the minimum voltage v represented by V_1V_4 produced a reading of 0·50 millivolt. Allowing for the constant 2·3, we have ·00115 volt as the value of v. It follows that $\sin\theta$ is ·00175, and θ is 0·10 degree for a primary current of 8 amperes. For a primary current of 10 amperes θ was found to be 0·088 degree, and for a current of 7 amperes 0·114 degree. Such values of θ yield values of $\cos\theta$ differing from unity by about one part in a million, and are such that the wave-forms of primary and secondary currents must correspond with almost absolute accuracy.

Several small instrument transformers as supplied commercially for wattmeters were tested under their normal working conditions, the noninductive resistance put in the secondaries being very small compared with the total secondary resistance, and only absorbing a few millivolts. In one case it was found that the phase-difference between the primary and secondary currents of a current-transformer was 2·32 degrees, and in another as much as 4·2 degrees.

A number of larger transformers for power purposes were tested for phase-differences between primary and secondary voltages. In several cases the phase-difference on open circuit was found to be of the order of a tenth of a degree, and on full load of the order of one degree. On an inductive load the phase-difference found was not so large as a rule as for the same secondary current through a noninductive circuit. The curve shown in fig. 4 represents the result of some tests on two transformers of 3 kilowatt capacity intended to work between voltages of 100 and 1000, and with currents of 100 cycles per second. One transformer was used to step up the volts from V_1 to V_2, the other to step down from V_2 to V_3. The arrangement of the circuits is shown in the diagram at the top of fig. 4. A noninductive resistance was put across V_1, and the phase-difference between V_1 and V_3 was tested by the two-voltmeter method already described. The phase-difference found is for the double transformation and is approximately twice that for each transformer. The load for the first curve was composed of lamps, and is noninductive; that

Fig. 4.


A
V1
High Tension Coils
V3
To Load
R.C.
v


12°
10°
8°
6°
4°
2°
φ
× Non-inductive Load
⊙ Inductive Load
0
10
20
30
40
50
Load in Amperes

for the second curve consisted of a hedgehog transformer, the secondary of which was, for the first test, open-circuited, but afterwards closed through a number of lamps in parallel. The primary took a current of nearly 25 amperes at a power factor of 0·09 when the secondary was open-circuited; but as the lamp-load on the latter was increased the primary current and power factor each rose. The voltages V_1 and V_3 used, instead of being about 100 at 100 cycles, for which the transformers were designed, were about 50 volts at about 50 cycles, so that the magnetic fluxes in the cores were about the same as if the normal voltages and frequencies had been used. The full-load current is 30 amperes for each of the low-voltage coils. The phase-difference between V_1 and V_3 for the two transformers with the second on open circuit, is only 0·109 degree or 0·054 degree per transformer. For a noninductive load it increases regularly with the current. For the full-load current of 30 amperes it is 9·4 degrees, and for 50 amperes it is 11·5 degrees. For the inductive load there was a remarkable, and sharply defined, minimum of 0·36 degree for a current of 25·5 amperes at low-power factor. This minimum was carefully tested. It occurred for a small load on the secondary which could be varied very gradually, the corresponding change in the primary current being only just measurable. For larger loads on the secondary, and therefore for power factors approaching unity as the current increased, the phase-difference curve is seen to approach that corresponding with a noninductive load. The portion of the curve for small inductive currents was not tested owing to the absence of a suitable load, but two special tests were made for lower currents at a power factor of about 0·1, as shown on the dotted portion of the curve, which of course must have the same starting-point for zero current as the curve for noninductive load. In all probability the dotted part of the curve denotes a "leading" condition of current, and the rest of it a "lagging" state of current.

In conclusion the writer desires to express his thanks to Mr. David Owen for much valuable assistance in carrying out several of the measurements above referred to.

DISCUSSION.

Dr. A. HAY congratulated the Author, and said that the determination of power factors was a matter of importance. He had found that alternators tested with water resistances gave better results than when tested with lamps. The phase-differences between the volts and the amperes in the water resistances were generally greater than those indicated by Dr. Sumpner. He asked the Author what current-density he used, and pointed out that the result would depend upon the current-density at the electrodes. A minimum phase-difference could be obtained by loading up the secondary of an ordinary transformer, and was due to an increase in the power factor up to a certain point and then a decrease. We were dealing with lag in both cases.

Mr. A. CAMPBELL said that some years ago (Inst. of Elect. Engineers, April 1901) he published a description of the method of measuring a phase-difference, close to 180°, between two independent voltages by the first method mentioned by Dr. Sumpner, viz., by shunting the larger by a high resistance, picking off from it a part as nearly as possible equal to the other, and measuring the distance resultant. A test on a small transformer gave a phase-difference of about $(180^\circ - 0^\circ{\cdot}15)$. Dr. Sumpner, however, had considerably improved the method; he ensured the sufficient equality of the components by adjusting until the resultant was a minimum,—a simple and beautiful device. In his discussion of the three-voltmeter method, Dr. Sumpner assumed that three instruments were necessary. Mr. Campbell had found that in many cases a single instrument with a quick change-over switch was quite sufficient provided a number of readings were taken in rotation and averages obtained. With respect to Dr. Sumpner's method of measuring small alternating voltages by rectifying them, although it might give good results when the wave-form was known, he thought that in the very cases where such measurements were wanted the wave-form was not known. Small differences of "power" phase may be as much due to small difference in wave-form as to small displacements of similar waves: thus the "difference" resultant was probably nearly always of very uncertain wave-form.

Mr. T. H. **Blakesley** expressed his interest in the paper, and said that many years ago he made use of commutation in order to work with direct-current instruments. He had not used a high-resistance galvanometer, so that considerations of the self-induction and the capacity of the coils of the instrument were avoided.

Mr. W. **Duddell** supported Mr. Campbell in his contention that small differences of "power" phase might be due to small differences in wave-form. He had used water resistances in testing large alternators, but had never found any phase-difference between the voltage and the current. He had usually worked with high voltages, so that condenser effects due to polarization (of the order of one volt) would have little effect.

Dr. C. V. **Drysdale** said he had given a good deal of attention to the measurement of small phase-differences, and had found it advantageous to measure the sine of the phase-difference instead of the cosine. This could be accomplished by putting a condenser in series with the pressure circuit of a wattmeter so as to produce a phase displacement of nearly 90°. In these circumstances he had previously shown that with ordinary conditions the watts were equal to the wattmeter reading *plus* a constant multiplied by the sine of the angle of phase-difference. When, as was usually the case, sin θ was nearly equal to unity it was only necessary to add a constant to the wattmeter reading.

Dr. W. E. **Sumpner**, in reply to Prof. Hay, said he had worked with low current-densities and was unable to say what would happen if the current-density was increased. With regard to rectification, he was aware of the effect of the shape of the wave in modifying the numerical value of the ratio of the virtual to the mean voltage. He had made experiments with currents of very different wave-forms, and the difference in the value of this ratio for normal and extreme cases was only 20 per cent. His methods were suggested as a means of getting an *approximate* value of very small phase-differences, which could not be determined in other ways. The difficulties referred to by Mr. Blakesley were not encountered, as he had worked with a moving coil galvanometer of small capacity and self-induction. He was afraid that the observations of Dr. Drysdale raised questions which it would take too long to discuss at the present meeting.

XXXVI. *A High-Frequency Alternator.*

By W. Duddell *.

[Plates XVII. & XVIII.]

In view of the increasing interest taken in high-frequency alternating currents, especially in connexion with wireless telegraphy, a description of a small alternator designed by the writer in 1900 and of the difficulties encountered in its construction may prove of use to other experimenters on this subject.

The alternator was made for some experiments on the resistance of the electric arc †. As the experiments progressed, and it was found that with each increase of frequency the solid arc behaved more and more like an ordinary resistance, the alternator was altered until finally the very high frequency of 120,000 ~ per second was attained. The inductor type of alternator was chosen; the final design is shown in plan and elevation in Pl. XVII. figs. 1 and 2.

The inductor, 6 cm. in diameter, was built up of 53 disks cut out of ferrotype plate, the iron being about 0·007 inch thick, enclosed between two plates 0·019 inch thick, the whole clamped together on a tool-steel spindle. The inductor alone weighed about 200 grammes. As at first constructed, the edge of the inductor had 30 V-shaped notches cut in it so as to have 30 flat-topped teeth.

Surrounding the inductor is a laminated soft-iron ring having two inwardly projecting poles; the clearance between these and the teeth of the inductor, which formed the air-gap of the machine, being less than 0·1 mm. The ring itself is wound so that a direct current flowing round the winding tends to produce lines of force from one pole to the other through the inductor. The number of lines of force varies from a maximum when the teeth on the inductor are opposite the poles to a minimum when the pole are in front of spaces. This variation in the total number of lines of force produces alternating E.M.F.'s in any winding either on the pole-pieces or on the ring itself; the movement of the inductor through the distance between two consecutive teeth producing one complete period. One winding on the ring could serve to carry both the direct field-current which magnetizes the ring and the alternating current produced. In practice, owing to the great importance of any self-induction in the alternate-current circuit, it was found advisable at high frequencies to use separate windings for the two currents. The ring was, therefore, wound in three sections as follows:—Two sections, each consisting of a single layer of 113 turns of No. 33 double cotton-covered wire wound next the core on each side of the ring which could be connected in series or parallel, were used as the armature. The third section consisting of 4 layers wound on top of the above, having 430 turns on each side of the ring, the whole being permanently connected in series, was used as the field-coil. At a later date coils on the pole-tips themselves were added.

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* Read December 9, 1904.

† Phil. Trans. A. vol. cciii. p. 305.

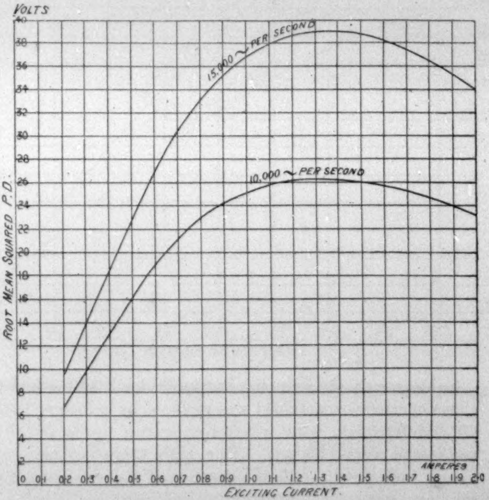
In whatever way the windings are arranged or used, it is necessary to prevent the induced E.M.F.'s from sending alternate currents round the exciting circuit, outside the alternator, as these currents would tend to annul the changes in lines of force through the coils and so prevent any useful alternating current from being produced. To prevent this the exciting circuit contained a choking-coil.

As no steam turbine or other high-speed motor was available, the alternator was at first driven by means of a figure-of-8 belt-drive, illustrated in fig. 3, from a ½ H.P. direct-current motor D fixed, with its shaft vertical, to a heavy wooden framework. Two bicycle-wheels 24½ inches actual diameter, No. 14 tangent-spokes, and "Palmer's" jointless hollow rims, were employed: the one fixed direct to the motor-shaft as the driving-wheel; the other, which was carried in an ordinary front-fork and hub with ball-bearings, acting as a tension-pulley to balance the pull on the alternator-spindle.

The pulleys on the alternator-spindle were of steel, having a diameter of only 10 mm. at the bottom of the V-grooves in them. The ratio of the belt-drive, that is the number of turns of the alternator-spindle for each revolution of the motor, was found experimentally to be 42·5.

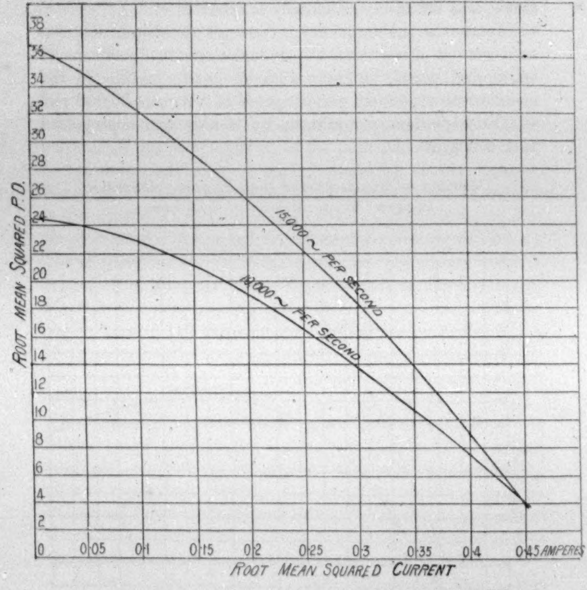
To measure the frequency, a Young's speed-indicator was used geared down from the inductor-spindle by means of worm and wheel, having a ratio of 20 : 1. Although the spindle was constantly run at speeds of between 30,000 and 40,000 revolutions per minute, no trouble was experienced with this gear.

Curves I.—Characteristics of High-Frequency Alternator on Open Circuit. Inductor with 30 teeth.



With this method of driving, frequencies up to 18,000 ~ per second could be obtained. Curves I. are characteristic curves for the alternator; and Curves II. are between R.M.S. current and P.D. with constant exciting current at frequencies of 15,000 and 10,000 ∼ per second.

Curves II.—Between P.D. and Current. High-Frequency Alternator. Constant Exciting Current = 1 ampere. Inductor with 30 teeth.



Subsequently the inductor having 30 teeth with which the above curves were obtained was replaced by inductors having 60 and 90 teeth, with which frequencies up to 50,000 ∼ per second were obtained. This high frequency being still hardly sufficient to settle the question of the resistance of the solid arc, some means for still further increasing it had to be devised.

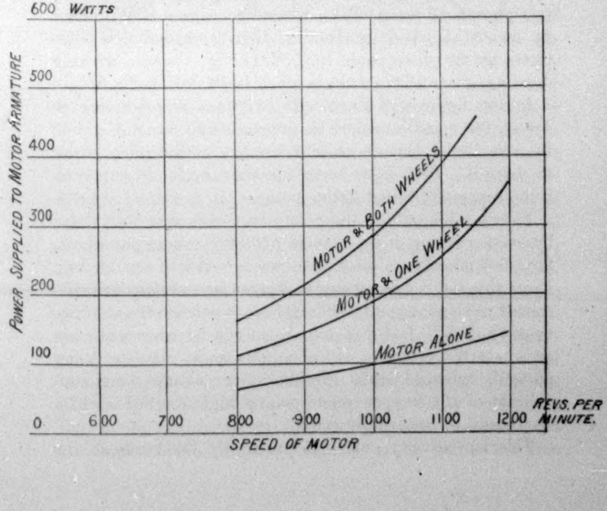
As the highest speed of the inductor was only 600 revolutions per second, whereas the limit at which it was considered safe to run the inductor, owing to considerations of bursting, was 1000 revolutions per second, attempts were made to increase its speed; but these failed owing to the large power required seriously overloading the motor.

In order to find out why such a large increase of power was necessary to drive the motor and alternator at higher speeds, the power supplied to the motor-armature was measured. At a motor speed of 800 revolutions per minute, which would correspond, if the alternator were connected, with a speed of its spindle of 567 revolutions per second, the power required to drive each part was as follows.

Table I.

	Watts.
Motor alone, no bicycle-wheels or alternator	63
" with one bicycle-wheel only	111
" " both bicycle-wheels bolted together...	162
" " " " and alternator....	532

Curves III.—Between Speed and Power required to drive:—Motor alone; Motor and one Bicycle-wheel; Motor and both Bicycle-wheels.



The power required to drive the motor alone and with one or two wheels at higher speeds is given in Curves III., which shows how very rapidly the power required to drive the bicycle-wheels increases with the speed even without the alternator. Practically, the whole of the difference between the power required to drive the motor and one wheel and that required to drive the motor alone, was expended in overcoming air-friction on the wheel, as the increase of the bearing friction due to the extra weight of the wheel cannot be large, the armature being many times heavier than the wheel; and the extra C^2R loss due to the extra armature current is only 22 watts at a speed of 1200 revolutions per minute.

The air-friction on this single bicycle-wheel running at 1200 revolutions per minute, or at a rim-velocity of about 85½ miles per hour, required an expenditure of energy at the rate of about 200 watts; so that for a cyclist to attain this speed he would have to develop over ½ H.P. to overcome the air-friction on his two wheels alone.

The motor and bicycle-wheels were therefore replaced by two phosphor-bronze disks driven from an 8 K.W. motor by means of ¼-inch gut bands. This arrangement is illustrated by the photograph Pl. XVIII. fig. 4, and a working drawing of one of the disks is given in Pl. XVII. fig. 5.

In the figure-of-8 drive with only one wheel acting as driver, the spindle cannot be relieved of all pressure due to the belt, if the tension-wheel takes any considerable power to drive it. Both disks were therefore made drivers, each being separately belted to the motor.

Before taking these disks into use, each one was tested by twice running it up to about 6500 revolutions per minute for a few minutes to see if there was any risk of its bursting. From this test it was concluded that, as far as safety was concerned, any speed up to 5000 revolutions per minute, or a rim-velocity of 327 feet per second, might be employed; but owing to the excessive vibration, due to the disks not being perfectly balanced, 4000 revolutions per minute, or a rim-velocity of 262 feet per second, was the highest speed at which they could be continually run.

This rim-velocity, which is practically the same as the linear belt-velocity, being so much higher than that attained with the bicycle-wheels, the 10 mm. steel pulleys on the alternator spindle were removed and replaced by larger ones of red fibre, as it was thought that this change would reduce the tension in the belt necessary to drive the alternator. It had, however, quite the opposite effect, owing to the inertia forces lifting the belt out of contact with the pulleys at the higher belt speeds necessary to obtain the same speed of the alternator-spindle. The extra tension required to make the belt wrap round and drive the pulleys was much larger than the saving in tangential pull on the pulleys, owing to their larger radius. It is evident that there is in any given case a best diameter of pulley which will give the least strain on the belt; this diameter was found by experiment in the present case to be the size of pulleys shown on the spindle in Pl. V. fig. 1. The object of making the new pulleys of red fibre was to prevent the heat developed by any slip of the belt being conducted to the spindle, and so expanding it and causing it to seize in the bearings.

At these high speeds the centrifugal force on the belt introduced a serious difficulty, as it tended to lift the belt right out of contact with the small pulleys, so that the slip became excessive. If, on the other hand, the belt was tightened up sufficiently to prevent this, then the strain on it became too great, and it broke after a short run. The joint in the belt was also a source of trouble owing to the inertia forces acting on it. Many different kinds of belts and ways of joining them were tried, such as: gut with connectors or splices; leather both round and flat with different kinds of wire and sewn joints; hempen cord with various splices; common cotton-tape twisted up and sewn together at the ends; and $\frac{3}{16}$-inch diameter cotton-cord joined with a long splice. Of all these, the last gave the best results. In order to get sufficient friction between the belt and the pulleys, a small quantity of thick solution of resin in turpentine was applied to it; and if it was then found to be too sticky, a trace of oil was put on the belt with the finger whilst running. The life of one of these belts was not very long, being only about four hours continuous use; but owing to their cheapness, the ease with which a new one could be made, and to the fact

TABLE II.—List of High-Frequency Alternators.

Alternator belonging to	Made by	Type of Alternator.	Method of driving.	Speed of Alternator-shaft. Revs. per min.	Peripheral Speed. ft. per sec.	Frequency ~ per sec.	Output on N.I. Load. Watts.	P.D. Volts.	Current. Amperes.
Prof. LODGE. Birmingham University.	Messrs. C. A. Parsons & Co.	Revolving field with radial poles. The armature-coils are wound on teeth projecting inwards from a laminated ring.	Electrical. Alternator on the same shaft as motor.	6000	250	3000	360	112	3·2
Prof. ELIHU THOMSON. The General Electric Co., Lynn, Mass., U.S.A.	Elihu Thomson, in mechanical laboratory of Thomson-Houston Electric Co.	Inductor, of Thomson design. Two stationary field-coils in iron frame surrounding ends of revolving inductor. Inductor-poles central. 50 poles=50 cycles per revolution.	Belt and pulley.	5000	150	4165	1000	1000	1·0
N. TESLA. New York, U.S.A.	....	Revolving armature. Zig-zag winding on edge 384 field-poles (1)	Belt.	1600	209	5100	......	100?	10·
M. WIEN. Physikal Institut der Technischen Hockschule, Aachen.	....	Inductor (2).	Direct, driven by a motor.	2040	185	8500 (2)	4	20	0·2
Sir D. SALOMONS.	Messrs. Pyke & Harris.	Field & armature both revolve in opposite directions. 174 pairs of poles on each (3).	Motor forming part of alternator.	1500	79	8700	200 (3)	200	1

C. P. STEINMETZ. The General Electric Co., Schenectady, U.S.A.	The General Electric Co.	Revolving armature single phase.	Induction motor or belt.	3750	310	10,000	2000	100	20
M. LEBLANC.	The Westinghouse Co.	Inductor, 200 polar projections (4).	Direct, coupled to motor.	3000	327	10,000	2000		
	Messrs. Siemens & Halske.	Inductor (5).	Ditto.	...	...	10,000			
	Thury.	" (6).	Ditto.	3000	...	10,000	1600	200	8
Prof. EWING. Cambridge University.	Messrs. C. A. Parsons & Co.	Special Disk field-magnet (7).	Steam turbine.	12,000	600	14,000	500	100	5
N. TESLA. New York, U.S.A.	....	Revolving armature Ferranti type, 480 polar projections.	Belt.	...	...	15,000			
Central Technical College, London.	The author and students at the Central Technical College.	Inductor with 30 teeth.	Belt.	30,000	310	15,000	5·5	22	0·25
		Inductor with 204 teeth		35,400	366	120,000	0·20	2·0	0·1
M. TURRETTINI, (8) Geneva, Switzerland.									

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(1) The 'Electrical Engineer' of New York, 1889, vol. xi. p. 338. Data as to output are wanting,

(2) Wiedemann's *Annalen*, 1901, vol. iv. p. 425. By resonating the second harmonic a frequency of 17,000 is said to have been obtained.

(3) The Journal of the Institution of Electrical Engineers, 1892, vol. xxi. p. 709. There seems a slight uncertainty as to whether this output was ever actually obtained.

(4) The American Institute of Electrical Engineers, 1904, pp. 405, 416.

(5) Exhibited St. Louis, 1904.

(6) *L'Eclairage Electrique*, 1904, vol. xli. p. 65.

(7) 'The Electrician,' 1892, vol. xxx. p. 65.

(8) Said to possess a high frequency alternator; no data available.

that they did no damage when they broke, this was not considered a very serious disadvantage.

In spite of every care, all attempts to run the inductor at 1000 revolutions per second failed, owing to the heating of the bearings; but on removing the inductor from the spindle, this latter could be run at 1000 revolutions per second with comparative ease. This was evidently due to the axis of inertia of the inductor not coinciding with its mechanical axis; all attempts to adjust this, such as making new inductors, &c., failed.

To give some idea of the importance of the two axes coinciding, suppose that they are parallel and 1 mm. apart; then at 1000 revolutions per second the pressure on the two bearings works out to about 0·8 metric ton; so that an error of only $\frac{1}{100}$ mm. would mean a prohibitive pressure on such small bearings.

Parallel bearings working in two loose sleeves were tried, and also combinations of conical and parallel bearing surfaces; but the simple conical bearings shown in Pl. XVII. fig. 1 were found to give the best results, probably owing to their shorter length allowing the inductor more freedom to spin on its axis of gyration rather than forcing it to spin on its mechanical axis. An ideal method would have been to allow the inductor to spin like a top and choose its own axis of rotation; but this could not be done with the actual alternator, as the clearance between the inductor and the pole-tips was less than 0·1 mm.

Experiments were next started to see whether the efficiency of the alternator could not be increased so as to allow the small amount of power required for the experiments to be obtained with an inductor having more than 90 teeth. An inductor having 204 teeth was therefore constructed.

By placing search-coils on different parts of the field-ring it was found that the E.M.F.'s induced were much larger in those coils placed near the poles than in those placed on any other parts of the ring, from which it seemed that the changes in the number of lines of force produced by the movement of the inductor were confined to the neighbourhood of the pole-tips, probably due to the magnetic leakage and to the eddy-currents set up in the laminations. Coils were therefore wound right on the pole-tips themselves. Using the coil on one pole-tip only, which consisted of 10 turns of No. 30 double cotton-covered wire, it was found that the output of the alternator was 10 times as great as could be obtained with the original armature-winding on the ring itself.

With this inductor and the coils on the pole-tips as shown in Pl. XVII. fig. 1, it was finally possible to obtain a R.M.S. current of 0·1 ampere at frequencies up to 120,000 ~ per second. The maximum P.D. at 100,000 ~ per second with 0·1 ampere flowing was 2 volts, and on open circuit 3·6 volts. As this output, though small, was ample for the purposes of the research, no attempt was made to further increase it. The wave-form of the current, when this inductor was driven so slowly that the frequency was only 2670 ~ per second, is given in Pl. XVIII. fig. 6.

An illustration will perhaps convey some idea of how high a frequency of 120,000 ~ per second really is. In plotting curves for ordinary frequencies of 50 to 100 ~ per second, a scale often adopted is 10 inches for 100 ~. If it were attempted to plot a curve up to 120,000 ~ per second to this scale the curve paper would require to be 12,000 inches, or nearly *one-fifth of a mile long*.

At the time of constructing the present alternator the owners of various high-frequency machines kindly furnished the author with data of their alternators. These data are collected in Table II. (pp. 438–39) together with three more recent alternators. The writer hopes that a description of other machines may be forthcoming in the discussion on the present paper.

In conclusion the author would like to point out that he has described this machine not with a view of suggesting it as the best, or even as a satisfactory, design, but rather to indicate some of the difficulties which are encountered in attempting to construct an alternator for high frequencies. The machine served its purpose and allowed the research for which it was designed to be completed, and no further work has been done on it, though there are several very obvious improvements which would greatly increase its output and utility. Firstly, the greater part of the difficulties were encountered with the belt-drive and the bearings, these could obviously be got over by mounting the inductor direct on the shaft of a steam turbine. A larger inductor, say 10 to 15 cm. diameter, could also probably be used with 200 teeth and a speed of 30,000 r.p.m. to give 100,000 ∼ per second. Secondly, the lamination of the iron must be carried to a very high degree to prevent eddy-current losses, and the armature-coils must be fixed as near as possible to the very tips of the poles. Thirdly, with accurately-spaced teeth on the inductor a large number of armature poles could be used.

All the experiments described in this communication were carried out at the Central Technical College. The author wishes to express his indebtedness to Prof. Ayrton and Mr. Mather for much valuable assistance and advice.

Discussion.

Dr. J. A. Fleming said he had felt surprised on hearing about Mr. Duddell's alternator that he had been able to reach any such frequency as 120,000 ∼ per second. It spoke highly of the excellent mechanical means employed. On the other hand, as far as its use in wireless telegraphy was concerned, it was no use to speak of a machine even if it gave this frequency, corresponding to a wave 1½ miles in length, if the output of the machine were only a few watts. If it were 150 H.P. it might be different. The difficulty with most high-frequency alternators was the magnetic leakage. Dr. Fleming then described some of the devices adopted by Tesla for obviating this defect. Many of the high-frequency alternators made had their power over-estimated. Hitherto the frequency reached had been only about 10,000 ∼ per second. A machine of this last frequency would be no particular assistance in wireless telegraphy. Such machines were, however, very useful for research and demonstration purposes.

XXXVII. *Note on a Rapid Approximate Method of Harmonic Analysis.* *By* SILVANUS P. THOMPSON, *D.Sc., F.R.S.*[*]

FOR the study of alternating electric currents and for several other applications, harmonic analysis is simplified by the consideration that all the even terms in the Fourier expansion are absent. In this case the second half-period is similar to the first half-period, but with the ordinates of the corresponding angles reversed in sign. Given a complicated harmonic curve containing constituents of the odd orders only, the zero-line can always be drawn so that the constant term vanishes from the Fourier series, the mean ordinate being zero; and it is then always possible to choose as origin a point for which the ordinates at 0° and 180° are zero.

There are many known methods for the approximate analysis of such curves. If a finite number of ordinates are given for a corresponding number of values of x, between $x=0^\circ$ and $x=180^\circ$, a solution is possible by algebraic reduction from the simultaneous equations thus afforded; the number of coefficients thus deducible being limited by the available number of simultaneous equations. Thus if there are 18 points on the curve given, it is possible to deduce the coefficients of the sine and cosine terms of the Fourier series up to the 17th term of each; *i.e.*, A_1 A_3 A_5 A_7 A_9 A_{11} A_{13} A_{15} A_{17} and B_1 B_3 B_5 B_7 B_9 B_{11} B_{13} B_{15} and B_{17}. But even with the aid of determinants the working is long and tedious, taking several hours for a single curve.

In 1892 Professor Perry[†] gave an arithmetical method of approximate analysis in which the ordinates of the curve to be analysed are taken as equidistant, the operation consisting in multiplying the values through by sin nx, where n is successively assigned the values (if odd harmonics only are present) of 1, 3, 5, &c., and the results are tabulated and averaged for each harmonic successively. Here again the labour becomes very great for even a small number of terms.

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[*] Read December 9, 1904.

[†] The Electrician, Feb. 5, 1892.

More recently Mr. S. M. Kintner * has given an extended tabulation for the same process for curves for which 36 equidistant ordinates in the half-period are given. Sine-values are set-down for every 5°, to be multiplied into the successive ordinates for each of the components up to the 17th. This requires no fewer than 630 separate multiplications by sines of angles to give the coefficients up to A_{17} and B_{17}.

The labour involved in these arithmetical processes is so great that graphic methods have come into favour, and have been extensively used. Those devised by Prof. Perry † and Mr. E. B. Wedmore ‡, have found favour in England, while on the Continent a similar method has been worked out by Fischer-Hinnen §. In the States another graphic method, due to Houston and Kennelly ‖, has been widely adopted. Perry's method, based on a suggestion of the late W. K. Clifford, consists first in a process of projection of the curve upon a cylinder, the apparent area being then measured by a planimeter. The method of Houston and Kennelly also requires the use of a planimeter, or else resort to some equivalent device such as drawing the curve on squared paper and taking areas by counting the enclosed squares. These methods are laborious, giving rise to many complications of overlapping lines, when more than a very few terms are to be got out; and they labour under the disadvantage, common to all graphic methods, of failing in accuracy when dealing with small quantities.

Nothing need be said here beyond a passing reference to the instruments known as harmonic analysers, of which the forms due to Lord Kelvin, Professor Henrici, Mr. A. Sharp, and Mr. Coradi are well known. There is nothing to be said against them beyond their great expense. They do not really come into the present consideration.

All the foregoing methods, except that of the simultaneous equations and the particular graphic methods of Wedmore

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* Electrical World and Engineer, xliii. p. 1023 (1904).

† The Electrician, June 28, 1895.

‡ The Electrician, August 16, 1895, and Journ. Inst. Elect. Engineers, xxv. p. 224 (1896).

§ *Elektrotechnische Zeitschrift*, 1901, Heft 19.

‖ Electrical World, xxxi. p. 580 (1898).

and Fischer-Hinnen, are based upon the two theorems that

$$\int_{t=0}^{t=\mathrm{T}} \sin^2 pt \, . \, dt = \tfrac{1}{2}\mathrm{T} \, ;$$

and

$$\int_{t=0}^{t=\mathrm{T}} \sin rpt \, . \, \sin spt \, . \, dt = 0 \, ;$$

where r and s are any two whole numbers.

According to these theorems the result of multiplying through by $\sin npt$ (where n is any whole number) and integrating over a whole period, eliminates everything except the particular harmonic of the nth order. This process of multiplying through by the sine of an angle, and integrating, is that which is sought to be approximately attained by the various devices, arithmetical, graphic, and mechanical, enumerated above.

The writer has recently found a very great saving of labour by adapting to the case under consideration an elaborate method of analysis published by Professor C. Runge, in 1903, in the *Zeitschrift für Mathematik und Physik*, xlviii. p. 443. This method also requires the multiplication by sines of angles, and the subsequent integration (here reduced to simple addition); but it shortens the process by a device which may be termed "grouping." In brief it groups the ordinates by sums and differences successively taken, in such a way that the work of multiplying through by the sines of angles is performed not on solitary ordinates but on assemblages of them. Runge's memoir treats the subject in a generalized manner, taking into consideration the even as well as the odd terms. The procedure is somewhat complicated, but is systematized by means of an elaborate schedule. When one tries to work out a corresponding method for odd terms only, the simplification is surprising, and the saving of labour as compared with all or any of the antecedent methods is great.

The procedure of the shortened process is as follows:—

(1) *Choice of Ordinates.*—The half-period must be divided into $2m$ equal parts, giving $2m-1$ equidistant ordinates.

(2) *Arrangement for taking Sums and Differences.*—These ordinates are then put together in pairs so that there then may

be found the sum and difference of the ordinate of any angle and the ordinate of its supplement. The sum will contain sine-components only, and the difference cosine-components only. For since $\cos n\theta = -\cos(180^\circ - n\theta)$, and $\sin n\theta = \sin(180^\circ - n\theta)$, it follows that adding the pair of supplemental ordinates eliminates all cosines, and subtracting them from one another eliminates all the sines. We therefore write down the ordinates in 2 rows under one another, the first row of m members from right to left, the second of $m-1$ members from left to right, and then take the sums and differences. This artifice separates all sine-terms from cosine-terms. We use the sums for getting out the coefficients of sine-terms, A_1, A_3, A_5, &c. ; and the differences for getting out those of the cosine-terms B_1, B_3, B_5, &c.

(3) *Grouping for Harmonics.*—A further grouping is possible for the higher harmonics of those orders for which the ordinal number is a factor of the numeric of the divisor m, for example the third and ninth terms, if $m=9$. For, when getting out the 3rd harmonic we have to multiply the ordinate at 15° by sin 45°, the ordinate at 45° by sin 135°, and the ordinate at 75° by 225°; and in doing this we may remember that $\sin 45^\circ = \sin 135^\circ = -\sin 225^\circ$. Hence the three ordinate-values at 15°, 45°, and 75°, as well as their three supplementals at 165°, 135°, and 105°, since they all have to be multiplied by sin 45° (+ or −) and then added (*i. e.* integrated) together, may just as well be grouped together by adding before being multiplied by sin 45°, so that one multiplication suffices instead of six.

(4) *Arranging in a Schedule.*—The numbers so assembled and grouped are then arranged in a convenient schedule, each number as it is entered being multiplied by the sine of the angle set over against it. By use of a slide-rule this is a rapid process. To get out, for example, the coefficients up to the 17th terms, using 36 ordinates, only 65 multiplications are needed, against Kintner's 630 ; and the time taken with the slide-rule in multiplying is not much more than half-an-hour. To facilitate operations the schedule shows where the numbers should be entered. Sine-terms are entered from the top opposite the angles 0° to 90° ; cosine-terms from the

bottom since cos 0° = sin 90°. The results for the harmonics will, as we shall see, come out in pairs; for example, if we have taken 12 ordinates the harmonics will come out 1st and 11th together, 3rd and 9th together, and so forth. Hence in entering the numbers in the schedule they are entered alternately in 2 columns side by side, in zig-zag order. For the 1st (and 11th) order of harmonics the entries fill successively every place opposite the sines. For the 3rd (and 9th) order they fill every third place. For the 5th (and 7th) order they fill every fifth place. The use of the schedule prevents slips in entering.

(5) *Integration by Addition and Subtraction.*—It may be remembered that the coefficients of the *n*th order are given by the integrals :—

$$A_n = \frac{2}{T}\int_0^T y \,.\, \sin npt \,.\, dt \,;$$

$$B_n = \frac{2}{T}\int_0^T y \,.\, \cos npt \,.\, dt.$$

Hence it remains to integrate and to take twice the means. The integration here becomes a mere summation of the products made during the preceding process. But since in the case where 12 ordinates are taken, the 11th coefficient would be got out by multiplying by the same sines as would yield the 1st coefficient, only with alternate + and − signs prefixed, the addition for both the 1st and the 11th harmonic may be effected by finding separately the sums of the alternate product-numbers, which are therefore set down alternately in 2 columns. The totals of the first column and of the second column are therefore made separately. The sum of them gives the "integral" for the 1st harmonic, the difference between them gives the "integral" for the 11th harmonic, and so forth. Then, finally, these sums and differences must be divided through by half the number of ordinates taken so as to give twice the mean. If 12 ordinates (including the zero at the origin) have been taken, we divide through by 6. If 18 ordinates have been taken, we divide through by 9. If 6 ordinates have been taken, we divide through by 3. The summation which here corresponds to integration is taken over the half-period. But as both half-periods are alike, and as the first process of adding up supplemental ordinates gives the doubles of their respective sine-terms, the result is the same as if we had divided the sum by two and had subsequently extended the summation from 0° to 360°.

The labour of the process, including the various addings and subtractings, varies approximately as the cube of the number of ordinates taken. If we want the coefficients up to the 17th order, using 18 ordinates the time taken is about 70 minutes to complete the whole analysis. If we want the coefficients up to the 11th order, using 12 ordinates, about 25 minutes are required. If the coefficients are required up to the 5th only, using 6 ordinates, the time needed is under 3 minutes, or, with practice, under 2 minutes.

Two schedules are given here: the first (**A**) to be used for finding the coefficients up to the 11th order, the second (**B**), the short form requisite for the coefficients up to the 5th order only, and a numerical example is worked out.

SCHEDULE A FOR THE ANALYSIS OF A PERIODIC CURVE IN WHICH ONLY ODD HARMONICS APPEAR UP TO THE ELEVENTH HARMONIC.

Note.—The two half-periods will be similar, so that if the mean line be taken between the highest and lowest points in the curve, there will be no constant term. For further simplification the origin should be taken where the curve crosses the zero line.

(1) Divide the half-period into 12 equal parts, and measure the 11 ordinates $y_1 y_2 y_3 \cdot \cdot \cdot \cdot y_{11}$; y_0 and y_{12} being each zero.

(2) Then arrange these ordinates as under:—

	y_1	y_2	y_3	y_4	y_5	y_6
	y_{11}	y_{10}	y_9	y_8	y_7	
Adding	s_1	s_2	s_3	s_4	s_5	s_6
Subtracting ...	d_1	d_2	d_3	d_4	d_5	d_6

Note.—s_1 stands for the sum of y_1 and y_{11}; d_1 for the difference between y_1 and y_{11}. Great care must be taken as to + and − signs throughout.

(3) Group numbers, to obtain values for use with third and ninth harmonics, as follows :—

$$s_1 + s_3 - s_5 = r_1$$
$$s_2 - s_6 = r_2$$
$$d_1 - d_3 - d_5 = e_1$$

(4) Then select from the above numbers and put them in their places in the table below, *multiplying each by the sine set down in the left-hand column before it is entered.*

Angle.	Sine-Terms.						Cosine-Terms.					
	1st and 11th Harmonic.		3rd and 9th Harmonic.		5th and 7th Harmonic.		1st and 11th Harmonic.		3rd and 9th Harmonic.		5th and 7th Harmonic.	
Sin 15° = 0·262......	s_1				s_5			d_5				d_1
Sin 30° = 0·500.....		s_2				s_2	d_4				d_4	
Sin 45° = 0·707......	s_3		r_1		$-s_3$			d_3		e_1		$-d_3$
Sin 60° = 0·866......		s_4				$-s_4$	d_2				$-d_2$	
Sin 75° = 0·966......	s_5				s_1			d_1				d_5
Sin 90° = 1·000......		s_6		r_2		s_6	...		$-d_4$		...	
Total first column .	...		...		...		...		...		...	
Total second column	...		...		...		...		...		...	
Sum	$6\ A_1$		$6\ A_3$		$6\ A_5$		$6\ B_1$		$6\ B_3$		$6\ B_5$	
Difference	$6\ A_{11}$		$6\ A_9$		$6\ A_7$		$6\ B_{11}$		$6\ B$		$6\ B_7$	

Result. $y = A_1 \sin \theta + A_3 \sin 3\theta + A_5 \sin 5\theta + A_7 \sin 7\theta + A_9 \sin 9\theta + A_{11} \sin 11\theta + B_1 \cos \theta + B_3 \cos 3\theta + B_5 \cos 5\theta + B_7 \cos 7\theta + B_9 \cos 9\theta + B_{11} \cos 11\theta.$

SCHEDULE B FOR THE ANALYSIS OF A PERIODIC CURVE IN WHICH ONLY ODD HARMONICS APPEAR UP TO THE FIFTH ORDER.

(1) Divide the half-period into 6 equal parts, and measure the 5 ordinates $y_1\, y_2\, y_3\, y_4\, y_5$; y_0 and y_6 being each zero.

(2) Arrange these ordinates as under :—

$$\begin{array}{lccc} & y_1 & y_2 & y_3 \\ & y_5 & y_4 & \\ \hline \text{Adding} \ldots\ldots\ldots\ldots & s_1 & s_2 & s_3 \\ \text{Subtracting} \ldots\ldots & d_1 & d_2 & \end{array}$$

(3) Grouping for third harmonic : $s_1 - s_3 = r_1$.

(4) Select from the above numbers, and put them in their places in the table below, *multiplying each by the sine set down in the left-hand column before it is entered.*

Sines of Angles.	First & Fifth.		Third.	First & Fifth.		Third.
Sin 30° = 0·500	s_1			d_2		
Sin 60° = 0·866		s_2			d_1	
Sin 90° = 1·000	s_2		r_1	...		$-d_2$
Total first column	...		...	...		...
Total second column ...	...			...		
Sum	$3A_1$		$3A_3$	$3B_1$		$3B_3$
Difference	$3A_5$			$3B_5$		

(5) Result. $y = A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta$
$+ B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta.$

(6) Check result by the rules that

$$A_1 - A_3 + A_5 = y_3\,; \quad B_1 + B_3 + B_5 = 0.$$

EXAMPLE. Given the curve having the following ordinates:—

$y_0 = 0 \quad y_1 = 4 \quad y_2 = 9{\cdot}5 \quad y_3 = 9 \quad y_4 = 3 \quad y_5 = 2 \quad y_6 = 0.$

Arranging :	4	9·5	9
	2	3	
Adding	6	12·5	9
Subtracting......	2	6·5	

Grouping for third harmonic : $6 - 9 = -3$.

Entering and multiplying by sines :—

Sin 30° = 0·500	3			3·25		
Sin 60° = 0·866		10·7			1·73	
Sin 90° = 1·900	9		−3	...		−6·5
Total first column	12		−3	3·25		−6·5
Total second column ...	10·7			1·73		
Sum	22·7		−3	4·98		−6·5
Difference	1·3			1·52		
Dividing by 3	$A_1 = 7{\cdot}57$ $A_5 = 0{\cdot}43$		$A_3 = -1{\cdot}0$	$B_1 = 1{\cdot}66$ $B_5 = 0{\cdot}51$		$B_3 = -2{\cdot}17$

RESULT. $y = 7{\cdot}57 \sin\theta - 1{\cdot}0 \sin 3\theta + 0{\cdot}43 \sin 5\theta$
$+ 1{\cdot}66 \cos\theta - 2{\cdot}17 \cos 3\theta + 0{\cdot}51 \cos 5\theta.$

DISCUSSION.

Dr. J. A. **FLEMING** said that he had found by experience that most alternating-current curves of current or electromotive force which occurred in connection with transformer working could be quite nearly represented by a simple sine curve added to the two first odd harmonics having wave-lengths $\frac{1}{3}$ and $\frac{1}{5}$ of the primary. Hence the analysis of any such curve as the magnetizing curve of a current-transformer resolved itself simply into the determination of the amplitude and phase-differences of these three constituents.

They were indebted to Professor Thompson for bringing before them such a simple arithmetical process of resolution which reduced the performance of the Fourier analysis of simple periodic curve to little more than mere arithmetic, even more simple than Horner's process for obtaining the arithmetic real roots of an equation of the *n*th degree. In addition to the analysis of a single periodic curve, it was useful sometimes to be able to perform the synthesis rapidly, and some time ago he had designed a simple form of instrument for effecting this synthesis. Everyone was familiar with the representation of simple periodic quantities by clock diagrams, but he was not aware that anyone had previously suggested a device for drawing what he would call a compound clock diagram. The instrument he exhibited was made in the following manner :—It consists of a wooden pulley 6 inches in diameter having two grooves in its circumference. The pulley has three pins on its lower surface by which it can be pressed down and held firmly on a drawing-board. On a fixed central pin a radial arm revolves, and this radial arm carries at its extremity another pulley revolving on a pin. This second pulley is half the diameter of the fixed pulley. This second pulley has fixed to it an arm of a certain length, which in turn carries at its end a third pulley, a quarter the size of the fixed central pulley. This third pulley has in like manner an arm attached to it which may end in a tracing-point or may carry another pulley, and so on. These pulleys are all driven by threads passed round the fixed pulley, thus the second pulley is belted to the first, and the third pulley to the first, by the intermediation of a jockey-pulley running freely on the axis of the second pulley. The arms therefore form a series of jointed vectors, and when the first arm is turned round through any angle the other radial arms are also revolved through angles respectively 3, 5, 7, 9, etc. times the first angle.

In dealing with a series of linked vectors of this kind, it was necessary to bear in mind that the angular displacement of any vector with regard to a line fixed in space is equal to the vector sum of all the angular displacements of the radial rods, with regard to one another. Hence, in virtue of the pulleys having sizes compared with the first respectively of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., the three radial arms revolve through angles which are 3 times, 5 times, etc. the angle through which the first radial rod is revolved. Hence, when the system is turned round, the tracing-point draws a compound polar curve. If we move the first radial rod through 180° by 12 steps of 15° each, and if at each step we draw lines horizontally through the tracing-point, and then at 12 equidistant intervals on a line drawn through the centre of rotation parallel to the above-mentioned lines, we erect ordinates to meet them, the intersections will give us the ordinary rectangular wave-form of the polar curve. The instrument exhibited by Dr. Fleming was constructed to draw the compound polar curve corresponding to a fundamental and the first two odd harmonics, and a diagram was exhibited showing the corresponding wave-form for one setting which was extremely similar to the no-load magnetizing current of a transformer. It is exceedingly easy to show from the principles of this compound clock diagram apparatus, that the rectangular wave-form so drawn must be a curve resolvable into the sum of 3 sine curves.

If A_1 is any vector, then it is well known that

$$A_1(\cos\theta + j\sin\theta)$$

represents the same vector turned through an angle θ, where $j=\sqrt{-1}$. Hence the radius vector of the compound polar curve drawn by the instruments must be represented by

$$r = A_1(\cos\theta + j\sin\theta) + A_2\{\cos(3\theta+\phi) + j\sin(3\theta+\phi)\}$$
$$+ A_3\{\cos(5\theta+\phi') + j\sin(5\theta+\phi')\}.$$

Therefore the projection of this radius on a vertical line gives us the ordinate y of the wave diagram, and this is therefore given by

$$y = A_1 \sin \theta + A_2 \sin \overline{3\theta + \phi} + A_3 \sin \overline{5\theta + \phi}.$$

Dr. Fleming exhibited diagrams showing how nearly the magnetizing current of a transformer could be represented in wave form by the use of such an apparatus. He said that the representation of alternating-current curves by polar diagrams had particular advantages, and it was much to be wished that some enterprising stationer would provide Polar ruled paper (analogous to squared paper) for plotting polar diagrams.

Mr. J. **Harrison** pointed out that the Author's system of Harmonic Analysis could very readily be performed entirely by graphical methods, and moreover, subject to certain restrictions, could be employed to find even terms as well as odd ones. He exhibited a graphical schedule for a particular case and demonstrated its use.

XXXVIII. *Action of a Magnetic Field on the Discharge through a Gas.* *By* R. S. Willows, *M.A., D.Sc.* (Communicated by Prof. J. J. Thomson, F.R.S.)*

When a discharge which is passing through a gas under reduced pressure is acted on by a magnetic field, it is known, both from theory and experiment, that if the field is parallel to the discharge it causes it to pass more easily, while if the field is transverse the opposite is the case. I have shown †, however, that below certain pressures which vary with the conditions of the experiment, a transverse field increases the current in the tube and diminishes the difference of potential at the terminals, provided it be applied near the cathode. At other points of the discharge a decrease in current is always produced by the field.

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* Read January 27, 1905.

† Phil. Mag. [6] i. p. 250 (1901).

Birkeland has shown * that with the lines of force parallel to the tube, supposed cylindrical, at pressures below ·012 mm., the potential-difference is made to diminish by the magnetic field, at first slowly as the intensity is increased, until a certain critical intensity is reached, when a large, abrupt diminution is obtained.

Almy † has studied these two effects, and has come to the conclusion that they are both due to the same cause. He concludes that the action of the magnet is simply to concentrate the discharge so that it passes through the gas by a sort of brush or arc rather than in the usual manner, and that this brings about an increase in the conductivity.

Further study of these effects was my object in starting the experiments described in the following paper.

When the negative glow is the part of the discharge acted on by a transverse field, an increase in potential at the terminals always takes place; but it has not been determined previously whether the results depend on whether the magnet acts at the surface of the cathode or at any other point in the dark space.

In the earlier paper I was fortunate enough to be able to use a large accumulator battery to produce the discharge, and so both voltage at the terminals and current through the tube could be measured. In the present case a coil, driven by a mechanical interrupter, was used, and the voltage only was measured by means of a multicellular voltmeter. Where the range of the voltmeter was not great enough, the tube was shunted by a liquid high resistance, and a fraction of the total voltage taken or the whole measured in steps. An electromagnet with pole-pieces formed so as to give a very local field was used, and the strength of the field at different distances from the line joining the poles, and for different currents, was found by means of a flat coil and ballistic galvanometer.

A long cylindrical tube of 25 mms. diameter, provided with disk electrodes, was pumped down until the Crookes dark space was 3 cms. long, and the transverse field applied

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* *Comptes Rendus*, cxxvi. p. 586 (1898).

† Proc. Camb. Phil. Soc. xi. p. 183 (1901).

successively at the cathode and at distances of 1, 2, 3, and 4 cms. away from it in the direction of the anode. The observations showed that when equal decreases in potential were produced, the field at the cathode surface was practically the same for all positions of the magnet. We may therefore conclude that it is at, or near, the surface of the cathode that the action arises. The field shortened the dark space from 3 cms. to about 2 mms.

If Almy's explanation is the correct one, the critical pressure, above which the field causes an increase and below which a decrease in the terminal voltage, might be expected to be more or less irregular and ill-defined. In any case, it was thought worth while to study this pressure more carefully with different transverse fields, and an observation of interest resulted. Starting from a pressure of 3–4 mms., the volts at the terminals were measured with the magnet off and on at the cathode. As the pressure is continually lowered, the magnet being off, it is, of course, known that the voltage decreases, reaches a minimum, and then rapidly rises. The value of the critical pressure corresponding to minimum volts depends on the diameter of the tube, the nature of the gas, and perhaps also on the current. The last point I could not investigate with a coil. It was found that *above this critical pressure the field caused an increase in the voltage at the terminals, while below it caused a decrease.* At pressures much above 5 mms. very little effect was noticed. This result was confirmed by experiments on tubes with lengths varying from 50 cms. to 10 cms., diameters from 25 mms. to 3 mms., and with disk or wire electrodes. By using hydrogen, carbon dioxide, and air in these tubes, the critical pressure could be shifted from ·3 mm. to about 1 mm. and the observation still held, although with hydrogen the results were not so regular as with the other gases.

This critical pressure is much more sharply marked in narrow tubes. The series of observations given below will serve as an example of the results obtained. They were taken with a tube 3 mms. in diameter, 10 cms. long, filled with air, and a field of about 600 lines per sq. cm. A fraction only of the voltage is given.

Volts.

Magnet off.	Magnet on.	Magnet off.	Magnet on.
290	305	218	200
262	272	248	218
252	260	268	235
208	198	270	242
190	197	285	265
184	191	295	275
186	187 (?)	305	285
195	190		

The pressure was gradually reduced from the beginning to the end of the series, the minimum voltage is 184–186; before this is arrived at the magnet causes an increase, after it is passed a decrease in the volts, but it never causes the voltage to fall below 186. The field therefore does not reduce the absolute minimum voltage causing the discharge. This is worthy of note, as Carr * has shown that the minimum sparking potential is a constant for the gas, and there is doubtless a close connexion between the minimum sparking potential and the minimum voltage required to maintain the discharge.

With tubes much shorter than 10 cms., the pressure had to be lower than the minimum voltage pressure before the magnet caused a decrease. This is no doubt due to the fact that the field could never be sufficiently localized to act at the cathode only, and where it acts at other points of the discharge it brings about a rise in voltage.

It would appear, therefore, from my experiments, that the transverse magnetic effect is closely connected with other features of the discharge, and hence is probably not due to a concentration of the luminous portion into a brush or arc.

I tried next the effect of a longitudinal field on several of the tubes, but, down to a pressure ·01 mm., a certain decrease in the terminal voltage was never obtained. The iron core of the magnet employed to produce the field was of greater diameter than the largest cathode used; the field had a

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* Phil. Trans. cci. p. 403 (1903).

maximum value of 900, it was therefore more intense than that used by Almy or Birkeland. I am unable to account for the discrepancy between my results and those of these physicists.

One tube was about 2 cms. long, so that the longitudinal field could act along the whole length of the discharge. Still no effect could be found, but if the same field was applied transversely at the cathode it produced a fall of 30 per cent., and this although it was not specially concentrated. The transverse and longitudinal effects are therefore of a different nature.

The dependence of the fall in voltage on the strength of the field when the pressure in the tube is kept constant, can be seen from the following series of readings. The gas-pressure was ·052 mm.

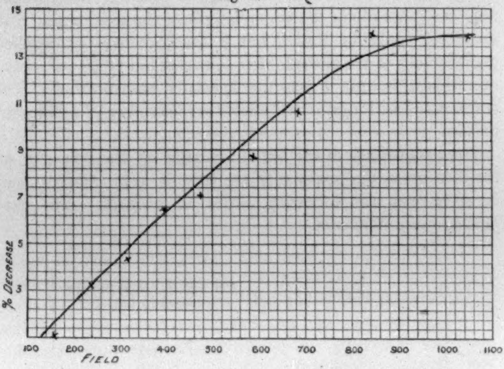
Field.	Per cent. decrease in volts caused by field.
159	1·1
238	3·2
318	4·3
398	6·5
477	7·1
589	8·7
685	10·6
844	13·9
1050	13·8

These numbers are shown graphically in fig. 1 (p. 458). The effect reaches a limiting value in this case ; in others it reached a maximum, and then slightly decreased as the field continually increased; this usually took place in the wider tubes.

The fall in volts does not increase indefinitely as the pressure is decreased. In all the cases tried, it appeared either to reach a limiting value or much more generally to fall off, and in some cases, at the lowest pressures reached, ·002 mm., a rise in volts was brought about by the field.

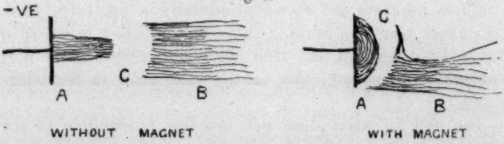
The results, however, at the lowest pressures were very irregular, and it may be that Almy's supposition as to the concentration of the discharge may have some weight here.

Fig. 1.



When the transverse field is applied, the Crookes dark space is greatly shortened on that side of the tube to which the discharge is deflected, very little alteration in its length is seen on the other side. The glow at the surface of the cathode (called by German writers the first negative layer) is also affected. The figures adjoining (fig. 2) show this.

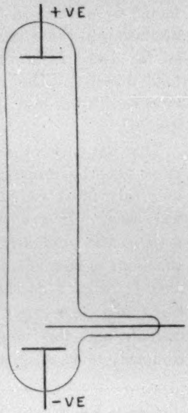
Fig. 2.



When the field is off, the first negative layer, A, covers the centre only of the cathode and is cylindrical in section; with the field on, its length is considerably less and it covers much more of the cathode area.

Since this layer is supposed to be due to ionization caused by canal-rays striking the cathode, an experiment was performed to see whether the path of these rays was altered by the field. A tube, as shown in fig. 3, with an aluminium wire projecting across the surface of the cathode was used.

Fig. 3.



The surface of the cathode behind this wire is devoid of all glow as shown by various investigators. The magnet caused no shift of this shadow, so that the fields used were not sufficient to deflect the canal-rays.

Townsend * has given a complete explanation of the fact that there exists a pressure for which the sparking potential between two fixed electrodes is a minimum. The fact that there exists a pressure for which the E.M.F. required to *maintain* the discharge is a minimum yet awaits a detailed explanation. The following observation on this point is of interest. If a curve be plotted showing the relation between terminal voltage and pressure of the gas in the discharge-tube, then, as the latter is diminished, starting from a few mms., the relation between the two is a linear one. As soon, however, as striæ appear at the cathode end of the positive column the voltage decreases less quickly than before; when the whole length of the positive column is striated, the voltage reaches its minimum value. If the pressure is further decreased, the voltage required to maintain the discharge rapidly increases. This observation was confirmed on tubes up to 50 cms. in length, of diameters ranging from 3–25 mms., and on air, carbon dioxide, and hydrogen. The last gas was irregular.

It appears, therefore, that the positive column striates because, by that means, the discharge is enabled to pass most readily. After it is completely striated, the magnetic field causes the current to pass more easily.

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* Phil. Mag. Nov. 1903, p. 598.

The field always causes more striæ to appear. Measurements were made on the longer tubes to see if it caused an alteration in the distance between the striæ when they were so far away from the cathode as to be unaffected by the field directly. The results obtained with a coil as source of current were too irregular to admit of conclusions being drawn.

That the positive column is differently affected in different gases may be inferred from the following experiment. A long tube filled with air giving striæ at a suitable pressure was used. If a current was then sent through the electromagnet and gradually increased, from each of the original striations a smaller one emerged on the side remote from the anode; with a further increase of field, this smaller one eventually coalesced with the original stria next nearest the cathode, and this could be repeated several times by continuously increasing the field. Below is a set of observations.

Current in magnet in amperes.	Appearance of striæ.
0	Fairly clear.
1·4	Very distinct double, one small one appearing exactly midway between two large ones.
1·65	No doubling.
2·4	Clear.
2·9	Double.
5·8	Very clear.
6·6	Very double.
7·3	Not so double.

Crookes * has shown that the double striæ frequently met with in hydrogen are due to the current being carried partly by the hydrogen and partly by mercury vapour. A similar explanation would probably apply to the above results.

The appearance with hydrogen in the tube was very striking at certain pressures. When the magnet was off, the bright parts of the striæ were very narrow and pink in colour, the spaces between being hazy. On putting on the magnet the

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* Proc. Roy. Soc. lxix. p. 399 (1902).

voltage was reduced, the striæ became very distinct and steady, double as in air and the interspaces much less hazy. After a few seconds each small stria retreated into the adjoining large one next nearest to the anode. After another interval of a few seconds the voltmeter became very unsteady, the positive column changed from pink to grey, the striæ assumed the form they had before the magnet was put on, and the voltage rose to its original value. If the coil was stopped and turned on again after the lapse of 15 seconds or more, the whole of the changes were reproduced; if turned on at a shorter interval, any particular stage was at once produced according to the time that had elapsed since the stoppage.

The magnet evidently causes the current to be carried entirely by one gas; this change occupies several seconds, and the tube must rest for 15 secs. in order to allow the ions of one gas to disappear so that the other may carry its share of current.

The lower the pressure the smaller the field required to produce double striæ.

I next sought to determine whether the decrease in potential brought about by the transverse field takes place at the cathode, or whether some of this might not be found in the positive column, owing to the increased number of striæ found there when the field is on. A tube as shown in fig. 4 was used.

The side electrodes B, C, were thin pointed wires fastened in with sealing-wax joints. B was always in the positive column when A was the anode; at the lower pressures C was in the Faraday dark space or the negative glow.

The difference of potential between A and the other electrodes was measured with the voltmeter both when the magnet was on at D, and when it was off. A large fall in potential between A–B and A–C was always brought about by the field below the critical pressure, the percentage fall over these portions of the tube being much greater than over the whole. By far the greatest potential fall (magnet off) takes place at the cathode, so that the actual fall in volts between A–C brought about by the magnet was not sufficient to account for the whole decrease for the tube. When the magnet caused the potential between A–D to increase, it frequently effected a fall between A–B or A–C if the gas-pressure was not above about 2 mms.

The results given by this tube are, however, not altogether trustworthy, for the sealing-in of the electrodes B, C caused the magnet to influence the tube as a whole at a much lower pressure than previously; that is to say, the positive column was striated for some time before the magnet caused the voltage to drop. This may be due to the distortion of the discharge by the wires, but it is much more likely on account of B and C acting as secondary anodes and cathodes according to which is connected with the voltmeter. It is certain that a dark space was developed round them in some instances.

Fig. 4.

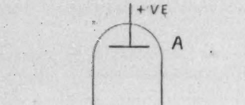
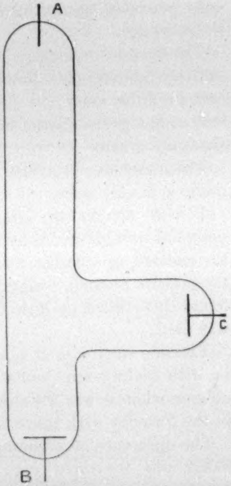


Fig. 5.



A tube was next tried in which the development of the positive column could be controlled while everything else remained the same. This is shown in fig. 5.

A was a pointed aluminium wire and was always made the anode. B, C were disks, and either could be made the cathode at will; the limbs containing them were made as nearly as possible equal. At low pressures, when B was cathode, the positive column was very short; when the glass round B gave a green fluorescence scarcely any positive column was present. Under similar conditions with C cathode the column extended to the side limb. With the field on at either cathode, the positive column extended nearly the whole length of the tube. In spite of this great difference in development, no difference could be found in the behaviour of B or C with the magnet, at a pressure when the latter brought about a large decrease in the voltage.

We may therefore conclude that the fall in potential has its chief seat near the cathode.

As to the cause of this fall, it may be suggested that the negative ions are caused to travel over greater paths in the dark space, because the magnetic field causes them to move obliquely through it. In this space, therefore, they produce more ions by collision, and the discharge being previously poor in ions at this point, as shown by numerous experiments, this causes the fall in voltage between the terminals.

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XXXIX. *Action of Radium on the Electric Spark.* *By* R. S. WILLOWS, *M.A., D.Sc., and* J. PECK, *B.Sc.* (Communicated by Prof. J. J. THOMSON, F.R.S.) *

IT is well known that radioactive substances, when placed near a spark, in general cause the discharge to pass more readily by the formation of ions in the electric field. The action on a long spark is, however, different, and appears, as far as our knowledge goes at present, somewhat irregular. In some cases the discharge passes more readily, in others with greater difficulty †.

The present paper describes experiments which were undertaken with the object of studying the origin of this influence.

The spark was produced between two brass spheres of unequal diameters, one 27 mm. the other 48 mm., by means of a Wimshurst machine carrying three pairs of plates of 2 feet diameter. The radium bromide, 5 mgm., was contained in a small capsule closed with mica. The strength of the sample was not known, but in most of the experiments it was not of the highest activity.

The action was altogether different according to the direction of the discharge. Using a spark-gap longer than 2 cms., and making the larger knob positive, the radiations had practically no influence. With the smaller sphere positive the radium in most cases extinguished the spark, the visible portion of the discharge being confined to a glow at the surface of both electrodes. With the machine used a torrent of sparks could be kept up with a gap of 3 cms.; in these circumstances the spark was least sensitive to the radium.

As the knobs were gradually pulled apart, a good deal of brush discharge became mixed with the spark, and the effect was more easily observed. With still longer discharges there appeared only a positive brush, beyond this a dark space, and on the surface of the cathode a faint glow. In this condition the discharge is very susceptible to influence, it may be extinguished instantaneously by the radium held at a distance of 1 metre.

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* Read January 27, 1905.

† See abstract of a paper by D. M. Sokoltzow, 'Science Abstracts,' vol. vii. p. 343 (1904), and also one by A. Stefanini & L. Magri on the same page.

The following series of observations and the accompanying curve (fig. 1, p. 466) plotted from the numbers, show how the sensibility increases with the spark-length. The two spheres were set apart so that their nearest points were distant from each other the lengths given in the first column, the machine was put in action and the radium gradually approached along a line at right-angles to the discharge. The numbers in column 2 give the distances at which the discharge was just extinguished; in each case the numbers are the mean of several observations.

Distance apart of knobs. Smaller sphere positive.	Distance of radium from spark-gap when the discharge is just put out.
cms.	cms.
4	6*
5	12
6	15
7	17·5
8	20
9	24
10	28
11	29·7
13	39
15	66

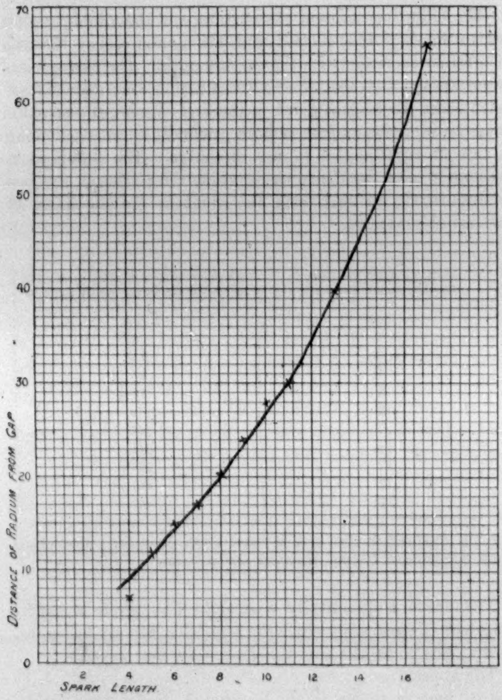
* The discharge in this case is mixed brush and spark. Most of the brush is stopped at 10 cms., the spark also at 6 cms.

The curve is practically a straight line up to a spark-length of 12 cms., after which the sensibility rapidly increases. It is at this point also that the dark space first appears.

The appearance of the discharge with spark-lengths 4, 7, and 13 cms. is shown in the sketches A, B, &c. respectively of fig. 2 (p. 467).

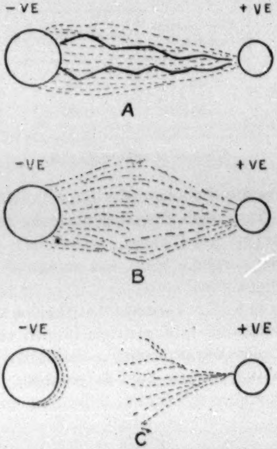
When the anode consisted of a piece of No. 22 wire, with which the discharge passed nearly entirely in brush form, the radium had no visible effect. Both this and the fact that the radiation has no influence when the discharge passes from the larger to the smaller sphere, are consistent with the theory that a minimum discharge potential is necessary for the effect to be produced. In each of these cases, the mean potential gradient is less than it is when the spark is in its sensitive condition.

Fig. 1.



Although the visible discharge is extinguished, it does not follow that the current actually passing is diminished; the conductivity might be so largely increased that a non-luminous discharge takes place. It was found possible to measure this current, the numbers, although variable, being sufficiently accurate for the purpose of the experiment. To do this, the positive side of the spark-gap was connected directly to the machine, the other side to one terminal of a dead-beat D'Arsonval galvanometer, the other terminal of which was earthed. The negative pole of the machine was also earthed. The current thus passed from the anode across to the cathode, and through the galvanometer to earth.

Fig. 2.



It was found that the extinction of the spark corresponded with a decrease in the current. Further, by encasing the radioactive salt in a thick lead case, so that a narrow beam only of rays issued, it was possible to investigate the effect at different points of the discharge. The lead case was closed by a thick lid of lead, working on a hinge, which could be opened or closed by the observer at a distance.

The discharge had the form shown in C, fig. 2. The two sets of numbers given below will show the results and the order of accuracy obtained.

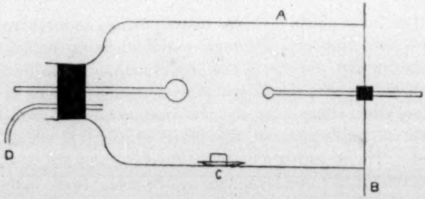
	Near positive knob.	Middle of discharge.	Near negative knob.
Per cent. decrease of current.	52	45·5	33
	50·5	41	30

Distance of radium, 20 cms.

The numbers are the mean of a large number of separate observations. It is seen therefore that the positive end is the most sensitive under these conditions. With shorter discharges, the currents were too irregular to admit of conclusions being drawn.

A reference to C, fig. 2, shows that the discharge near the anode is concentrated into a single branch very narrow in section. It hence seems probable that the most sensitive part is where the electric field is greatest. We found it impossible to measure this field at different points of the gap.

Fig. 3.



The apparatus shown in fig. 3 was used to investigate the effect of altering the pressure of the gas in which the discharge took place. The bell-jar A was 25 cms. long and 12 cms. in diameter. One end was closed with a brass plate, B, through which passed a rubber bung; D was connected with a water-pump and gauge. The radium capsule, C, was fastened to a piece of soft iron which allowed of it being shifted to any part of the jar by means of an external magnet. The spark-gap was about 6 cms. long. The results are as follows:—

Pressure in cms. of mercury.	Effect of radium.
77	Visible discharge (spark and brush) stopped.
51	Discharge (all brush) stopped.
40	Discharge (all brush) concentrated but not extinguished.
28	No effect.

The action of the radium is therefore less marked as the pressure is reduced. This may be due to the reduction of the voltage required to maintain the current.

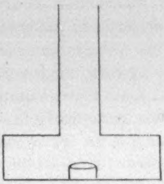
A very much weaker sample of the salt enclosed in a glass capsule showed quite an appreciable effect when the discharge was in a fairly sensitive condition. Thus the action cannot be due to the α rays, since these would be absorbed by the glass. It must be due to either the β or γ rays. Since the latter are supposed to be identical with Röntgen rays, the influence of these on the spark was tried, but they brought about no stoppage of the discharge. The ionizing powers of the Röntgen rays and of the rays from the radium on the air in the spark-gap were of altogether different orders. Thus the radium at 50 cms. readily extinguished the spark, a Röntgen-ray tube 25 cms. away had no effect; but if the gap was replaced by the knob of a charged electroscope, the latter was completely discharged in less than 1 sec. when the tube was working, while with the radium complete discharge took 5 min. to accomplish.

The result obtained cannot therefore be ascribed to the γ rays, nor does it depend directly on the number of ions produced by the ionizing agent, but rather is a special action of the β rays.

The negative result obtained with Röntgen rays also precludes the following explanation:—The ions in the discharge are produced in two ways, directly by the field and by collision of ions already present with neutral atoms if the field is sufficiently intense. If the ionizing agent, rays of any kind, produces sufficient ions, the voltage between the electrodes may be so reduced that the second method of formation is impossible, and the final result may be a decrease in the current which passes. The effect would then be more marked as the external agent produces more ions, but, as is seen, this is not the case.

It remains only to show directly that the effects observed are due to the β rays. The radium was enclosed in a lead capsule of the shape shown in fig. 4, and was placed near enough to the spark to extinguish it. The capsule was further placed between the poles of a strong electromagnet, so orientated that the β rays would be driven into the lead by its action. Immediately the magnetic field was created, the discharge passed again with undiminished intensity, while when the current in the magnet was stopped the visible discharge also stopped, showing clearly that it was the easily deflectable β part of the rays that was concerned in the extinction.

Fig. 4.



As these rays are of the same nature as Lenard rays, the action on the spark of the latter was also tried. With the aluminium window of the tube distant from the spark about 10 cms., an appreciable effect could be seen; when the tube was in good working order and the spark in its most sensitive condition, extinction of the latter could be readily brought about. If the cathode rays inside the tube were deflected from the window by means of a magnet, so that no Lenard rays existed outside, no action on the spark could be seen. In fact the discharge from the Wimshurst could be started and stopped quite readily by simply approaching a magnet to the cathode of the Lenard tube.

The ionizing action of the rays as tested by an electroscope 10 cms. away from the window, was more than that of the radium 50 cms. away, but less than that of the Röntgen rays when the tube emitting the latter was 25 cms. away.

This experiment also shows that the failure of the Röntgen rays to put out the spark cannot be ascribed to their *intermittence*, as both Lenard and Röntgen tubes were worked by the same coil. It would also tend to prove that the effects are dependent on the high velocity of the β rays, rather than directly on the number of ions they produce in the gap; for the ionization produced by the Lenard rays is here greater than that produced by the radium, while it is well known that their velocity is much less than that of the β rays that the radium emits.

In conclusion we must express our gratitude to Mr. J. J. Vezey for the loan of some radium bromide, with which the preliminary experiments were performed, and also to the Governors and Principal of the Institute for providing us with the radium bromide necessary to continue those experiments. We are also indebted to Mr. F. G. Bratt for considerable help in the experimental part of the investigation.

Sir John Cass Technical Institute, Jewry St., E.C.
November 10, 1904.

DISCUSSION.

Prof. F. T. TROUTON expressed his interest in the paper, and referred to the fact that the action of the radium depended upon the direction in which the current was passing. He could see no obvious explanation of this.

XL. *Some Methods for Studying the Viscosity of Solids.* *By* BORIS WEINBERG, *Private-docent of Physics at the University of Odessa.*

1. As the investigations of Prof. Trouton and Mr. Andrews "On the Viscosity of Pitch-like Substances," which were presented to the Society on June 12, 1903, bear on the same subject as my independent * experiments, I hope that the Society will receive the following brief account of my investigations, which differ in detail from those of Prof. Trouton and Mr. Andrews.

The "coefficient of viscosity" of solid materials being a function of—at least—the angular velocity of shear, I turned my attention rather to work out methods for studying the viscosity of solids and find the laws governing the phenomena than to obtain precise numerical data. I worked with substances much more "viscous" than the substances studied by Prof. Trouton and Mr. Andrews—principally with lead at ordinary temperatures (μ being of the order 10^{14}–10^{16} C.G.S.). Such substances show more complicated behaviour than the less viscous ones—*e. g.*, the initial stage, before the

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* Without in any way touching the question of priority I would like to give proofs of the independency of my researches, as this is one more sign of the fact that the questions relating to the viscosity of solid materials are rapidly passing from the stage of qualitative observations (Tresca, Spring) to the stage of quantitative measurements (*vide* Segel, *Phys. Zeits.* iv. pp. 493-494, 1903; Werigin, Lewkojeff and Tammann, *Ann. d. Phys.* (4) x. pp. 647-654, 1903).

I have used the method of parellelepipedon (I) as a lecture experiment for illustrating the physical meaning of the coefficient of viscosity and for determining this quantity for shoemaker's wax as long ago as 1901, in lectures on molecular physics (printed in May 1903). Preliminary accounts on this and other two (II and III) methods and some numerical data relating to the order of magnitude of the coefficient of viscosity of several solid materials were presented to the "Mathematical Section of the Society of Naturalists of New Russia" in Odessa, 31st October, 1903 (see "Proceedings" of the Society, xxvi. p. xli, 1904), and to the "Physical Section of the Russian Physico-Chemical Society" in St. Petersburg, 10th February 1904 (printed in the Journal of the Society, xxxvi. pp. 47-48 & 105, 1904).

"steady velocity" under constant strain is reached, lasts several weeks.

2. The following three methods were used :—

I. One base of a parallelepipedon of the substance is fixed, the opposite base is subjected to a stress parallel to its plane and the variation of the angle of shear with time is observed. The deformation being uniformly distributed over the body, this method is subject to all the complications arising from the variability of the "coefficient of viscosity" with angular velocity, which renders other methods less adapted for the calculation of this quantity. It is the only method which can serve to determine the coefficient of viscosity—if any—of a crystalline substance. Nevertheless this method is not sensitive enough for working with comparatively little stresses and is not sufficiently precise.

If F denotes the force, S the section, ρ the angle of shear, and t the time, the coefficient of viscosity μ can be calculated from the formula

$$\mathrm{F} = \mu \mathrm{S} \frac{d\rho}{dt}. \qquad (1)$$

It may be observed that in all the formulæ of the theory of elasticity containing the modulus of rigidity the latter can be replaced by the coefficient of viscosity with simultaneous replacement of the angle of shear by the angular velocity of shearing.

II. A rod or, better, a tube is subjected to a twist and the variations in time of the relative angular displacements of different sections are observed.

This method is very sensitive and precise : it easily allows to ascertain angular velocities of shearing of superficial layers as low as $\frac{1}{2} . 10^{-11} \frac{\text{radian}}{\text{second}}$ and to measure these velocities with equal precision. It can give reliable results only for relatively thin tubes: as the angular velocity of shearing is different for different coaxial layers, it is only for thin tubes that the rectilineal generating lines are transformed into true helices and that the quantity μ can be assumed constant and calculated from the formula

$$T = \mu \frac{\pi(R_1^4 - R_2^4)}{2L} \frac{d\rho}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In formula (2) T is the torque, R_1 and R_2 the external and internal radii of the tube, and ρ the relative angular displacement of two sectional layers distant L cms.

The method is adapted still less, than the previous, for experiments with substances of small viscosity for the reasons already pointed out by Mr. Andrews and Prof. Trouton.

III. The substance to be investigated fills the space between two coaxial glass tubes, the inferior part of this space being filled with mercury to avoid complications arising from the friction at the bottom.

This method can be made only a few times more sensitive than the method I $\left(\text{precision not exceeding } 10^{-10} \frac{\text{radian}}{\text{second}}\right)$, but is the best of the three for substances of small viscosity yielding under their proper weight. We have here

$$T = \mu \frac{4\pi L}{\frac{1}{R_2^2} - \frac{1}{R_1^2}} \frac{d\rho}{dT}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R_1 and R_2 are the outer and inner radii of the cylindrical layer, and L its length.

3. All the measurements are made by the scale and mirror method.

The substances were examined under the following *regimes*:

a. Constant stress;
b. No stress after a finite temporary strain;
c. Constant strain;
d. Constant rate of variation of the strain.

Most of the experiments were and are performed with lead; some with shoemaker's wax, annealed copper, Iceland spar, soft and hard steel.

The laws which govern all these phenomena and their theory I postpone for further publications.

XLI. *Radiation Pressure.*

By Prof. J. H. Poynting, *Sc.D., F.R.S.**

A HUNDRED years ago, when the corpuscular theory held almost universal sway, it would have been much more easy to account for and explain the pressure of light than it is today, when we are all certain that light is a form of wave-motion. Indeed, on the corpuscular theory it was so natural to expect a pressure that numerous attempts were made † in the eighteenth century to detect it. But the early experimenters had a greatly exaggerated idea of the force they looked for. Even on their own theory it would only have double the value which we now know it to possess, and their methods of experiment were utterly inadequate to show so small a quantity. But had these eighteenth-century philosophers been able to command the more refined methods of today, and been able to carry out the great experiments of Lebedew and of Nichols and Hull, and had they further known of the emission of corpuscles revealed to us by the cathode stream and by radioactive bodies, there can be little doubt that Young and Fresnel would have had much greater difficulty in dethroning the corpuscular theory and setting up the wave theory in its place.

The existence of pressure due to waves, though held by Euler and used by him 160 years ago to explain the formation of comets' tails by repulsion, seems to have dropped out of sight, till Maxwell, in 1872, predicted its existence as a consequence of his Electromagnetic Theory of Light. It is remarkable that it should have been brought to the front through the investigation of such a special type, such an abstruse case, of wave-motion, and that it was not seen that it must follow as a consequence of any wave-motion, whatever the type of wave we suppose to constitute Light. I believe that the first suggestion that it is a general property of waves is due to Mr. S. Tolver Preston, who in 1876 * pointed out the analogy of the energy-carrying power of a beam of light with the mechanical carriage by belting, and calculated the pressure on the surface of the Sun by the issuing radiation, obtaining a value equal to the energy-density in the issuing stream, without assumption as to the nature of the waves. But though the analogy is valuable, I confess that Mr. Preston's reasoning does not appear to me conclusive, and I think it still remains an analogy. There is, I suspect, some general theorem yet to be discovered, which shall relate directly the energy and the momentum issuing from a radiating source. It seems possible that in all cases of energy transfer, momentum in the direction of transfer is also passed on, and therefore there is a back pressure on the source. Such pressure certainly exists in material transfer, as in the corpuscular theory. It exists too, as we now know, in all wave transfer. From the investigation below (p. 479) it appears to exist when energy is transferred along a revolving twisted shaft. In heat-conduction in gases, the kinetic theory requires a carriage of momentum from hotter to colder parts ; so that there is some ground for supposing the pressure to exist in all cases.

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* Presidential Address, delivered at the Annual General Meeting, February 10, 1905.

† Some account of these methods is given by Nichols and Hull in "The Pressure due to Radiation," Proc. Am. Ac. xxxviii. No. 20, p. 559. See also Priestley, "On Vision," p. 385.

Though we have not yet a general and direct dynamical theorem accounting for radiation pressure, Professor Larmor † has given us a simple and most excellent indirect mode of proving the existence of the pressure, which applies to all waves in which the average energy-density for a given amplitude is inversely as the square of the wave-length. Let us suppose that a train of waves is incident normally on a perfectly reflecting surface. Then, whether the reflecting surface is at rest, or is moving to or from the source, the perfect reflexion requires that the disturbance at its surface shall be annulled by the superposition of the direct and reflected trains. The two trains must therefore have

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* Engineering, 1876, vol. xxi. p. 83.

† *Encyc. Brit.* xxxii. Radiation, p. 121.

equal amplitudes. Suppose now that the reflector is moving forwards towards the source. By Doppler's principle, the waves of the reflected train are shortened, and so contain more energy than those of the incident train. This extra energy can only be accounted for by supposing that there is a pressure against the reflector, that work has to be done in pushing it forward. When the velocity of the reflector is small, the pressure is easily found to be equal to $E\left(1+\frac{2u}{U}\right)$, where $\frac{E}{2}$ is the energy-density just outside the reflector in the incident train, U is the wave-velocity, and u the velocity of the reflector. If $u=0$, the pressure is E; but it is altered by the fraction $\frac{2u}{U}$ when the reflector is moving, and the alteration changes sign with u. A similar train of reasoning gives us a pressure on the source, increased when the source is moving forward, decreased when it is receding.

It is essential, I think, to Larmor's proof that we should be able to move the reflecting surface forward without disturbing the medium except by reflecting the waves. In the case of light-waves it is easy to imagine such a reflector. We have to think of it as being, as it were, a semipermeable membrane, freely permeable to æther, but straining back and preventing the passage of the waves. In the case of sound-waves, or of transverse waves in an elastic solid, it is not so easy to picture a possible reflector. But for sound-waves I venture to suggest a reflector which shall freeze the air just in front of it, and so remove it, the frozen surface advancing with constant velocity u. Or perhaps we may imagine an absorbing surface which shall remove the air quietly by solution or chemical combination. In the case of an elastic solid, we may perhaps think of the solid as melted by the advancing reflector, the products of melting being passed through pores in the surface and coming out to solidify at the back.

Though Larmor's proof is quite convincing, it is, I think, more satisfying if we can realize the way in which the pressure is produced in the different types of wave-motion.

In the case of electromagnetic waves, Maxwell's original mode of treatment is the simplest, though it is not, I believe, entirely satisfactory. According to his theory, tubes of electric and of magnetic force alike produce a tension lengthways and an equal pressure sideways, equal respectively to the electric and magnetic energy-densities in the tubes. We regard a train of waves as a system of electric and magnetic tubes transverse to the direction of propagation, each kind pressing out sideways—that is, in the direction of propagation. They press against the source from which they issue, against each other as they travel, and against any surface upon which they fall. Or we may take Professor J. J. Thomson's point of view *. "Let us suppose that the reflecting surface is metallic; then, when the light falls on the surface, the variation of the magnetic force induces currents in the metal, and these currents produce opposite effects to the incident light, so that the inductive force is screened off from the interior of the metal plate: thus the currents in the plate, and therefore the intensity of the light, rapidly diminish as we recede from the surface of the plate. The currents in the plate are accompanied by magnetic force at right angles to them; the corresponding mechanical force is at right angles both to the current and the magnetic force, and therefore parallel to the direction of propagation of the light." In fact, we have in the surface of the reflector a thin current-sheet in a transverse magnetic field, and the ordinary electrodynamic force on the conductor accounts for the pressure.

In sound-waves there is at a reflecting surface a node—a point of no motion, but of varying pressure. If the variation of pressure from the undisturbed value were exactly proportional to the displacement of a parallel layer near the surface, and if the displacement were exactly harmonic, then the average pressure would be equal to the normal undisturbed value. But consider a layer of air quite close to the surface. If it moves up a distance y towards the surface, the pressure is increased. If it moves an equal distance y away from the surface, the pressure is decreased, but by a slightly smaller quantity. To illustrate this, take an extreme case, and for simplicity suppose that Boyle's law holds. If the layer advances half way towards the reflecting surface, the pressure

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* Maxwell's 'Electricity and Magnetism,' 3rd edition, vol. ii. p. 441, footnote.

is doubled. If it moves an equal distance outwards from its original position, the pressure falls, but only by one-third of its original value ; and if we could suppose the layer to be moving harmonically, it is obvious that the mean of the increased and diminished pressures would be largely in excess of the normal value. Though we are not entitled to assume the existence of harmonic vibrations when we take into account the second order of small quantities, yet this illustration gives the right idea. The excess of pressure in the compression half is greater than its defect during the extension half, and the net result is an average excess of pressure—a quantity itself of the second order—on the reflecting surface. This excess in the compression half of a wave-train is connected with the extra speed which exists in that half, and makes the crests of intense sound-waves gain on the troughs.

Lord Rayleigh*, using Boyle's Law, has shown that the average excess on a surface reflecting sound-waves should be equal to the average density of the energy just outside ; and I think the same result can be obtained by his method if we use the adiabatic law. But the subject is full of pitfalls, and I am by no means sure that the result is to be obtained so easily as it appears to be. It is perhaps worth while to note one of these pitfalls, of which I have been a victim. It is quite easy to obtain the pressure against a reflecting surface by supposing that the motion just outside it is harmonic. But the result comes out to $(\gamma+1)$ energy-density, where γ is the ratio of the specific heats. Lord Rayleigh kindly pulled me out of the pit into which I fell, pointing out that when we take into account second-order quantities the ordinary sound equation does not hold. In fact we cannot take the disturbance as harmonic, and the simple mode of treatment is illusory.

The pressure in transverse waves in an elastic solid is, I think, to be accounted for by the fact that when a square, ABCD, is sheared into the position *a*BC*d* (fig. 1) through an angle *e*, the axes of the shear, *a*C and B*d*, no longer make 45° with the planes of shear AD, BC. Since $ACa=\frac{e}{2}$, the pressure-line *a*C is inclined at $45^\circ \quad \frac{e}{2}$ to the direction of propagation, and the tension-line at $45^\circ+\frac{e}{2}$ to that line. The result is a small pressure perpendicular to the planes of shear, that is, in the direction of propagation; and this small pressure is just equal to the energy-density of the waves.

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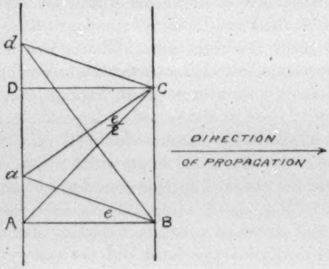
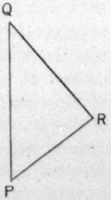
* Phil. Mag. iii. 1902, p. 338, "On the Pressure of Vibrations."

Fig. 1.



For let PQR (fig. 2) be a small triangular wedge of the solid, PQ being a plane of shear perpendicular to the direction of propagation. Let this wedge have unit thickness perpendicular to the plane of the figure. Let PR be along a pressure-line and QR along a tension line, and let pressure and tension each be P. Resolve the forces on PR and QR perpendicular to PQ. Then we have a force from right to left,

Fig. 2.



$$\mathrm{P.\,QR}\cos\mathrm{PQR}-\mathrm{P.\,PR}\cos\mathrm{QPR}$$

$$=\mathrm{P.\,PQ}\left(\cos^2\left(45^\circ-\frac{e}{2}\right)-\cos^2\left(45^\circ+\frac{e}{2}\right)\right)$$

$$=\mathrm{P.\,PQ}\,.\,e.$$

Thus, to prevent motion in the direction of propagation there must be a pressure on PQ equal to $\mathrm{P}e=ne^2$, where n is the rigidity modulus. But the strain-energy per unit volume is $\frac{ne^2}{2}$, and the kinetic energy is equal to it. The total energy-density is therefore ne^2, and the pressure is equal to this.

The pressure of elastic solid waves appears to be beyond experimental verification at present. But that of sound-waves has been demonstrated most successfully by Altberg*, working in Lebedew's laboratory at Moscow.

A small wood cylinder, 21 mm. diameter, was suspended at one end of a torsion arm, with its axis horizontal and transverse to the arm. One end of the cylinder occupied a circular hole in the middle of a board, there being just sufficient clearance to allow it to move; and the plane end was flush with the outer surface of the board. When very intense sound-waves 10 cm. in length, from a source 50 cm. distant, impinged on the board, the cylinder was pushed back, the pressure sometimes rising to as much as 0·24 dyne/sq. cm. The intensity of the sound was measured independently by the vibrations of a telephone-plate, in a manner devised by M. Wien; and through a large range it was found that the pressure on the cylinder was proportional to the intensity indicated by the telephone manometer.

Just lately Professor Wood † has devised a strikingly simple experiment to illustrate sound-pressure. The sound-waves from strong induction-sparks are focussed by a concave mirror on a set of vanes like those of a radiometer, and when the focus is on the vanes as they face the waves the mill spins round.

Theory and experiment, then, justify the conclusion that when a source is pouring out waves, it is pouring out with them forward momentum as well as energy, the momentum being manifested in the reaction, the back pressure against the source, and in the forward pressure when the waves reach an opposing surface. The wave train may be regarded as a stream of momentum travelling through space. This view is most clearly brought home, perhaps, by considering a parallel train of waves which issues normally from a source for one second, travels for any length of time through space, and then falls normally on an absorbing surface for one second. During this last second, momentum is given up to the absorbing surface. During the first second, the same amount was given out by the source. If it is conserved in the meanwhile, we must regard it as travelling with the train.

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* *Ann. der Physik*, xi. 1903, p. 405.

† *Phys. Zeitschrift*, 1 Jan. 1905, p. 22.

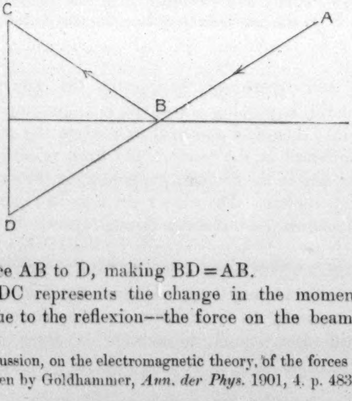
Since the pressure is the momentum given out or received per second, and the pressure is equal to the energy-density in the train, the momentum-density is equal to the energy-density ÷ wave-velocity.

This idea of momentum in a wave train enables us to see at once what is the nature of the action of a beam of light on a surface where it is reflected, absorbed, or refracted, without any further appeal to the theory of the wave-motion of which we suppose the light to consist [*].

It is convenient to consider the energy per linear centimetre in the beam, and the total pressure force, equal to this linear energy-density, so as to avoid any necessity for taking into account the cross section of the beam.

Thus, in total reflexion, let a beam AB (fig. 3) be reflected along BC, and let AB = BC represent the momentum in each in length V equal to the velocity of light.

Fig. 3.



Produce AB to D, making BD = AB.

Then DC represents the change in the momentum per second due to the reflexion—the force on the beam, if such language is permissible; and CD is the reaction, the total light-force on the surface.

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[*] A discussion, on the electromagnetic theory, of the forces exerted by light is given by Goldhammer, *Ann. der Phys.* 1901, 4. p. 483.

If there is total absorption, let AB (fig. 4) represent the momentum of the incident beam. Resolve AB into AE parallel and EB normal to the surface. Then, since the momentum AB disappears as light-momentum, there must be a normal force EB on the surface and a tangential force AE parallel to the surface. I have lately[*] described an experiment which shows the existence of the tangential force AE.

Fig. 4.

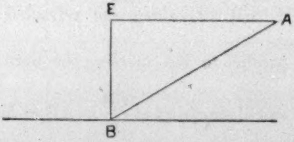
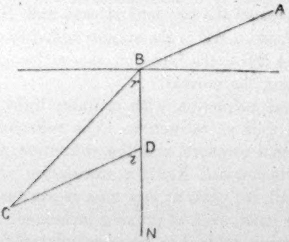


Fig. 5.



If there is total refraction, let AB (fig. 5) be refracted along BC. If E is the energy in unit length of AB, and if E′ is the energy in unit length of BC, the equality of energy in the two beams is expressed by

$$VE = V'E'.$$

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[*] Phil. Mag. Jan. 1905, p. 169.

But if M is the stream of momentum passing per second along AB, and if M′ is that along BC,

$$M = E \quad \text{and} \quad M' = E'.$$

Whence $$VM = V'M'$$

and $$M' = \frac{V}{V'} M = \mu M.$$

Let $AB = M$, and BC along the refracted beam $= M' = \mu M = \mu AB$.

Draw CD parallel to BA, meeting the normal BN in D. Then

$$CD = CB \sin r / \sin i = \frac{CB}{\mu} = AB = M.$$

Hence by the refraction, momentum DC has been changed to momentum BC, or momentum BD has been imparted to the light. There is therefore a reaction DB on the surface. The force DB may be regarded as a pull-out or a pressure from within, and it is along the normal *.

If the refraction is from a denser to a rarer medium, CB will now represent the incident stream and BA or CD the refracted stream. BD is the stream added to CB to change it to CD, and DB is the force on the surface, again a force outwards along the normal.

In any real refraction with ordinary light, there will be reflexion as well as refraction. The reflexion always produces a normal pressure, and the refraction a normal pull. But with unpolarized light, a calculation shows that the refraction pull, for glass at any rate, is always greater than the reflexion push, even at grazing incidence.

The following table has been calculated from Fresnel's formula for unpolarized light by Dr. Barlow :—

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* It has been pointed out by J. J. Thomson, 'Electricity and Matter,' p. 67, "that even when the incidence of the light is oblique, the momentum communicated to the substance is normal to the refracting surface." The change of momentum of a beam of light is, it may be noted, the same on the wave and on the corpuscular theory.

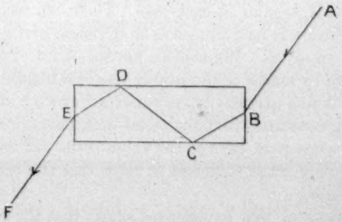
P = total pull on surface.
M = momentum per second in incident beam.
R = reflexion coefficient for angle *i*.
$\mu = 1{\cdot}5$.

i.	R.	P/M.
0	·0400	·4000
20	·0402	·4240
40	·0458	·4925
50	·0572	·5310
60	·0 93	·57 0
65	·1205	·5771 Maximum.
70	·1710	·5683
75	·2551	·5329
80	·3878	·4521
89	·9044	·0738
$90-d\theta$	......	$2\mu^2 d\theta$
90	1·0000	·0000

If a ray of light passes obliquely through a parallel plate, there is a normal pull outwards at incidence and a normal pull outwards at emergence; and if the refraction were total, this would result in a couple. But since some of the light returns into the first medium, it is easy to see that the net result is a normal repulsion and a couple.

An experiment which I have lately made in conjunction with Dr. Barlow will serve as an illustration of the idea of a beam of light regarded as a stream of momentum. A rectangular block of glass, 3 cm. × 1 cm. × 1 cm., was suspended

Fig. 6.—Plan.

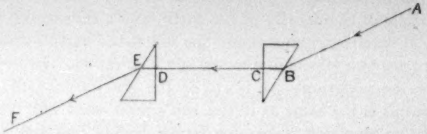


by a quartz fibre so that the long axis of the block was horizontal. It hung in a case with glass windows, which was exhausted to about 15 mm. of mercury. A horizontal beam of light from either a Nernst lamp or an arc was directed on to one end of the block so that it entered centrally as AB in fig. 6, and at an angle of incidence about 55°. After two internal reflexions it emerged centrally as EF from the other end. Thus a stream of momentum AB was shifted parallel to itself into the line EF, or a counter-clockwise couple acted on the beam. The reaction was a clockwise couple on the block. Using mirror, telescope, and a millimetre-scale about 184 cm. distant, a very small deflexion could just be detected with the strongest light and in the right direction. But the quartz fibre was rather coarse, indeed needlessly strong; and as the time of vibration was only 39 seconds, the deflexion was very minute. To render the effect more evident we used intermittent passage of the beam, sending it in during the half-period of vibration while B was moving from A, and shutting it off while B was moving towards A. The swings then always increased. When the beam was sent in during the approaching half and shut off during the receding half, the swings always decreased, and always rather more rapidly than they increased during the first half. For in the first case the natural damping acted against the light couple, and in the second with it. In one experiment the average increase was ·55 scale-division and the average decrease ·61 per period, and was fairly regular in each case. The mean was ·58. The steady deflexion is half this, or 0·29 division, giving a couple 11×10^{-6} cm.-dyne. We made a measurement of the energy in the beam by means of the rate of rise of a blackened silver disk; but it was necessarily very inexact, as we had no means of securing constancy in the arc used in this experiment. This energy measurement gave as the value of the couple 6×10^{-6}, and the agreement is sufficient to show that the order of the result is right.

An analysis of this experiment shows that the couple was really due to the pressures at the two internal reflexions; for, as we have seen, the forces at incidence at A and emergence at E are normal and produce no twist.

Another experiment which we have made is, I think, more interesting, in that it brings into prominence the pull outwards or push from within occurring on refraction. Two glass prisms, each with refracting angle 34°, another angle being a right angle, and with refracting edge 1·6 cm. long, were arranged as in fig. 7 (which shows the plan) at the ends of a thin brass torsion-arm suspended at its middle point from a quartz fibre in the same case as that used in the last experiment. The two inner faces were 3 cm. apart, and their width was 1·85 cm. A mirror gave the reflexion of a millimetre-scale 171·4 cm. distant. The moment of inertia of the system was 48 gm.-cm.^{2}, and the time of vibration was 317 seconds. The air-pressure was reduced as before. When a beam of light from a Nernst lamp was sent through the system, as shown in the figure, it was shifted parallel to itself through a distance about 1·64 cm. The torsion-arm moved round clockwise by an easily measurable amount. In one experiment the deflexion was 3·3 scale-divisions, indicating a couple $1·84 \times 10^{-5}$ cm.-dyne. The same beam directed on to the blackened silver disk gave the linear energy-density as $9·8 \times 10^{-6}$, which should have given a couple $1·6 \times 10^{-5}$. Though the agreement is perhaps accidentally close, yet, as we could use a Nernst lamp, the measurements were much more trustworthy than in the last experiment.

Fig. 7.—Plan.



The interesting point here is that the effect could only be produced by a force outwards at B and E. Whatever forces exist at C and D would be normal to the surfaces and would give no twist.

A very short experience in attempting to measure these light-forces is sufficient to make one realise their extreme minuteness—a minuteness which appears to put them beyond consideration in terrestrial affairs, though I have tried to show* that they may just come into comparison with radiometer-action on very small dust particles.

In the Solar system, however, where they have freer play and vast times to work in, their effects may mount up into importance. Yet not on the larger bodies; for on the earth, assumed to be absorbing, the whole force of the light of the sun is only about a 50 million-millionth of his gravitation-pull. But since the ratio of radiation-pressure to gravitation-pull increases in the same proportion as the radius diminishes if the density is constant, the pressure will balance the pull on a spherical absorbing particle of the density of the earth if its radius is a 50 billionth that of the earth—a little over a hundred-thousandth of a centimetre—say, if its diameter is a hundred-thousandth of an inch.

We may illustrate the possible effects of radiation pressure without proceeding to such fineness as this. Let us imagine a particle of the density of the earth, and a thousandth of an inch in diameter, going round the sun at the earth's distance. There are two effects due to the sun's radiation. In the first place, the radiation-push is $\frac{1}{100}$ of the gravitation-pull; and the result is the same as if the sun's mass were only 99/100 of the value which it has for larger bodies like the earth. Hence the year for such a particle would be longer by $\frac{1}{200}$, or about 367 instead of $365\frac{1}{4}$ days. In the second place, the radiation absorbed from the sun and given out again on all sides is crushed up in front as the particle moves forward and is opened out behind. There is thus a slightly greater pressure due to its own radiation on the advancing hemisphere than on the receding one, and this appears as a small resisting force in the direction of motion. Through this the particle tends to move in a decreasing orbit spiralling in towards the sun, and at first at the rate of about 800 miles per annum.

Further, if there be any variation in the sun's rate of emitting energy, there will be a corresponding variation in the increase of the year and the decrease of the solar distance, and the particle, if we could only observe it, would form a perfect actinometer.

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* 'Nature,' Dec. 29, 1904, p. 200.

Though, unfortunately, we cannot observe the motion of independent small particles circling round the sun at the distance of the earth, there is good reason to suppose that some comets at least are mere clouds of dust. If we are right in this supposition, they should show some of these effects. Encke's comet at once suggests itself as of this class; for, as everyone knows, it shortens its journey of 3½ years round the sun on every successive return, and on the average by about 2½ hours each revolution. Mr. H. C. Plummer * has lately been investigating this comet's motion; and he finds that if it were composed of dust particles, each of the earth's density and about $\frac{1}{50}$ mm. or rather less than a thousandth of an inch in diameter, the resisting force due to radiation pressure would account for its accelerating return. But the sun's effective mass would be reduced by about 1/80; and on certain suppositions he finds that the assumed mean distance as calculated from Kepler's law, without reference to radiation, is greater than the true mean distance by something of the order of 1 in 400, and he thinks such a large error is hardly possible. So that radiation pressure has not yet succeeded in fully explaining the eccentricities of this comet. But comets are vague creatures. As Mr. Plummer suggests, we hardly know that we are looking at the same matter in the comet at its successive returns; and I still have some hope that the want of success is due to the uncertainty of the data.

There is one more effect of this radiation pressure which is worthy of note: its sorting action on dust particles. If the particles in a dust cloud circling round the sun are of different sizes or densities, the radiation accelerations on them will differ. The larger particles will be less affected than the smaller, will travel faster round a given orbit, and will draw more slowly in towards the sun. Thus a comet of particles

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* Monthly Notices R.A.S., Jan. 1905, "On the Possible Effects of Radiation on the Motion of Comets, with special reference to Encke's Comet."

of mixed sizes will gradually be degraded from a compact cloud into a diffused trail lengthening and broadening, the finer dust on the inner and the coarser on the outer edge.

Let us imagine, as an illustration of this sorting action, that a planet, while still radiating much energy on its own account, while still in fact a small sun, has somehow captured and attached to itself as satellite a cometary cloud of dust. Then, if the cloud consists of particles of different sizes, while all will tend to draw in to the primary, the larger particles will draw in more slowly. But if the larger particles are of different sizes among themselves, they will have different periods of revolution, and will gradually form a ring all round the planet on the outside. Meanwhile the finer particles will drift in, and again difference in size will correspond to difference in period and they too will spread all round, forming an inner fringe to the ring. If there are several grades of dust with gaps in the scale of size, the different grades will form different rings in course of time. Is it possible that here we have the origin of the rings of Saturn ?

The Radiation Theory is only just starting on its journey. Its feet are not yet clogged by any certain data, and all directions are yet open to it. Any suggestion for its future course appears to be permissible, and it is only by trial that we shall find what ways are barred. At least we may be sure that it deals with real effects and that it must be taken into account.

Hovestadt's 'Jena Glass and its Scientific and Industrial Applications,' which brought him some interesting correspondence from practical opticians. Optics was perhaps his favourite subject, but his interests were wide. A paper on "Geometrical Illustrations of the Theory of Rent" appeared in the Journal of the Royal Statistical Society, vol. lxii. 1900. Some investigations of the Algebra of Difference Tables in the 'Quarterly Journal of Mathematics' were followed by his discovery of a new interpolation formula, an account of which was published at the request of the Institute of Actuaries in their Journal (vol. xxxv. 1901). He also wrote the article "Interpolation" for the Supplement to the 'Encyclopædia Britannica.'

During the last months of his life he was actively engaged in very varied investigations. At the first meeting of the Research Committee of the Royal Geographical Society, in Nov. 1903, he described "A Flat Model which solves Problems in the Use of the Globes." He read two papers before the Royal Microscopical Society on "Microscopic Resolution" (Nov. 1903), and "A Short Proof of Abbe's Theorem" (July 1904). He gave a good deal of time to questions related to normal piling, and two papers which he read on the subject before the Mathematical Society were published together in the Society's 'Proceedings' under the title "A Calculus of Point Assemblages." As an expert on Underground Temperature, he was summoned as a witness before the recent Royal Commission on Coal Supplies, and at the time of his death he had just finished correcting the proofs of his evidence, and also his last British Association Report. He left no unfinished papers.

Professor Everett was elected a Fellow of the Physical Society in 1875, a Member of Council in 1899, and a Vice-President in 1900. He took a most keen interest in the Society, and rarely failed to attend the meetings, where he usually took an active part in the discussions. He was a man of kindly and genial disposition, ever willing to place his time and knowledge at the disposal of others, and taking genuine pleasure in the successes of his old students and friends. Though he was approaching his seventy-third year, Professor Everett's mental and physical energies were unflagging, and his sudden death from heart failure on the 9th August, 1904, came as a great shock to his family and friends.

FIG. 1.

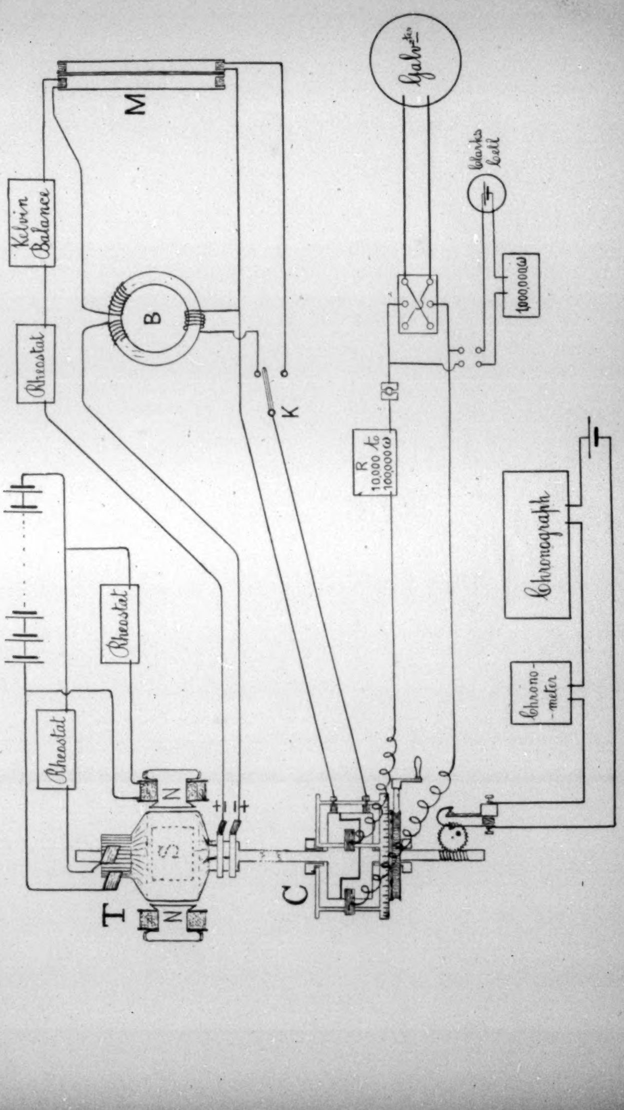
Kelvin Balance
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Fig. 1.

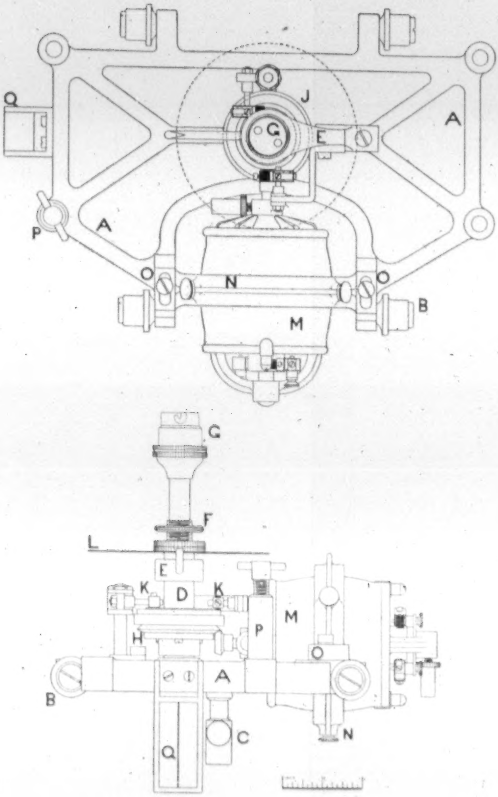
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Fig. 3.

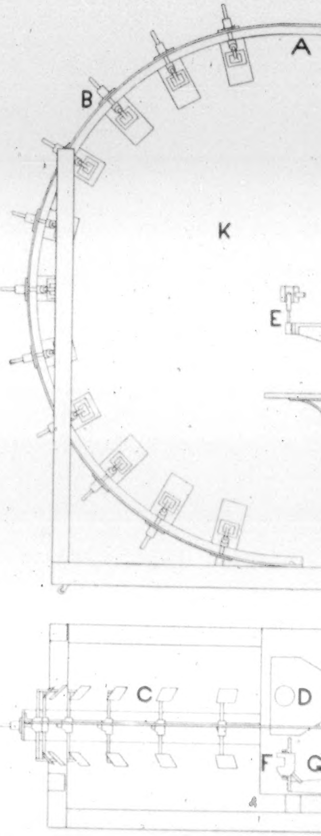

Fig. 2.

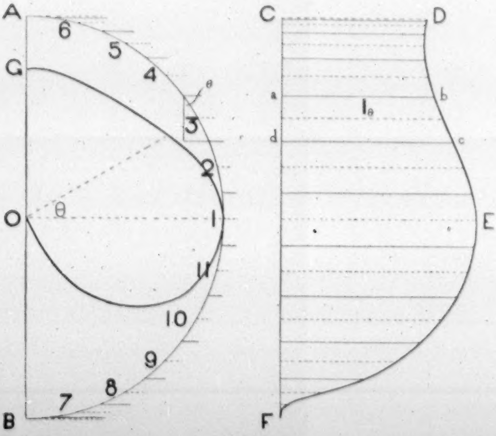
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Fig. 4.

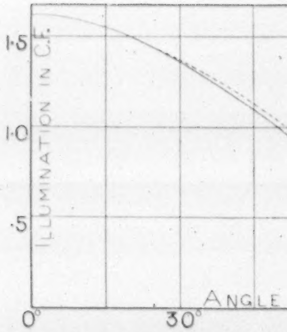
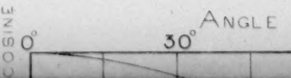
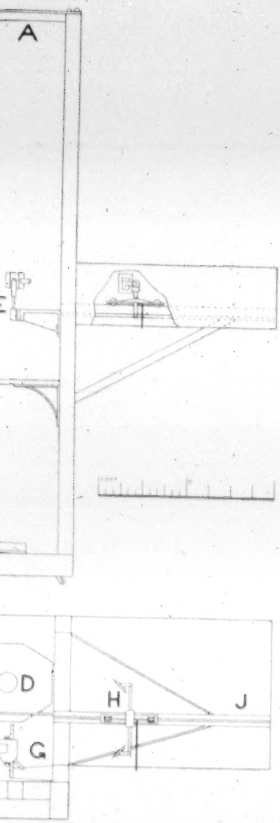


Fig. 5.



IG. 3.



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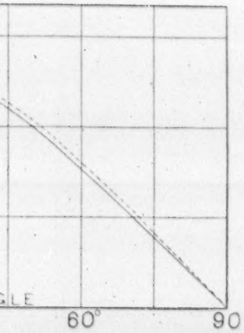


. 5.

Fig. 6.

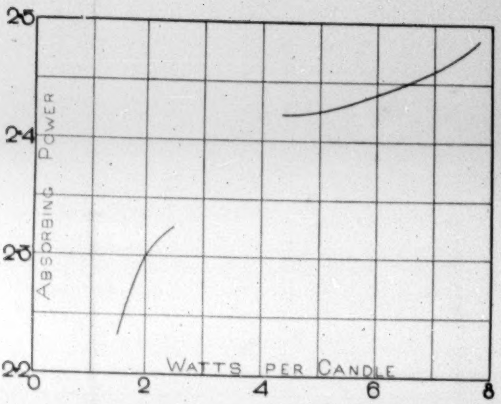


Fig. 7.

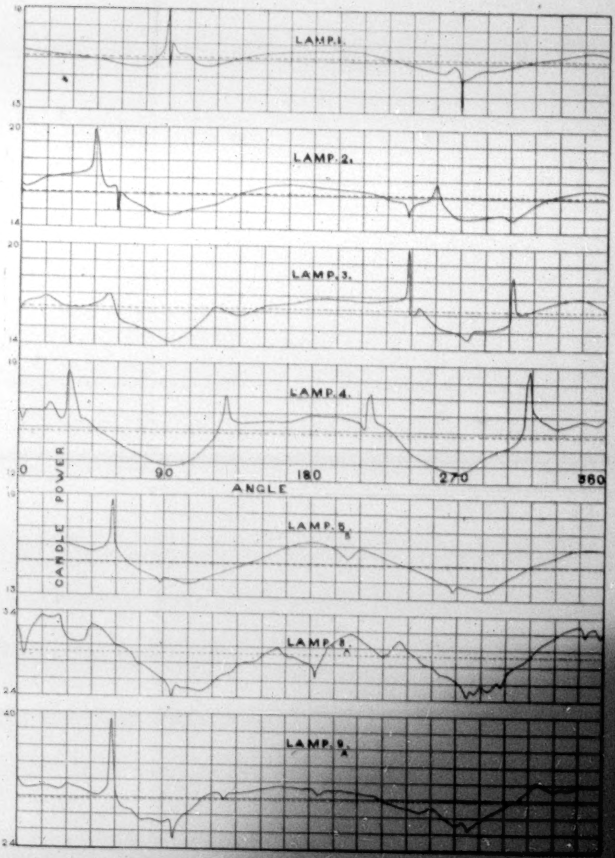
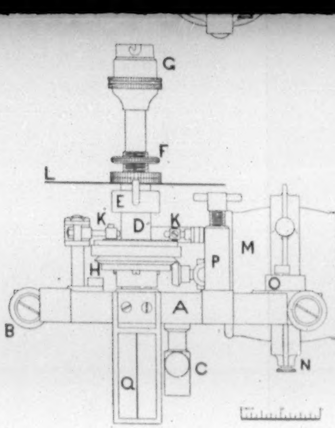


Fig. 8.

Fig. 8a.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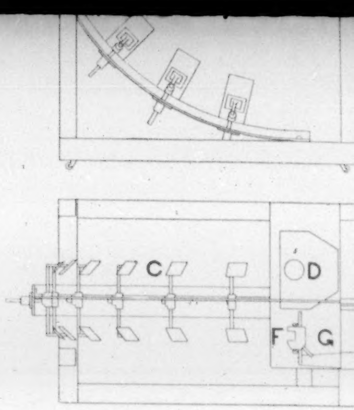

Fig. 2.

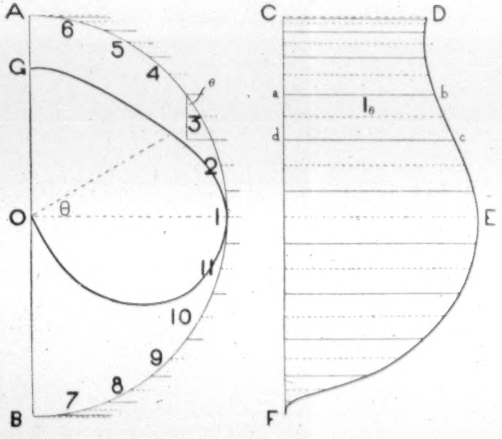
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Fig. 4.

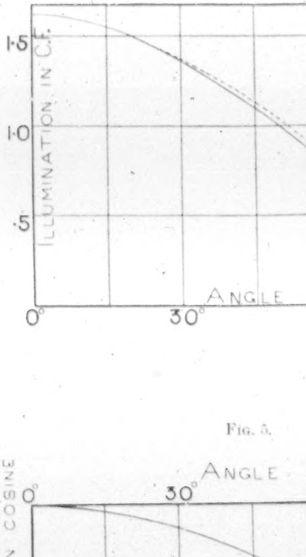
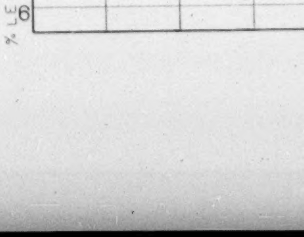
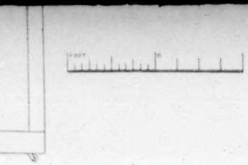
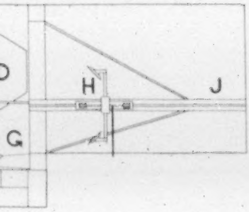


Fig. 5.







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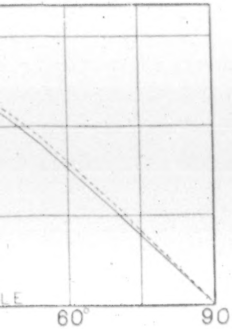

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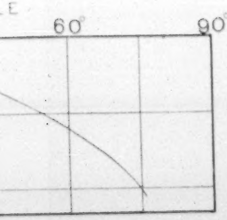

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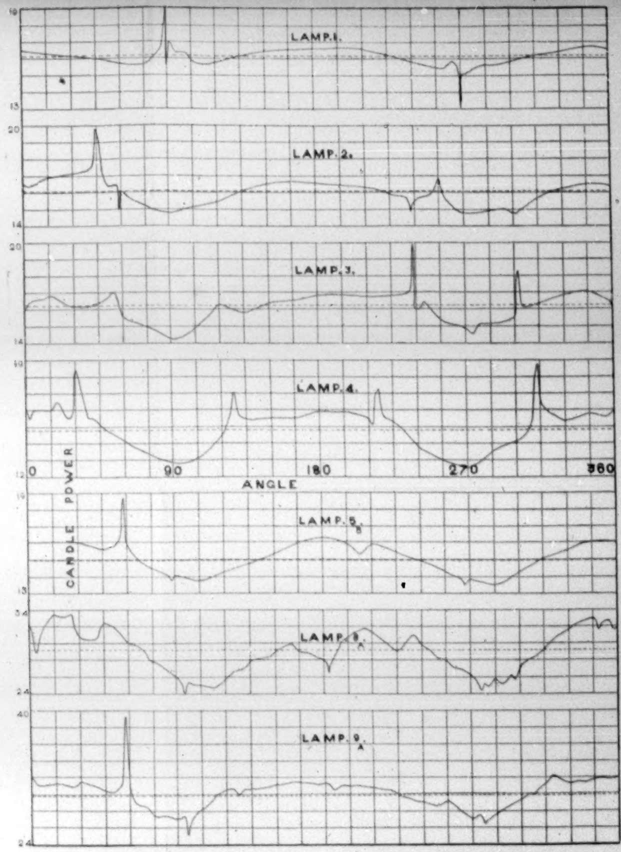
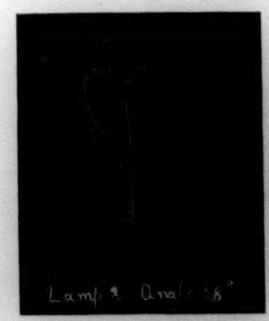


FIG. 8.

FIG. 8 *a*.


Lamp 2 Angle 28°

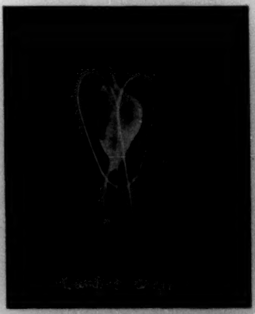
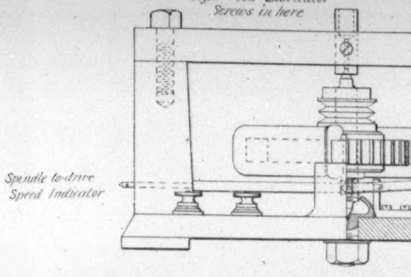
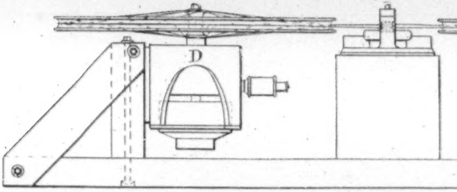


Fig. 1.—High-Frequency Alternat

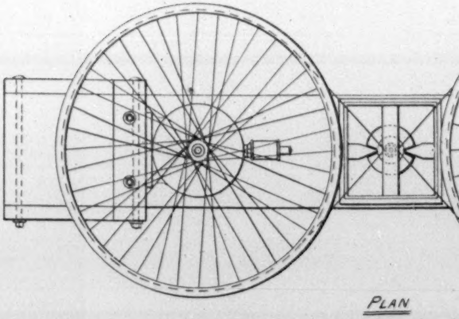

Sight Feed Lubricator
Screws in here
Spindle to drive
Speed Indicator

PART SECTIONAL ELEVAT

Fig. 3.—First Method of driving High-Frequ



ELEVATION



PLAN

lternator. Two-fifths full size.

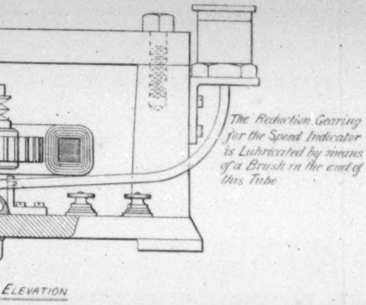
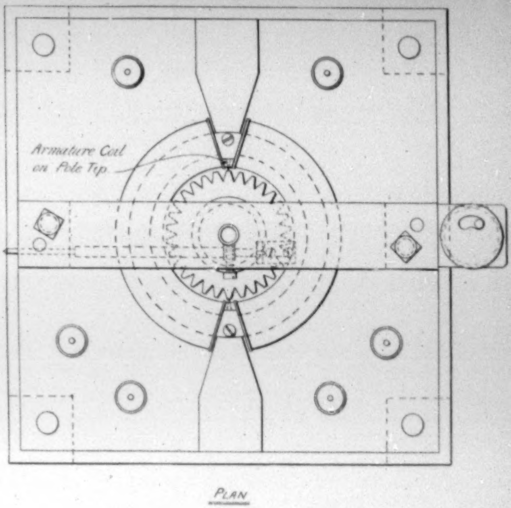
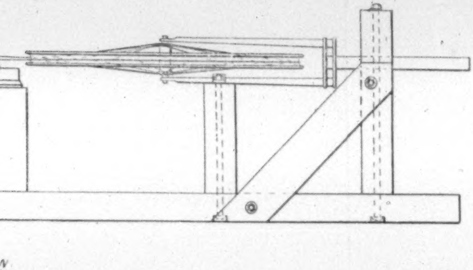

The Reduction Gearing for the Speed Indicator is Lubricated by means of a Brush in the end of this Tube
ELEVATION

Fig. 2.—High-Frequency Alternator. Two-fifths full size.


Armature Coil on Pole Tip
PLAN

-Frequency Alternator. 1 inch = 1 foot.



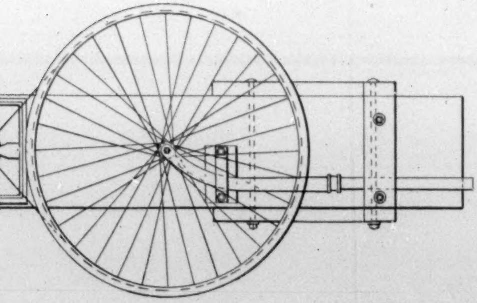


Fig. 5.

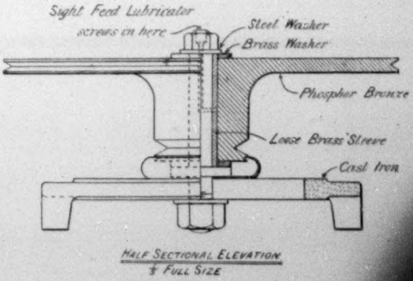

Sight Feed Lubricator screws in here
Steel Washer
Brass Washer
Phosphor Bronze
Loose Brass Sleeve
Cast Iron
HALF SECTIONAL ELEVATION
⅓ FULL SIZE

FIG. 4.

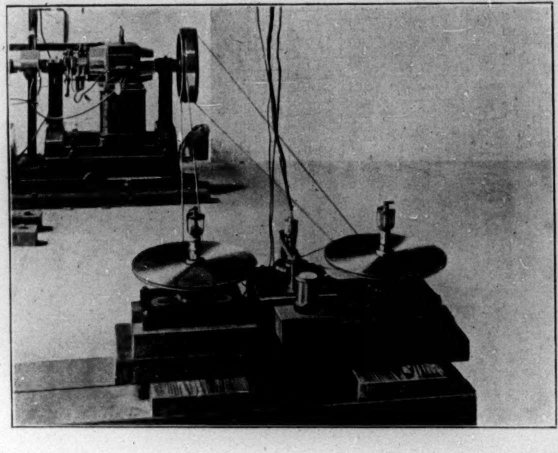


FIG. 6.

XLII. *The Slow Stretch in Indiarubber, Glass, and Metal Wires when subjected to a Constant Pull.* By PERCY PHILLIPS, *M.Sc.* (Communicated by Prof. POYNTING.)*

The Slow Stretch in Indiarubber.

IN an investigation of the adiabatic and isothermal elasticities of indiarubber, it was observed that whenever indiarubber was subjected to a sustained pull it continued to yield slowly after the completion of the first large stretch. This slow yielding, being quite considerable in size, was very easily investigated, and the following general results were obtained.

Firstly, with a particular pull the stretch is a linear function of the logarithm of the time elapsed after the establishment of the pull. Thus, if x is the stretch and t the time since the pull was established,

$$x=a+b\log t,$$

where a and b are constants for the particular pull. This obviously does not hold for very small values of t. No doubt t is to be reckoned from an instant a little before the establishment of the pull, but if t has any value above a few seconds it is sufficiently exact to measure it from the instant of establishment of the pull.

Secondly, the constant b is proportional to the pull, and when t is measured in minutes the constant a is also roughly proportional to the pull.

Thirdly, if a pull is established for a short time and then removed, the indiarubber returns to its original length, according to the law $x=b\log\frac{t}{t_0}$, where x and b denote the same things as before, t represents the time which has elapsed since the pull was established, and t_0 the time since the pull was removed.

Fourthly, if indiarubber is stretched to a certain length and retained at that length, then the pull required to keep it so stretched decreases for about the first day, according to the law

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* Read January 27, 1905.

$$P = a - b \log t,$$

where P is the pull, t is the time which has elapsed since the initial stretch was established, and a and b are constants for a particular extension. b is proportional to the amount of the initial extension.

The apparatus used to investigate the slow stretching in indiarubber was very simple. One end of a strip of indiarubber bandage 30 cms. long was attached by a clip to a wall bracket, and a light scale-pan and a photographic scale with divisions of ·210 mm. were attached by another clip to the other end. A telescope with cross wires in the eyepiece was used to view the scale.

The mode of procedure was as follows:—

1st, the reading in the telescope was noted.

2nd, the required load was placed very gently in the scale-pan at a given time.

3rd, as each division of the scale passed the cross wire of the telescope the time was taken.

The following is a typical set of readings:—

Pull.	Reading on Scale.	Stretch in Scale-divisions.	Time.	log t.
0	30·12			
500 grams	...	...	12 0 0 noon	
	76·8	46·68	2 m. 59 s.	·47
	76·9	46·78	3 50	·58
	77·0	46·88	4 42	·67
	77·1	46·98	6 0	·78
	77·2	47·08	7 40	·88
	77·3	47·18	9 24	·97
	77·4	47·28	11 40	1·07
	77·52	47·4	15 36	1·19
	77·61	47·49	20 18	1·31
	77·73	47·61	27 45	1·44
	78·43	48·31	2 30 0	2·18

Plotting the scale-readings against log t, as in Curve No. I., we get a straight line, *i. e.* we have the relation

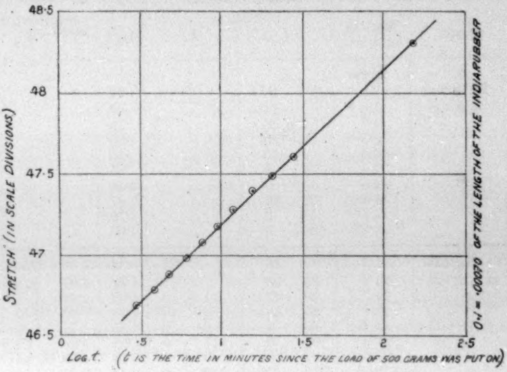
$$x = a + b \log t \quad \text{or} \quad \frac{x-a}{\log t} = b,$$

the symbols having the same meaning as before.

If x is measured in scale-divisions and t in minutes, then for this case

$$b = 1{\cdot}006 \qquad \text{and} \qquad a = 46{\cdot}2.$$

Curve No. I.



The constancy of $\dfrac{x-a}{\log t}$ is exhibited in the following table.

x.	$\log t$.	$\dfrac{x-46{\cdot}2}{\log t}$.
46·68	·47	1·02
46·78	·58	1·00
46·88	·67	1·02
46·98	·78	1·00
47·08	·88	1·00
47·18	·97	1·01
47·28	1·07	1·01
47·4	1·19	1·01
47·49	1·31	·98
47·61	1·44	·98
48·31	2·18	·97

If x is measured as stretch per unit length

$$b = {\cdot}0073 \qquad \text{and} \qquad a = {\cdot}3235.$$

On removing the load from the indiarubber it returned gradually back to its original length according to the law

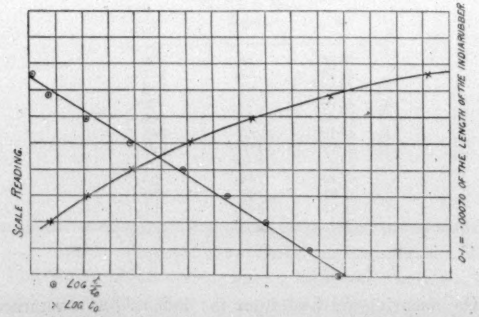
$$x = a + b\log t - (a + b\log t_0)$$

i. e. $$x = b\log\left(\frac{t}{t_0}\right),$$

where t is the time which has elapsed since the load was put on, and t_0 that since the load was removed. The following is an ordinary set of readings :—

Load.	Reading on Scale.	t.	t_0.	log t.	log t_0.	log $\frac{t}{t_0}$.
0	22·24					
200 grams at 3 P.M.	12·0	20 s.	...	$\bar{1}$·52		
	11 8	50 s.	...	$\bar{1}$·92		
	11·7	1 m. 25 s.	...	·15		
	11·6	2 20	...	·37		
	11·5	4 15	...	·63		
	11·4	7 5	...	·85		
	11·3	14 30	...	1·16		
	11·2	29 30	...	1·47		
	11·08	58 0	...	1·76		
	11·02	80 0	...	1·90		
0 grams at 4.23 P.M.	21·5	83 23	23 s.	1·92	$\bar{1}$·58	2·34
	21·6	83 38	38	1·92	$\bar{1}$·80	2·12
	21·7	84 23	1 m. 23	1·93	·14	1·79
	21·8	85 40	2 40	1·93	·43	1·50
	21·9	89 0	6 0	1·95	·78	1·17
	22·0	100 0	17 0	2·00	1·23	·77
	22·09	132 0	49 0	2·12	1·69	·43
	22·18	277 0	194 0	2·44	2·29	·15
	22·26	1153 0	1070 0	3·06	3·03	·03

Curve No. II.



On Curve No. II. these results are exhibited, both log t_0 and log $\left(\frac{t}{t_0}\right)$ being plotted against the stretch so that they may be compared.

The relations of the constants a and b to the stretching force were then found by plotting the curves of stretching for the different loads in exactly the same way as is shown for the 500 gram load. The following results were obtained:—

Load.	a.	b.	$\frac{b}{\text{load}}$.
100 grams	6·8	·176	·00176
200 ,,	14·7	·385	·001925
300 ,,	23·5	·580	·001933
400 ,,	33·4	·800	·002000
500 ,,	46·2	1·006	·00201

These numbers show that b is nearly proportional to the load.

The effect of superposing two loads was then investigated, but this showed that the effect was by no means the sum of the effects which each would produce separately. *E.g.*, suppose the stretch due to the first load be given by

$$x = a_1 + b_1 \log t_1,$$

t_1 being the time which has elapsed since the first load was put on, and suppose the effect of the second load be given by

$$x = a_2 + b_2 \log t_2,$$

t_2 being the time since the second load was put on. Then the total stretch should be

$$x = a_1 + a_2 + b_1 \log t_1 + b_2 \log t_2.$$

No such relation was found to hold however, and the only conclusion which was drawn from these experiments is that when t_2 is small compared with $t_1 - t_2$ the indiarubber stretches almost as if the second load were the only one acting, and when t_2 is large compared with $t_1 - t_2$ the stretch is the same as if both loads had been put on at the same time. This shows that unless the duration of an experiment is small compared with the time which has elapsed since the indiarubber was last strained, an effect will be produced by that previous strain.

The alteration in pull required to keep a piece of indiarubber stretched to a certain length was next investigated. This was done by means of a balance so arranged that the beam was lifted just off the supports. From one end of the beam the scale-pan was removed, and just under this end was placed a heavy weight to which the lower end of a piece of indiarubber bandage was attached by a clip. To the upper end of the bandage a hook was attached by means of another clip. At first the indiarubber was too short to reach to the beam of the balance, but at a certain time it was stretched just far enough to hook on to the beam, and weights were placed in the pan at the opposite end of the beam so as to just balance the pull of the indiarubber. The weight had to be altered from time to time as the stress altered, but as the beam of the balance was only just raised above the supports, the indiarubber was stretched by a practically constant amount all the time.

The readings in one case were :—

Pull in grams.	Time after the indiarubber was stretched.	log *t*.
1472·5	1 m. 30 s.	·18
1467·5	2 15	·35
1462·5	3 15	·51
1457·5	4 35	·66
1452·5	7 30	·88
1448·5	10 0	1·00
1444·5	14 45	1·17
1440·5	19 24	1·29
1436·5	26 45	1·43
1430·5	48 0	1·68
1427·5	58 0	1·77
1412·5	210 0	2·32
1384·5	1436 0	3·16

If the pull be plotted against log *t* we get a straight line, so that the stress obeys the law

$$S = a - b \log t.$$

From the curve we immediately deduce a to be 1477 grams, and the constancy of $\frac{a-S}{\log t}$ is exhibited in the following table.

Little notice should be taken of the first few figures, for $a - S$ and $\log t$ are very small, and so a small error in measuring S is largely exaggerated.

Pull in grams.	$\log t$.	$\frac{a-S}{\log t}$.
1472·5	·18	25·0
1467·5	·35	27·2
1462·5	·51	28·5
1457·5	·66	29·5
1452·5	·88	36·0
1448·5	1·00	35·3
1444·5	1·17	36·1
1440·5	1·29	35·4
1436·5	1·43	35·3
1430·5	1·68	36·2
1427·5	1·77	35·8
1412·5	2·32	36·0
1384·5	3·16	34·3

Subsequent experiments have shown that after about a day the decay of stress becomes much more rapid until S becomes very small, and then S gradually approaches a zero limit. In the experiment of which the observations are given the pull after twelve months had fallen to 22 grams.

Stretching by different proportions of the unstretched length shows that b is proportional to the stretch.

All the ordinary assumptions which one can make, such as considering the stretch as consisting of elastic and viscous portions, seem to lead to the conclusion that the stretch should be a function of e^{-t} and not of $\log t$.

Consequently it was thought that the amplitude of the up and down oscillations of a weight on the end of a length of indiarubber bandage might obey some other law than the usual one. Observations showed, however, that the usual law is obeyed, *i. e.* that the amplitude decreases in geometrical progression as the time increases in arithmetical progression.

In order to give some idea of the effect of alternations of temperature in the laboratory in producing errors, the temperature coefficients of the indiarubber were found while the indiarubber was in tension. The coefficient of expansion on cooling proved to be much larger than the coefficient of contraction on heating. Consequently the alternations of temperature in the laboratory would produce a slow creep on their own account: thus when the creep is very slow the effect of the alternations of temperature may be much greater than the true logarithmic creep. In all cases it was found that when the creep became very slow, *e.g.* a few days after the load was put on, it was a great deal faster than the logarithmic law would warrant.

To sum up the results for indiarubber :—

1. When indiarubber is stretched by a given load it stretches according to the law

$$\text{stretch} = a + b \log (\text{time}),$$

and b is directly proportional to the load.

2. When the load is removed the stretch still remaining is equal to

$$b (\log t - \log t_0),$$

where t is the time since putting on the load, and t_0 is the time since removing the load.

3. When two loads are put on successively their effects are not exactly superposable, but just after the second load is put on, the effect is as if the last load only were acting, and then it gradually alters till finally the effect is the same as if both loads had been put on simultaneously.

4. When indiarubber is stretched to a fixed length and retained there, the load required to maintain that stretch alters according to the law

$$\mathrm{S} = a - b \log t,$$

where S is the stress and t is the time since the rubber was stretched ; b is proportional to the initial stretch.

5. The temperature coefficients of expansion and contraction of indiarubber in tension are quite different.

6. The decrement of the amplitude of vibrations in indiarubber obeys the ordinary law.

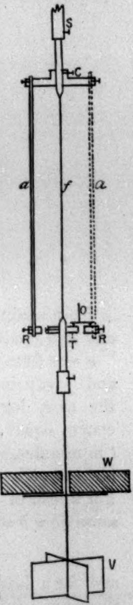
The Slow Stretch in Glass.

Having established these results for indiarubber, it was thought desirable to investigate whether the properties were possessed by other substances.

Glass being a substance which is known to exhibit a slow creep when under stress, it was next investigated. As the motion is so much smaller in glass than in indiarubber, it was necessary to use a much more exact measuring device.

A diagram of the apparatus is given in fig. 1. *f* is a glass fibre made by drawing out a glass rod to a fairly thin neck and then heating the neck and drawing it out to a length of about 1 foot. By this means a fairly uniform fibre was obtained, but it was still a little thinner in the centre than it was elsewhere. One of the thick ends of the fibre was attached as shown to a firm support S.

Fig. 1.



To the other end was attached a little table T, and a weight carrier W which had a system of vanes V dipping into oil to damp vibrations.

To the upper end was also attached a circular piece of brass C from which depended three rods, *a*, of glass, and to the lower ends of *a* a ring of brass R was fixed. An optical lever O rested with two of its legs on the ring R and one on the table T. A telescope viewed the reflexion of a millimetre-scale in the mirror of the optical lever O.

This arrangement is that used by Mr. G. A. Shakespeare in this laboratory, the rods *a* serving to compensate for temperature alterations and for any give of the support. The length of the optical lever was ·820 cm. and the distance of the scale was 3 metres during some experiments, and 461 cms. in others.

In the same way as for the indiarubber experiments the weight was put on by hand as gently as possible, and the reading in the telescope taken at intervals. At first no consistent results could be obtained, but when the fibres were put on one side for a few weeks after their manufacture the subsequent results for stretching were quite concordant.

In the case of glass, as in that of indiarubber, the law of stretching was

$$x = a + b \log t,$$

the symbols having the same significance as before; and when the load was left on for along time and then removed the creep back was proportional to the logarithm of the time which had elapsed since the removal of the load if the duration of the observations was short compared with the time since the load was put on.

The numbers given were obtained for a load of 779 kilograms per sq. cm. cross section.

Stretch in Scale-divisions.	log t.	$\frac{x-a}{\log t}$.
289·7	·38	2·63
290·2	·50	3·0
290·6	·66	2·88
291·0	·86	2·68
291·3	·94	2·77
291·75	1·15	2·66
292·3	1·32	2·73
292·5	1·40	2·72
292·9	1·54	2·73
293·2	1·65	2·73
293·8	1·90	2·69
296·7	2·92	2·74

Here again, for the first few numbers $x-a$ is rather small, and so a small error in x would make a large error in $x-a$.

a was found as before by plotting the stretch against log t, and producing the straight line so obtained till it cut the axis, $\log t=0$. The value of x at $\log t=0$ is of course equal to a. If x is measured in scale-divisions and t in minutes, we see that b in this case is equal to 2·74 : if x is the stretch per unit-length and t in minutes, $b=1{\cdot}55\times10^{-5}$. For a load of 599 kilograms per sq. cm. cross section on the same fibre b was found to be

$$1{\cdot}16\times10^{-5},$$

and for a load of 422 kilograms $b={\cdot}780\times10^{-5}$. Plotting these three values of b against the load we get a straight line, and when the load is 284 kilograms b becomes 0. This is shown by the approximate equality of the three values of $\frac{\text{Load}-284}{b}$:—

$$313\times10^{5}$$
$$273\times10^{5}$$
$$300\times10^{5}$$

The Slow Stretch in Copper, Platinum, Silver, and Gold Wires.

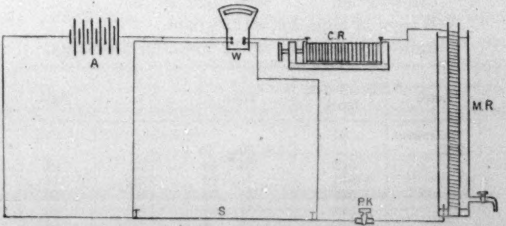
It was next thought very desirable to find out whether metals behave in a similar way to glass and indiarubber with regard to slow stretch.

To investigate this an arrangement almost identical with that used for glass was employed, the rods *a* being of course replaced by wires of the same kind as that being stretched. Until the load used reached a certain value no slow stretch at all approaching in size to that occurring in glass could be detected, but above this value there was a creep obeying the same law as for the indiarubber and glass. This creep, however, caused a permanent extension, and when the load was removed there was little, if any, slow creep back.

The inquiry into the slow stretch of wires reveals astonishing differences in behaviour, even in specimens from the same roll of wire, and no consistent results were obtained until wires were used which came from the same roll, and which were annealed at the same temperature for the same length of time, and were cooled at the same rate.

Having no convenient annealing furnace, I annealed the wires separately by an electric current. The whole annealing apparatus is represented diagrammatically in fig. 2.

Fig. 2.



A represents the accumulator-cells which supply the current.

W is a Weston standard ammeter and voltmeter.

S is the wire which is to be stretched. It is enclosed in a glass tube to prevent air draughts, and when copper wires were used the glass tube was filled with coal-gas to prevent the oxidation of the wire.

CR is a carbon rheostat for adjusting the current.

MR is a rheostat for cutting off the current at the same rate at each annealing. It consists of two concentric tubes, the outer of glass and the inner of porcelain. The lower end is closed with a bung, and at the botton of the glass tube is a side tube with a tap. Round the inner tube is wound a long spiral of iron wire of large resistance compared with the rest of the circuit, and connected up as shown. The space between the two tubes is filled with mercury while the annealing continues. At the required time the tap is turned on and the mercury slowly flows out, thus putting in gradually more and more resistance.

PK is a plug-key.

It was found that the way in which the load was put on made some difference to the result.

Ultimately uniformity was obtained by using a flotation method which put the full load on in about 5 seconds.

Before the load was put on, the telescope was placed to view the scale: then the load was put on and the times were taken as the successive divisions on the scale passed the cross-wire of the telescope.

The following is a fairly typical set of readings for a copper wire:—

Length of wire = 28·2 cms.
Diameter of wire = ·0390 cm.
∴ Cross-section = ·001195 cm^2.
Weight of clips, &c. = 122 grams.
Annealing current = 12 amps. for 5 minutes.

Load.	Reading on Scale.	Time.	*t*.	log *t*.
122 grams.	84			
722 "	...	11h 30m 0s		
	542	31 13	1·22	·09
	543	31 23	1·38	·14
	544	31 36	1·6	·20
	545	31 48	1·8	·26
	546	32 8	2·13	·33
	547	32 26	2·43	·39
	547·5	32 36	2·6	·41
	548	32 48	2·8	·45
	548·5	32 58	2·97	·47
	549	33 12	3·2	·50
	549·5	33 24	3·4	·53
	550	33 38	3·63	·56
	550·5	33 56	3·93	·59
	551	34 15	4·25	·63
	551·5	34 34	4·57	·66
	552	34 52	4·87	·69

t, as in the case of indiarubber, is the time in minutes which has elapsed since the load was put on. If $\log t$ is plotted against the scale-reading we again get a straight line, and therefore copper obeys the same law $x=a+b\log t$.

If x is measured in scale-divisions, b for this case is equal to 16·28, and $a=456{\cdot}7$.

In the following table the constancy of $\dfrac{x-456{\cdot}7}{\log t}$ is exhibited.

x.	$\log t$.	$\dfrac{x-456{\cdot}7}{\log t}$.
458	·09	14·4
459	·14	16·4
460	·20	16·5
461	·26	16·5
462	·33	16·1
463	·39	16 2
463·5	·41	16·6
464	·45	16·2
464·5	·47	16 6
465	·50	16·6
465·5	·53	16·6
466	·56	16·6
466·5	·59	16·6
467	·63	16·4
467·5	·66	16·4
468	·69	16·4

To bring b to absolute units we must know the value of the scale-division. This was found by removing the wire, and so placing a micrometer-screw gauge that the leg of the optical lever, which previously rested on the table t, was now resting on the top of the movable jaw of the screw-gauge. As the screw was turned the readings in the telescope and in the gauge were taken :—

Scale.	Micrometer.	Scale.	Micrometer.	Scale Difference.	Micrometer Difference.
137·1	12·50	543·6	15·0	406·5	2·5
217·5	13·00	626·0	15·5	408·5	2·5
299·0	13·50	709·4	16·0	410·4	2·5
379·8	14·00	794·0	16·5	414·2	2·5
461·4	14·50	879·9	17·0	418·5	2·5
				2058·1	12·5

$\therefore$ 1 scale-division = ·00608 cm.

Expressing everything in absolute units we have

Load per sq. cm. cross section = 604 kilos,

and when x is measured as stretch per unit length

$$b = \cdot 000351.$$

Exactly the same process was gone through with different copper wires from the same coil. They were annealed with the same current (12 amps.) for the same time (5 mins.), and then their curves of stretching with various loads were obtained. It appears that the logarithmic law obtains right up to the load which causes the wire to pull out quite rapidly and break.

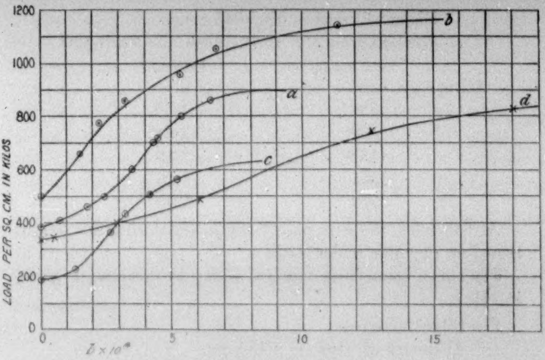
The following is the series of values of b which was obtained for this particular coil of copper wire treated in this manner.

	Load per sq. cm. in kilos.	b.
All up to	380 kilos.	0
	399·2	·000073
	458·0	·000174
	498·0	·000237
	531·0	·000297
	604·0	·000351
	700·0	·000428
	720·0	·000442
	795·0	·000542
	850·0	·000658

At 900 the wire drew out so far that the scale went right out of view, and finally the wire broke.

If we observed the form of the curve connecting b with the load (Curve No. III.a) we see that there appears to be a very definite point below which there is no very considerable slow creep, and there is a somewhat indefinite load above which the wire begins to stretch very rapidly. The range of load between these points is from 380 kilos to 850 kilos per sq. cm. cross section.

Curve No. III.



Exactly similar series of experiments have been conducted with platinum, silver, and gold wires. All these substances show the same general characteristics that between wide limits (to nearly three times the smallest load necessary to produce a creep) they stretch according to the law $x = a + b \log t$, and that this creep starts at a very definite load and becomes very rapid at a less definite point.

For the Platinum wire:—

Diameter = ·0506 cm. ∴ cross-section = ·00202 sq. cm.
Annealing current = 8·5 amps. for 5 minutes.

	Load per sq. cm. cross section.	$b \times 10^4$.
All up to	500 kilos.	0
	654	1·525
	771	2·265
	854	3·22
	952	5·39
	1050	6·73
	1141	11·35
	1247	26·40
	1354	46·40
	1455	138 0
	1560	wire pulled out and broke.

The curve connecting b with the load is shown in III. b.

For Gold wire :—

Diameter of wire = ·043 cm.
Annealing current = 7·5 amps. for 5 minutes.

	Load per sq. cm.	$b \times 10^4$.
All up to	180 kilos	0
	222	1·30
	362	2·61
	432	3·19
	501	4·16
	570	5·03

At 640 kilos. the wire pulled out so much that the weight rested on the table underneath.

The curve connecting the load with *b* in this case is given in III. *c*.

For Silver wire :—

Diameter of wire = ·0486 cm.
Annealing current = 16 amps. for 5 minutes.

	Load per sq. cm cross section.	$b \times 10^4$.
All up to	330 kilos	0
	335	0·479
	396	2·81
	486	6·08
	731	12·68
	827	18·08

At 900 kilos the wire pulled out and broke.

The curve connecting *b* with the load for silver is given on Curve No. III. *d*. All these curves show the characteristics mentioned in the case of copper. No great importance must be attached to the absolute value of these numbers, for besides the great effect which is produced by the annealing temperature, by the length of time during which the annealing continues, and by the rate at which the wire is cooled down, there is an effect produced by the way in which the weight is put on. Added to this, different specimens of wire give very different results, though when pure wires were obtained from Messrs. Johnson & Matthey these differences were fairly small.

A fundamental difference between the slow stretching of indiarubber and glass and that of metal wires is that the former is reversible, *i. e.*, the creep back when the load is removed follows the same law, while in the case of the metals there is no appreciable slow recovery.

In the case of the stretching of these metals, it is just as difficult to make a reasonable assumption which shall give the law as it is with indiarubber. If we assume that the permanent stretch is produced by cleavage of crystals, we require that the rate of breakdown of the crystals shall be given by

$$\frac{dx}{dt}=\alpha e^{-\beta x}$$

where x is the stretch. Or, as it seems reasonable to suppose that the stretch is proportional to the number (n) of crystals broken down,

$$\frac{dn}{dt}=\alpha' e^{-\beta' n}$$

or, if N be the number of unbroken crystals,

$$\frac{dn}{dt}=\alpha'' e^{+\beta'' N}.$$

All previous assumptions have led to a limit, or else have attributed the characteristics of a liquid, and so have made the stretch a linear function of the time.

We see, however, that

$$\frac{d^2x}{dt^2}=-\alpha\beta e^{-\beta x}\frac{dx}{dt}$$
$$=-\alpha^2\beta e^{-2\beta x},$$

i. e. $\frac{d^2x}{dt^2}$ is proportional to $\left(\frac{dx}{dt}\right)^2$.

We might therefore assume that after the very beginning of the stretch no further crystals are broken, and that after this the resistance to the load is provided by ordinary viscous force between the faces of the cloven crystals according to the law which sometimes obtains in very viscous liquids, that the viscous force is proportional to the square of the velocity. This would suggest that even in pure metals there may be a non-crystalline matrix in which the crystals are embedded, and that this matrix acting as a very viscous liquid is responsible for this law of stretching. In the Phil. Mag. Oct.

1904, Prof. Trouton and Mr. Rankine have published a research in which they find this logarithmic law of decay of stress and strain in the case of lead wires. They did not require to anneal in any way, probably because any internal strain would decay very rapidly in such a soft metal. In the case of the metals which I used the logarithmic law did not obtain unless they had been annealed.

Prof. Trouton and Mr. Rankine also found that after about the first hour the stretching became a linear function of the time, and this also is probably due to their using such a soft metal.

In the case of gold, I found that after about a day the stretch approximated to a linear function of the time, but of course the linear stretch was a very much smaller proportion of the logarithmic stretch than was found with lead. The copper, platinum, and silver wires made no attempt towards approximation to a linear function, even after two days or more, but when the stretch became so slow that it was difficult to measure, there were considerable deviations from the logarithmic law.

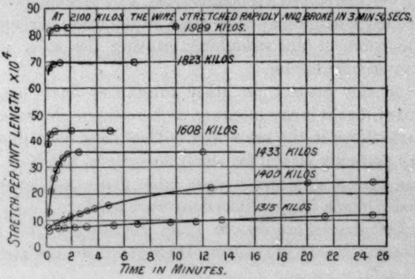
There do not appear to be any absolute values of the stretch given in their paper, so that I cannot compare the size of the stretch in lead wires with that in copper, platinum, silver, and gold.

They also indicate that the assumptions which one usually makes with regard to the stretching of solid bodies do not give the stretch as a logarithmic but as an exponential function of the time, and it seems difficult to make any reasonable assumption which shall give the experimental law.

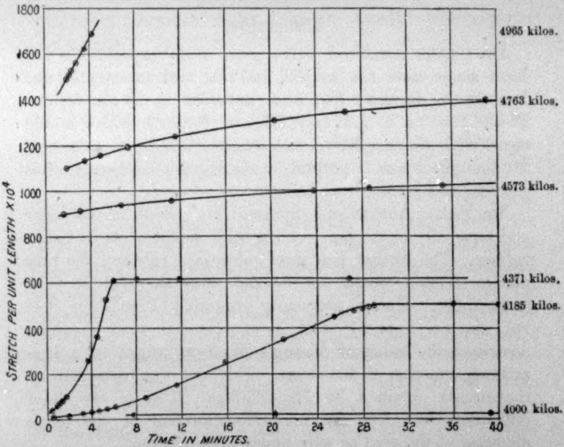
The Slow Stretch in Iron and Steel Wires.

In the case of iron and steel wires a bundle of each kind of wire was annealed for me in one of the annealing furnaces at Messrs. Perry & Co.'s pen works, but in all other respects the wires were experimented upon in exactly the same way as the other wires. The results, however, are entirely different. As before, there seems to be no slow stretching until a load large enough to cause a permanent extension is used, but when such a load is used the stretching is no longer a linear function of the logarithm of the time. As there seems to be no simple connexion between the curves for stretching for different loads, a series of curves for iron wires is given on curve No. IV., and a series for steel on

Curve No. IV.



Curve No. V.



curve No. V. Opposite each curve is given the load per sq. cm. used in the particular stretching. It will be seen at a glance how entirely different these curves are from those for the other metals investigated. I have no explanation for this difference, but it is well known that iron and steel are anomalous in many other properties. The steel wires after having been stretched are magnetized, and this suggests that a molecular change may take place in the steel, perhaps at a definite point in the stretching, causing the steel at that point to become harder.

Up to the present no other substances have been investigated, but it is my intention to investigate the behaviour of quartz fibres in the same way. They have been shown to be very perfectly elastic for small distortions, but it is quite possible that a slow creep does take place and could be detected if much larger distortions were used.

My best thanks are due to Prof. Poynting for his many valuable and kindly suggestions throughout the research.

Birmingham University,
Oct. 1904.

Discussion.

Dr. Chree remarked that a great many experiments had been made upon the subject, and that both exponential and logarithmic formulæ had been proposed to fit the results. It was possible to get very different formulæ which would agree with the results of experiment with equal accuracy. He thought it was important to distinguish between viscous yielding and elastic creep followed by recovery.

Mr. Rollo Appleyard expressed his interest in the paper and especially in the part dealing with experiments on indiarubber. The Author had used vulcanized rubber; the properties of this would not be the same as those of pure indiarubber. He had previously suggested (Proc. Phys. Soc. vol. xix. pt. i. p. 57) that in experiments such as those described, the value of Young's Modulus might be a logarithmic function of the stress. This idea was borne out by the results obtained by Mr. Phillips. One of the chief difficulties was the accurate determination of the mean diameter of the cord or wire under investigation.

Prof. F. T. Trouton said the experiments described threw a good deal of light upon the subject, and he was pleased that they confirmed the law which Mr. Rankine had obtained for lead. The logarithmic law also applied if the strain was kept constant and the stress varied. This method had the advantage that changes of diameter were obviated. He thought the law was most important and that there was something fundamental in it.

Mr. A. **Campbell** referred to the analogy between the experiments described and those on the residual charge of a condenser. The more heterogeneous the mixture which formed the dielectric the more pronounced were the phenomena of the residual charge. Possibly something of a similar nature might hold in the cases of stretching, and he asked the Author if he had performed any experiments on alloys.

Mr. **Phillips** in reply said the alteration in the cross section of the rubber during the experiments was very small. There were many things which might affect the accuracy of the numbers which he had obtained, and he did not wish to lay too much stress upon the absolute values given in the paper.

XLIII. *Determination of Young's Modulus (Adiabatic) for Glass. By* CHICHESTER A. BELL, *M.B. With an Appendix by* C. CHREE, *F.R.S.*[*]

ALTHOUGH the elastic constants of glass have been the subject of many investigations, it is not easy to make use of the published results in any experimental inquiry into which these enter. The characteristics of the specimens examined are usually so vaguely given, and the composition of the material is so variable, that it is never safe to assume the identity of any sample purchased in the shops with one of ascertained properties.

This difficulty was acutely felt by the writer in the course of a series of experiments in which a tolerably exact knowledge of the value of Young's Modulus (adiabatic) for different kinds of glass was all-important. The results of these experiments showed variations which it was impossible to account for by any justifiable assumptions, and accordingly it became necessary to determine the required constant by experiments on specimens of the glass actually employed, the method selected for this purpose being the simplest of all—the acoustical.

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[*] Read January 27, 1905.

So far as can be judged from a cursory examination of the literature of the subject, this method has been but little used in the case of glass, doubtless because of the well-known difficulty of obtaining perfectly uniform rods of this material of any considerable length. The original plan of these experiments was simply to determine the tones given by longitudinal vibration of a number of rods (or tubes) of each kind of glass; to cut each of them into ten approximately equal segments; and by weighing and measuring the segments to construct for each rod a diagram representing the variations in its cross-section from end to end. It was expected that, of the more nearly uniform of the original rods, a few might thus be selected, from the tones of which Young's Modulus might be calculated.

Inspection of the diagrams showed, however, that of the many rods examined not one showed such an approach to uniformity as to be suitable for the object in view. But, taking as a standard for comparison the mean value of the product, number of vibrations per second × twice length of rod, for each bundle, it was possible in a general way to connect the variation from this mean exhibited by each rod with its most marked irregularities; and this observation suggested that it might be possible, by the aid of the diagrams, to calculate for each rod an "addition to length" which would enable the velocity of sound through it to be calculated from its rate of vibration as if the cross-section were constant—a correction analogous to that given by Lord Rayleigh* for fluid columns of slightly variable section. Trial showed that this was certainly possible; and on mentioning the matter to Dr. Chree, to whom I am indebted for his cordial interest, he kindly undertook to look into it, and has given in the Appendix to this Note the mathematical proof that the correction, somewhat empirically applied, was strictly in accordance with theory. The necessary correction is, in fact, given by Lord Rayleigh's formula *with reversed sign.*

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* 'Theory of Sound,' 1st edition, vol. ii. p. 61.

In the experiments quoted below, tuning-forks of high frequency were not available; the product $n \times 2l$ for the glass rods was therefore determined as follows. Bessemer steel rods of remarkable uniformity as regards diameter and texture are now easily procurable. Three such rods from the same drawing, $\frac{3}{8}$ of an inch in diameter, when cut to exactly the same length, about 11 feet, and clamped centrally by means of corks in a vertical position *, gave, on being struck endwise, notes so nearly identical in pitch that beats were scarcely perceptible. This trial was carried out in the manner described farther on, and every precaution taken to secure uniformity of temperature. In the course of subsequent experiments many opportunities presented themselves of comparing the notes given by pieces of these bars of approximately equal length, and the differences not accounted for were always so small as to be unimportant, the only serious difference noted amounting to about six vibrations per second in the case of two pieces of exactly equal length, about 40 cms.

To determine the velocity of sound in the steel rods, the length of one of them was cautiously altered by cutting and filing until its note nearly coincided with that of a König fork giving 1024 double vibrations per second; and, subsequently, taking advantage of a spell of cold weather, when room temperature had been long constant at 0°, rod and fork were compared together, beats being counted for 20 seconds. According to König the rate of his forks, standardized at 20° C., increases by ·01118 per cent. for each degree below standard temperature. With these data, the product $n \times 2l$ for the steel bar was found to be

516040 cms. per second at 0°.

Similarly favourable opportunities did not occur for testing the other bars at 0°; but repeated comparisons of the bars with each other, and with 768 and 896 König forks at other temperatures, left no doubt of the substantial accuracy of this determination.

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* The note given by a rod of these dimensions is distinctly raised in pitch when it is fixed horizontally.

Some very careful determinations of the velocity of sound in rods of various metals and of St. Gobain glass, at temperatures ranging from 0° to 100° C., have been made by Mayer*, who found for a rod of Bessemer steel at 0° V=515090 cms. per sec. In view of possible variations in the composition of the metal, the agreement must be considered satisfactory.

According to Mayer also the velocity of sound in such a rod diminishes by ·2 per cent. when the temperature rises from 0° to 20°, or by nearly 50 cms. per sec. for each degree.

With these data the rates of vibration of the steel rods here employed were calculated, measurements of length being throughout made by a steel metre-scale, the very trifling errors of which had been determined at Kew.

In the experiments with glass, a steel rod was first cut giving a note higher in pitch than any one of the bundle of rods or tubes under examination; and the lengths of these were gradually reduced by cutting and grinding, until each gave a note differing from that of the steel by about 4 or 5 vibrations per second. In cutting an ordinary glass knife was used, but the trimming of the ends, and small alterations of length, were effected by lightly pressing the ends against a sheet of moderately coarse emery-cloth clamped on the face of a circular wooden disk which was rapidly rotated in a lathe. During the comparison of tones, the steel rod was simply balanced on the corner of a table, and the glass rod held at its centre in the hand, both rods being struck endwise by a light wooden mallet.

After this preliminary adjustment, the bundle of rods was laid aside until its temperature might be assumed to be that of the room. Each rod was then passed through a hole in a short cylindrical cork placed at such a point—not necessarily the centre—that the vibration was of maximum persistence, and mounted on a board side by side with the similarly supported steel rod.

A cone or funnel was fixed on the board so that its wider end embraced one pair of the rod ends, while its narrower end was connected by rubber tubing with a short length of glass tube pushed well into the ear-cavity. Beats due to the rod vibrations were thus heard with the greatest distinctness. To throw the rods into vibration, two small "pneumatic hammers" were arranged to deliver light blows on their distant ends. Each hammer was formed of a cylinder of wood about 5 cms. in length, cut from an ordinary pen-handle, lying within a slightly wider tube of glass in which it could just slide freely. A perforated cork or a section of rubber tubing, forced into the glass tube, kept the cylinder lying normally just within one of its ends; the other drawn-out end was connected by thin rubber tubing with a bag such as is used for pneumatic bells, which could be placed in any convenient position. By *tapping* on this bag with the finger the little wooden cylinder was sharply projected from, and as sharply drawn back into, the glass, after impact on the rod end; whilst by *pressing* gently on the bag the hammer was caused to rest against the rod, at once checking its motion. The rods could thus be thrown into vibration either alternately or simultaneously, the force of the blows delivered being under complete control. This little device, which I have found superior to any other mechanical or electrical arrangement, has proved of eminent service in a number of acoustical experiments. Thus in working with tuning-forks it makes it unnecessary for the operator to approach them—sometimes a matter of importance; whilst in experiments on rods, hammer and rod may be enclosed in a tube, which is then immersed in a water or steam bath.

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* "Researches in Acoustics," by Alfred M. Mayer, Phil. Mag. [5] vol. xli. p. 168.

The beats produced by the simultaneous vibration of the glass and steel rods were, as a rule, simply estimated by comparison with those of a watch. The most successful of more exact methods tried consisted in timing the oscillations of a heavy ball attached to a string to coincide with the period of the beats, by altering the length of the string, the rate of the pendulum being subsequently ascertained by means of a watch. But a moderate amount of practice enables one to make a pretty exact estimate of the frequency of the beats when this is in the neighbourhood of four or five per second. It is not advisable to attempt to bring steel and glass rod more closely into unison: as a rule the vibrations of thin glass rods, especially if these are rather irregular, die out so rapidly that the apparent absence of beats is no criterion of exact synchronism.

In every case, the sense of the difference between steel and glass rod was positively ascertained in the usual way by means of a spring-brass rider which could be fixed on the steel bar. This was more than usually important, since the fundamental tone of an irregular rod is certain to be accompanied by anharmonic overtones.

In the way described, the product $n \times 2l$ is easily and rapidly found for a number of thin rods. The correction, Δl, to be applied to the observed length l of a free-free irregular rod vibrating longitudinally in its gravest mode, is (see Appendix)

$$\Delta l = \int_0^l \frac{\delta S}{S_0} . \cos \frac{2\pi z}{l} dz,$$

where δS is the difference, positive or negative, between the cross-section at the point z and its mean value, S_0, for the whole rod.

Now an exact determination of S throughout even a single rod of considerable length would be a matter of some difficulty; and as rods and tubes are never quite free from knots and streaks, very often not homogeneous throughout, and only exceptionally properly annealed, would in the end lead to no high degree of accuracy. In these experiments I have, as a rule, been satisfied to divide each rod into *ten* nearly equal lengths; after suitable trimming of the ends of each piece on the emery disk, to determine its length to the nearest tenth millimetre, and its weight to within one milligramme; and to take $\frac{\text{weight}}{\text{length}}$ as a measure of S, supposed to be *uniform* throughout the segment, S_0 being similarly determined for the whole rod. The values of the integral

$$\int \cos \frac{2\pi z}{l} . dz$$

between the limits $z=0$, $z=\frac{1}{10}l$; $z=\frac{1}{10}l$, $z=\frac{2}{10}l$, and so on, are, to the degree of approximation considered necessary, and in order, $\frac{l}{2\pi}$ multiplied by the fractions

$$\overset{+}{\cdot 59} \quad \overset{+}{\cdot 36} \quad 0 \quad \overset{-}{\cdot 36} \quad \overset{-}{\cdot 59} \quad \overset{-}{\cdot 59} \quad \overset{-}{\cdot 36} \quad 0 \quad \overset{+}{\cdot 36} \quad \overset{+}{\cdot 59}.$$

The total correction thus becomes

$$\Delta l = \frac{l}{2\pi}\left(\frac{\cdot 59(\mathrm{S}_1-\mathrm{S}_0)+\cdot 36(\mathrm{S}_2-\mathrm{S}_0)-\cdot 36(\mathrm{S}_4-\mathrm{S}_0)\ldots+\cdot 59(\mathrm{S}_{10}-\mathrm{S}_0}{\mathrm{S}_0}\right)$$

the third and eighth sections being omitted. Inspection of the diagrams will show at once if the omission is justified.

The lists below include three rods in which the subdivision was subsequently made into *twenty* parts. In two cases the final result was practically unchanged ; in the third the value of a was raised only 3 metres per second. On the whole, further subdivision did not appear to be necessary.

Finally, we get for the " velocity of sound "

$$a = 2n(l + \Delta l),$$

and for Young's Modulus (adiabatic)

$$\mathrm{E}' = a^2\rho,$$

where ρ is the density referred to water at 4° C.

In the tables are given the values of a and E′ reduced to 0°. According to Mayer (*loc. cit.*) Young's Modulus for St. Gobain glass ($\rho = 2{\cdot}545$) diminishes by 1·16 per cent. for a rise of temperature from 0° to 100° C. In a special experiment with the Gallenkamp soft German glass, a diminution of 1·12 per cent. from 5° to 100° was observed, closely agreeing with Mayer's determination. With Powell's flint-glass, on the other hand, the curious result was obtained that Young's Modulus is *sensibly constant* between the same temperature limits. This observation is quite in harmony with the results obtained by Winkelmann and Schott * for glass of the same general character. In fact, if two rods of this glass be " tuned " to the same pitch, one of them may be made very hot by passage through a gas-flame before any distinct alteration of its note can be observed. Owing to the brilliance and persistence of the tones obtained from stout and fairly uniform rods of this glass, the low value of a and the absence of a temperature-coefficient, it is admirably adapted for standards of high pitch. The end of the rod on which blows are delivered may be fused, to diminish the risk of chipping, and the tuning effected by grinding the other end.

* " Ueber die Elasticität, etc., verschiedener Gläser," Wiedemann's *Annalen*, Bd. li. p. 697 (1894).

All results are given in absolute units. The density at 0° has been determined from selected samples. As it was found to be slightly variable, especially in the case of flint-glass, it is given to only four figures.

It is perhaps well to point out that the occasional close agreement between the corrected and uncorrected values of $n \times 2l$ does not imply approximate uniformity in the corresponding rod. This is far indeed from being the case.

Soft German Glass.

1. From Messrs. Gallenkamp & Co.

Mean $n \times 2l$ for eight rods at $14° = 527280$ cms. per sec. Of these five of the best were selected for examination. Lengths about 150 cms. ρ_0 (mean) $= 2{\cdot}508$.

$n \times 2l$ observed.	a (corrected) cms. per sec.
527620	525930
523920	526030
517240	525940
525850	526610
528100	526820

Mean $= 526260$ at 6° C.
$= 526400$ at 0°

Young's Modulus (adiabatic) or E′
$= 6{\cdot}95 \times 10^{11}$ dynes per sq. cm. at 0°.

Eight *selected* thin tubes of this glass gave a mean value of $n \times 2l$ at $6° = 525500$ cms. per sec. These were not cut up.

2. From Messrs. C. E. Müller & Co.

Mean product for 12 carefully chosen rods at $14° = 524800$. Of these six examined. Lengths about 150 cms. $\rho_0 = 2{\cdot}507$.

$n \times 2l$ observed.	a (corrected).
529000	525490
523020	524780
521270	525400
526020	524760
526020	524720
524580	525100

Mean $= 525050$ at 9°
$= 525250$ at 0°.

$E' = 6{\cdot}91 \times 10^{11}$ at 0°.

3. From Messrs. Baird & Tatlock.

Mean product for nine rods at 9° = 525990. Lengths about 143 cms. $\rho_0 = 2{\cdot}488$.

$n \times 2l$ observed.	a (corrected).
524310	527360
530640	527180
527900	526710
527480	526900
517660	526370

$$\text{Mean } a \text{ at } 6^\circ = 527040 \text{ cm. per sec.}$$
$$\text{at } 0^\circ = 527170.$$
$$E' = 6{\cdot}91 \times 10^{11} \text{ at } 0^\circ.$$

The last result here has been rejected. For this very irregular rod Δl was $+2{\cdot}36$ cms.

Bohemian Glass.

From Messrs. Baird & Tatlock. This glass was obtainable only in thin tubes, very irregular, much streaked and full of knots.

Mean product for six tubes = 513740 at 6°. Lengths about 138 cms. $\rho_0 = 2{\cdot}394$.

$n \times 2l$ observed.	a (corrected).
524580	518780
511690	515430
511740	516950
511040	517260
512050	516220
511360	515800

$$\text{Mean } a \text{ at } 6^\circ = 516740 \text{ cm. per sec.}$$
$$\text{at } 0^\circ = 516880.$$
$$E' = 6{\cdot}39 \times 10^{11} \text{ at } 0^\circ,$$
$$\text{or, rejecting the first result, } 6{\cdot}38 \times 10^{11}.$$

The agreement here is not nearly so good as usual, as indeed was anticipated.

Gauge-tube Glass.

A hard, ferruginous, light-green glass, almost infusible, manufactured by Messrs. Powell & Sons of Whitefriars, specially for gauge-tubes. Four carefully-chosen tubes, about

112 cms. in length, were examined. The external diameter of these tubes, about 1·3 cm., was rather greater, in proportion to length, than was desirable, but narrower or much longer tubes were not obtainable. $\rho_0 = 2·525$.

$n \times 2l$ observed.	a (corrected).
542380	539820
539690	539350
540570	539850
541420	539560

Mean a at 9° = 539650 cm. per sec.
at 0° = 539850.
$E' = 7·36 \times 10^{11}$ at 0°.

"Soft" Jena Glass.

Stout tubes, somewhat narrower than the foregoing, of length about 144 cms. $\rho_0 = 2·499$.

$n \times 2l$ observed.	a (corrected).
533120	535370
535650	535650
531510	535200
527800	534860

Mean a at 7° = 535270 cm. per sec.
= 535400 at 0°.
$E' = 7·16 \times 10^{11}$ at 0°.

Flint Glass.

From the Whitefriars Glass-works. Mean product for ten thin rods at 3° = 409910. Lengths about 91 cms. ρ_0 (mean) = 3·164.

$n \times 2l$ observed.	a (corrected).
405360	408740
410400	409550
412680	409820
407300	409780
410510	409610

Mean a = 409500 cm. per sec.
Not changing with temperature.
$E' = 5·30 \times 10^{11}$ at 0°.

For this glass, the value of E' depends greatly on the amount of lead contained. For example, in two other specimens from the same works, but of density 3·136 at 10°/4°, an indirect determination of E' gave $5{\cdot}39 \times 10^{11}$.

The foregoing list includes *all* the rods and tubes that have been examined, with only one exception, where some gross blunder had evidently been committed. Inspection shows that discrepancies between the uncorrected products $n \times 2l$ are reduced by 90 per cent. in the corrected values of a, and it is highly probable that the calculated values of E' are correct to three figures.

When a glass can be obtained in thin rods or tubes—a diameter of 5 or 6 mms. seems most desirable—the experiments are easily and rapidly carried out, since no great accuracy is required in the weighing or measurement of the segments. Thick rods or tubes naturally give more trouble, especially when of hard glass, and here some care in selecting suitable specimens is desirable. Tubes which do not give persistent notes when struck should be at once rejected. Perfect uniformity is not to be expected; failing this, the best results are obtained with rods or tubes which taper continuously from one end to the other.

APPENDIX *by* C. CHREE, *F.R.S.*

Taking the axis of the tube as axis of z, the origin being at one end, we have for the longitudinal displacement in the fundamental free-free vibration

$$w = A \cos kt \cos \pi z/l, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A is a constant, l the length, and $k/2\pi$ the frequency.

In reality w varies slightly over the cross-section, but for the purpose of the present correction we may neglect this variation and also the energy answering to the lateral motion.

If the section S were uniform we should have—neglecting the correction for finite area of section—

$$k/\pi = a/l, \qquad \text{where } a = \sqrt{E/\rho}.$$

Here ρ denotes the density, E Young's modulus, and a

the "velocity of sound" in the nomenclature employed by Lord Rayleigh.

S, it should be noticed, means the sectional area of the *material*; in a circular tube whose outer and inner surfaces are circles of radii r and r', $S=\pi(r^2-r'^2)$.

Let

$$S=S_0+\delta S, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where S_0 is constant, and $\delta S/S_0$ is supposed small everywhere along the tube.

Let T denote the kinetic, V the potential energy, then, as in Lord Rayleigh's 'Theory of Sound,'

$$T=\int_0^l \tfrac{1}{2}\rho\dot{w}^2 S dz,$$

$$V=\int_0^l \tfrac{1}{2}E\left(\frac{dw}{dz}\right)^2 S dz.$$

Thence from (1) and (2)

$$T=\tfrac{1}{2}\rho A^2k^2S_0 \sin^2 kt \int_0^l \cos^2 (\pi z/l)(1+\delta S/S_0)dz,$$

$$V=\tfrac{1}{2}A^2E(\pi/l)^2 \cos^2 kt \int_0^l \sin^2 (\pi z/l)(1+\delta S/S_0)dz.$$

But T+V is constant throughout the motion, and so the coefficients of $\sin^2 kt$ and $\cos^2 kt$ are equal. Equating them, we find

$$k^2=(E/\rho)(\pi/l)^2\frac{\int_0^l \sin^2 (\pi z/l)(1+\delta S/S_0)dz}{\int_0^l \cos^2(\pi z/l)(1+\delta S/S_0)dz},$$

or

$$k^2=(E/\rho)(\pi/l)^2\frac{1+\dfrac{2}{l}\displaystyle\int_0^l \sin^2 (\pi z/l)(\delta S/S_0)dz}{1+\dfrac{2}{l}\displaystyle\int_0^l \cos^2 (\pi z/l)(\delta S/S_0)dz} \quad . \quad . \quad (3)$$

As $\delta S/S_0$ is by hypothesis everywhere small, we may neglect squares of either integral as compared to unity, and so deduce

$$k^2=\frac{E}{\rho}\left(\frac{\pi}{l}\right)^2\left\{1+\frac{2}{l}\int_0^l\left(\sin^2\frac{\pi z}{l}-\cos^2\frac{\pi z}{l}\right)\frac{\delta S}{S_0}dz\right\}$$

$$=\frac{E}{\rho}\left(\frac{\pi}{l}\right)^2\left\{1-\frac{2}{l}\int_0^l\frac{\delta S}{S_0}\cos\frac{2\pi z}{l}dz\right\}. \quad . \quad . \quad . \quad . \quad (4)$$

Now suppose that the rod has the same period as if it possessed a uniform section but were of slightly different length $l+\delta l$. Then

$$k^2=\frac{E}{\rho}\,\frac{\pi^2}{(l+\delta l)^2}=\frac{E}{\rho}\,\frac{\pi^2}{l^2}\left\{1-\frac{2\delta l}{l}\right\}. \quad . \quad . \quad . \qquad (5)$$

Comparing the identities (4) and (5) we deduce at once Mr. Bell's formula

$$\delta l=\int_0^l \frac{\delta S}{S_0}\cos\frac{2\pi z}{l}\,dz. \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

This gives the correction δl necessary to obtain the length of the equivalent rod of uniform section.

The proof does not assume S_0 to be the *mean* area of the cross section. But the assumptions that $\delta S/S_0$ is everywhere very small, and that the squares of the integrals in (3) are negligible, will be in general most satisfactorily fulfilled when S_0 is the mean section.

When S is more than usually variable, a slight increase in accuracy would probably be obtained by determining k directly from (3). In this event it would be convenient that S_0 should be the mean section, as we should then have

$$\int_0^l\left(\cos^2\frac{\pi z}{l}+\sin^2\frac{\pi z}{l}\right)\frac{\delta S}{S_0}\,dz=0,$$

and so

$$\int_0^l \cos^2\frac{\pi z}{l}\cdot\frac{\delta S}{S_0}\,dz=-\int_0^l \sin^2\frac{\pi z}{l}\cdot\frac{\delta S}{S_0}\,dz=\frac{1}{2}\int_0^l \cos\frac{2\pi z}{l}\cdot\frac{\delta S}{S_0}\,dz.$$

We should thus have still really only one integral to evaluate.

If the tube were fixed-fixed instead of free-free, the origin being still at an end, we should obtain for the correction to the length

$$\delta l=-\int_0^l (\delta S/S_0)\cos(2\pi z/l)\,dz. \quad . \quad . \quad . \qquad (7)$$

This is numerically equal but opposite in sign to the correction obtained for the free-free rod; thus by combining results from the two species of vibrations we could eliminate the effects of variation of section, assuming them of course to be small.

In a fixed-free bar, $z=0$ being the fixed end, we find

$$\delta l = -\int_0^l (\delta S/S_0)\cos(\pi z/l)\,dz. \quad . \quad . \quad . \quad (8)$$

In all these cases an increase of S near a fixed end or node tends to make δl negative or raises the pitch, while an increase of S near a free end tends to make δl positive. In other words, an increase of material stiffens the bar elastically when near a node, but acts mainly as a load when occurring near a section of maximum amplitude of motion.

XLIV. *On the Curvature Method of Teaching Geometrical Optics.* *By* C. V. DRYSDALE, *D.Sc.**

NOTWITHSTANDING the fact that the wave theory of light has been employed to demonstrate some of the more simple problems in the domain of what is generally termed geometrical optics, and with manifest simplicity and convenience, this appears to have been done rather with the object of verifying the wave theory than of showing how the subject of optics can be completely dealt with from this standpoint ; and few men of science or teachers of optics appear to have realized the advantages of physical methods both for practical work and teaching, and that they should entirely supersede the geometrical or ray methods. This is unquestionably due to a very large extent to unfamiliarity, and to the cramping effect of our university curricula and text-books ; but a possible factor in the question is the impression which seems to be prevalent, even among optical specialists, that the physical methods have to be abandoned at a certain stage, and that the more complex problems relating to lens systems and aberrations must be treated by geometrical methods. Owing to the fact that British men of science have been actively engaged in extending the wave theory towards penetrating the more fascinating mysteries of interference, polarization, and electromagnetic theory, the practical applications of optics have passed for the last half century into the hands of the Germans, who took from us the geometrical methods then in vogue and have since extended them with such marked success as to give the impression that these geometrical methods are the most suitable for the purpose. At the present time a strong attempt is being made to revive the study of technical optics in this country, unfortunately with great difficulties, owing to the lack of satisfactory teaching; and the object of this paper is to show that not only are physical methods the most suitable at the outset, but that they are capable of being employed with the same increased simplicity in the whole domain of "geometrical optics."

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* Read February 24, 1905.

A few words are desirable at the outset as to what has been done in the application of physical problems to reflexion and refraction. The first step in this direction appears to have been made by Herschel in 1827 *, who seems to have dealt with ordinary lens problems fairly completely and to have devised a very satisfactory curvature notation. According to Mr. Cheshire †, in an article just published, Porro, in 1857 ‡, gave a very complete exposition of lenses and thin lens combinations from the physical standpoint, including suggestions for the study of spherical aberration and of thick lenses. In Preston's 'Light' § a short account of the elementary treatment of lenses by this method appears; and Lord Rayleigh ‖ has also devoted attention to the subject. It is, however, to Prof. S. P. Thompson ¶ that we owe nearly everything that has been done in recent years towards the rational teaching of optics, and in 1889 he wrote a paper on the curvature method of treating lens problems which he had rediscovered and had used in teaching for about eight years; and he has followed this up by several valuable papers, and his two important works—'Optical Tables and Memoranda,' and the translation of Lummer's 'Photographic Optics.'

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* Herschel, *Encyc. Metropolitana*, 1827.

† F. J. Cheshire, British Optical Journal, Nov. 1904.

‡ Porro, *Société Française de Photographie*, vol. iii. pp. 211–222, 1857.

§ Preston, 'Light,' pp. 99–106, Third edition, 1901.

‖ Lord Rayleigh, *Encyclopedia Britannica*, 1884. Article on "Optics."

¶ S. P. Thompson, Phil. Mag. 1889, vol. xxxiii., "Notes on Geometrical Optics," Part i. pp. 232–248.

It will perhaps be well to mention here that the present writer has unfortunately, until the last few days, been absolutely unconscious of practically all that has been done in the application of wave methods by others. Although a student under Prof. Thompson in 1888–90, for some reason none of this work was done, and the only thing that directed his attention to the subject was a few lectures given by Dr. Sumpner at the Central Technical College in 1892, dealing with the matter as in Preston's 'Light.' The interest roused by the manifest superiority of the wave method, however, led him afterwards to make some simple applications in practical work, and later to start a course of lectures on technical optics at the Northampton Institute, entirely based on physical optics. It is a singular fact that although we owe to scientific men the wave theory of light and its applications to optical theory, the optical trade have, apparently quite independently, adopted a system of lens and prism nomenclature which harmonizes completely with it, and makes its application to reflexion and refraction problems simple and logical. It was due to the writer becoming acquainted with this notation that he was led to take up the subject so fully, and he ventures to think that few teachers of optical science would find a knowledge of optical trade methods detrimental.

Before dealing with the curvature method, it may be interesting to note that there are several alternative methods of attacking lens problems, of which one only, and that probably the least suitable, has been given a fair trial in this country. They may perhaps be enumerated as follows:—

(*a*) The method of reckoning in conjugate distances—Geometrical or Gauss method.
(*b*) The physical or curvature method.
(*c*) The method of deviations—Von Seidel, Finsterwalder.
(*d*) The use of the characteristic function, or principle of least time—Hamilton, Thiesen, and Chalmers.
(*e*) { Thermodynamic method—Clausius.
      Employment of the "eikonal"—Bruns.
(*f*) Vector or Quaternion treatment.

Of these methods (*b*), (*c*), and (*d*) are the ones which the writer believes will be found most suitable, and it will be shown, as is almost self-evident, that they are essentially similar. Of the eikonal treatment he has no experience, but is inclined to think that its use is confined to problems of an advanced nature, and that it is unsuitable for a general elementary treatment. As to (*f*) it seems curious that no one has proposed the application of the modern vector calculus of Prof. Henrici, Mr. Oliver Heaviside, and Prof. Gibbs, to geometrical optics, as it should be capable of effecting considerable simplifications. For example, if $\hat{a}$ is a unit vector representing the direction of an incident ray, $\hat{b}$ the corresponding refracted ray, and $\hat{n}$ the vector normal to the surface, both of the ordinary laws of refraction are summed up in the simple vector relation

$$\mu_1 [\hat{a}\,\hat{n}] = \mu_2 [\hat{b}\,\hat{n}],$$

or in Heaviside's notation

$$\mu_1 \mathrm{V}\hat{a}\,\hat{n} = \mu_2 \mathrm{V}\hat{b}\,\hat{n}.$$

The writer has deduced a few interesting consequences of this fundamental expression, but has not yet had an opportunity of following it up completely; moreover, as the method is a purely geometrical one, it could only have advantages in a possible simplification of ordinary procedure, and would not have any other physical signification.

Practical Optical Units.—The simple device of opticians for spectacle-lens notation was to adopt two units which are now fairly generally known. The first of these may be used to express either the curvature of a surface or the power of a lens or mirror, and is termed a Dioptre *. This unit is a curvature corresponding to a radius of one metre; and we consequently say that a lens has a convergence of one dioptre when it has a focal length of one metre, or that a wave-front or surface has a curvature of one dioptre when it has a radius of curvature of one metre. The other unit adopted is one of angle or deviation and is termed the Prism Dioptre: it is defined as a deviation of one centimetre on a tangent line at a distance of one metre, and one prism dioptre therefore corresponds approximately to ·01 of a radian. The prism dioptre and the radian thus serve as two units of angular measurement, and the former is very conveniently related to the dioptre, as the deviation in prism dioptres produced by any thin lens at any zone is simply obtained by multiplying the convergence of the lens in dioptres by the radius of the zone in centimetres.

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* Variously spelt, Dioptrie, Dioptre, and Diopter. The author has hitherto preferred the spelling Dioptre, harmonizing with metre, but modern usage favours meter and diopter. It appears, however, that Dioptrie was proposed by Monoyer in 1872, and adopted by the Brussels International Congress in 1875.

In the prism dioptre and the radian we have two units of angular measurement which cover most requirements; but the curvature dioptre, although an exceedingly convenient unit for many purposes, is inconvenient to deal with when we are concerned with microscope lenses and others of short focal length. The writer has therefore recently proposed [*] the adoption of multiples and sub-multiples of this unit, using the ordinary prefixes as follows:—

Radius of curvature	Curvature.
Kilometre	Millidioptre.
Metre	Dioptre.
Centimetre	Hectodioptre.
Millimetre	Kilodioptre.

The great advantages of such a system is, that to any particular length measurement there is a corresponding curvature measurement, which enables one to have a physical idea of the magnitudes involved and avoids the use of unwieldy fractions which frequently occur otherwise. The range of curvatures may be very great, the radius varying from a wave-length to infinity.

Notation.—The writer has also found it convenient to adopt a standard notation in optical work, using small letters for the distances, radii of curvature, focal lengths, and thicknesses; large letters for the corresponding curvatures, and Greek letters for the angles [†]. The conjugate distances and focal lengths being denoted by u, v, and f respectively; we have U, V, and F as the corresponding curvatures or convergences. If the distances u, v, &c. are expressed in metres, we have $U = \frac{1}{u}$, $V = \frac{1}{v}$, &c., the curvatures in dioptres, while if u and v are in mm. U and V are in kilodioptres.

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[*] "On some Points in the Design of Optical Instruments." Proc. Optical Soc., December 18th, 1902.

[†] This appears to be almost identical with the notation employed by Herschel and by Dr. Thompson.

In like manner, if we denote the radii of the surfaces of a lens by r_1 and r_2, we should have R_1 and R_2 as the curvatures of the surfaces, and $C = R_1 - R_2$ the total curvature in the case of a thin lens. Curvatures of wave-fronts may be reckoned positive when they are convergent, and surfaces are said to have positive curvature when they are curved in the same direction as convergent emergent light. This harmonizes the formulæ for reflexion and refraction at curved surfaces.

Part I.—ELEMENTARY OPTICS.

In dealing with this subject it was the writer's first intention to give a fairly complete exposition of the methods he has adopted, but a subsequent perusal of Dr. Thompson's article in the Phil. Mag. has shown him that the procedure he has followed is so closely identical with that advocated by Dr. Thompson as to render this unnecessary. For the sake of completeness, however, the various steps in the development of an elementary optical course may be briefly given here.

Nature and mode of Propagation of Light.—The first important step is to familiarize the student with the notion of waves and their propagation. A beginner is easily convinced by a few simple illustrations that light must be either propagated by projectiles or undulations. The ripple tank can then be used to demonstrate the propagation of waves, and the effects of reflexion, &c. shown simultaneously by optical projection, and in the tank. The justification for the undulatory theory can be well shown by interference, ripples being excited from two simultaneously vibrating points, and the results compared with interference from a bi-prism. Even elementary students can quite appreciate that the wave theory is the only one which can satisfactorily account for the dark interference-bands. The finite rate of propagation of waves is then pointed out with references to determinations of the speed of light, and it is also shown that ripples of different frequencies can be excited and that the wave-lengths differ. Finally it is shown by analogy with the ripples that the wave-fronts diverge in expanding spheres, becoming less curved as they recede from the source, and ultimately plane, and that the direction of propagation is perpendicular to the wave-front; and the difference between plane, divergent, and convergent waves is illustrated. At the same time, it may be pointed out with advantage that if the medium were not isotropic, the speed of propagation would be different in different directions, the wave-front being ellipsoidal and not perpendicular to the lines of propagation. The writer has never found that elementary students have any difficulty over this, and it prepares the way for subsequent work in polarization, &c.

The next stage is to explain the formation of shadows and of an image by a pinhole, referring at the same time to diffraction. The formation of the penumbra may also be well illustrated by waves, and photometry explained by showing that any portion of the wave-front carrying a certain amount of light-energy expands proportionally to the square of the distance from the source.

The study of reflexion at plane surfaces comes next, first with plane and then with spherical waves, proving the laws of reflexion, application to various instruments, measurement of angles of a prism, &c., and formation of images by one or more plane mirrors.

Curvature and its Measurement.—At this stage the ideas and units of curvature are introduced, explaining the necessity for the measurement of the curvatures both of waves and of reflecting surfaces. Students will easily realize that a satisfactory definition of curvature is the reciprocal of the radius, but it may be proved from the ordinary definition of curvature if preferred. The dioptre with its multiples and submultiples may then be introduced and the relations:

$$\text{curvature in dioptres} = \frac{1}{r\ (\text{metres})} = \frac{100}{r\ (\text{cm.})} = \frac{39{\cdot}37}{r''},$$

and the corresponding conversions from curvatures to radii.

Practical curvature measurement may then be taken up, the equality of the products of the segments of any two chords of a circle being demonstrated either by geometry or by simple measurement, and from this the ordinary spherometer formula

$$r=\frac{c^2+h^2}{2h},\ \text{or}\ R=\frac{2h}{c^2+h^2}=\frac{2}{c^2}h$$

when the curvature is small. When c and h are in millimetres the resultant R must evidently be in kilodioptres, and the curvature in dioptres is then $\frac{2000\,h}{c^2+h^2}=\frac{2000}{c^2}h$ for small curvatures. It is then pointed out that if $c^2=2000$ or $c=44{\cdot}7$ mm. the sagitta or sag of the curve denotes directly the curvature in dioptres, and Prof. Thompson's "dioptrie" spherometer and other forms are explained. At the same time attention is called to the facts, first that the curvatures of all circular curves having the same chord are proportional to their sagittas, and secondly, that this is only true for chords of small lengths compared with the radius. This assists the explanation of spherical aberration later.

The properties of spherical mirrors follow immediately. From the fact that light striking the surface travels back the same distance as it would have gone forward in the absence of the mirror, it follows immediately that the curvature of the mirror is the mean between the curvatures of the incident and reflected wave-fronts. Hence $R=\frac{V-U}{2}$ or $V-U=2R$, where V and U are the convergences of the incident and reflected light and $2R=F$ is shown to be the convergence of the mirror. By considering the image of an object formed by the mirror when the latter is stopped down to a pinhole at the vertex, we at once find the magnification $m=\frac{v}{u}=\frac{U}{V}$, and the whole of the properties of mirrors are deduced from the two equations

$$V-U=2R=F$$

$$m=\frac{v}{u}=\frac{U}{V}.$$

The error of considering the sagitta proportional to curvature for large apertures is again pointed out, with reference to parabolic mirrors, caustics, &c.

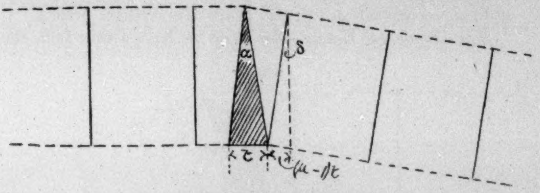
Refraction and Dispersion.—It is at this point that the value of the wave method begins to show itself most strikingly. Refraction at a simple plane surface may be shown by Huyghens' method. The writer has found it of the greatest assistance to students to illustrate the wave-fronts by parallel lines of men marching towards a river which they can only ford at some fraction of their marching speed. The swinging round of the line is easily realized ; and if at the same time the idea is introduced of the lines being made up of men of different heights, with uniforms ranging from violet to red according to size and strength, the reason for dispersion is at once grasped. It is at the same time explained that the more rapid violet light vibrations or shorter waves would naturally be more encumbered by dense or gross matter than the larger red waves. At the same time the general effects of refraction by parallel plates, prisms and lenses are illustrated by the crossing of rivers with parallel, oblique or curved banks. This paves the way for a complete study of refraction and of the properties of prisms and lenses. As it would take up too much space to give a detailed course here, it is proposed simply to give one or two examples as showing either original methods or ones which are less generally known and which the writer considers should be adopted.

It should be here mentioned that it has been found convenient when dealing with thin prisms and lenses, to consider the alteration of the wave-front by the prism or wave as a whole, and only to introduce the formulæ for single surfaces when thick lenses and prisms or refracting systems in general are considered.

Retardation by a Parallel Plate.—Since the refractive index $\mu = \frac{\text{velocity in air}}{\text{velocity in medium}}$, it follows that in passing through a thickness t of glass, the wave would have travelled forward a distance μt in air, and the retardation caused by the plate is $\mu t - t = (\mu - 1)t$.

Refraction by a Thin Prism.—In the case of a thin prism (fig. 1) having a thickness t at its base and of zero at its apex, the retardation of the wave at the base being $(\mu - 1)t$, we have evidently $\delta=(\mu-1)\alpha$. It is pointed out that this cannot hold for thick prisms, as in that case the distance traversed by the light depends on the direction of entry, and it is not justifiable to take the angles as being proportional to the bases.

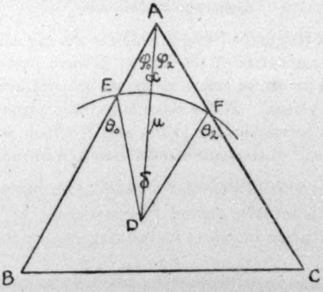
Fig. 1.



Refraction by Thin Prism.

Thick Prisms.—Here we have only to take a section of the prism ABC (fig. 2), and indicate the direction of the three wave-fronts before incidence, in the glass, and after emergence by DE, DA, and DF respectively. The angles θ_0 and ϕ_0 are obviously the angles of incidence and of refraction with reference to the left face of the prism, while θ_2 and

Fig. 2.

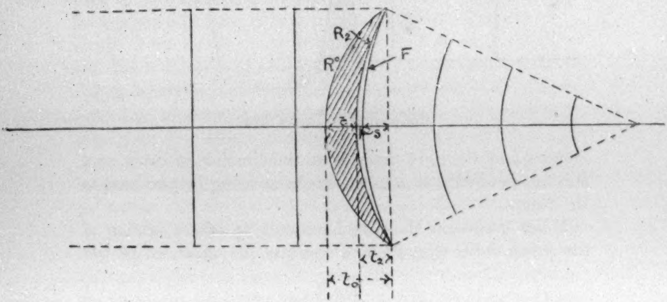


Refraction by Thick Prism.

ϕ_2 are those relating to the right face. We thus have immediately $\alpha = \phi_0 + \phi_2$ and $\delta + \alpha = \theta_0 + \theta_2$, and since $\sin\theta_0 = \mu \sin\phi_0$ and $\sin\theta_2 = \mu \sin\phi_2$ we have the distances DE, DF, and DA in the ratio of $1 : 1 : \mu$, and we are immediately led to the ordinary construction for, and properties of thick prisms.

Thin Spherical Lenses.—In fig. 3 we have a thin lens, the thickness at its centre being t, which is the sag corresponding to the total curvature of the lens. A wave passing through it will have its centre retarded by an amount $s = (\mu - 1)t$ as with a thin prism. If the wave is initially plane it receives a curvature corresponding to the sag S, which we term the convergence F of the lens, and we therefore have immediately $F = (\mu - 1)C$ corresponding to $\delta = (\mu - 1)\alpha$ for a thin prism. If the wave is initially curved or convergent by an amount U, the retardation increases its convergence by the amount

Fig. 3.



Refraction by Thin Lens.

$$V - U = (\mu - 1)C = F.$$

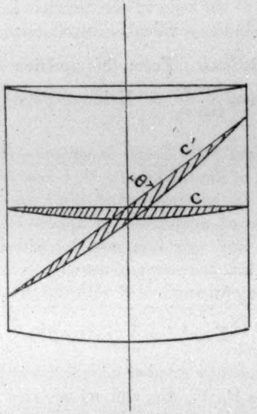
The thickness t of the lens can be made up of the two sags t_0 and t_2 to the curvatures R_0 and R_2 of the faces.

Hence
$$V - U = F = (\mu - 1)C = (\mu - 1)(R_0 - R_2).$$

By considering a pinhole stop in contact with the lens we immediately have $m = \frac{v}{u} = \frac{U}{V}$ as with the mirror; and the properties of lenses and formation of images follow immediately.

Thin Cylindrical Lenses.—For a principal section of a cylindrical lens perpendicular to its axis, we have obviously the same relation as for a spherical lens. For an oblique section the retardation of the wave-front is obviously the same,

Fig. 4.



Cylindrical Lens.

but the breadth of the beam is increased (fig. 4). From the spherometer formula $R = \frac{2h}{c^2}$ we have

$$R' = R\left(\frac{c}{c'}\right)^2 = R\sin^2\theta,$$

and hence also $F' = F\sin^2\theta$. In a meridian perpendicular to the former we must similarly have $F'' = F\cos^2\theta$, hence $F' + F'' = F$, or the sum of the convergences in two meridians at right angles is constant and equal to the convergence of the lens, analogous to a well-known theorem for curved surfaces.

Deviation and Decentring of Lenses.—By multiplying the equation $V-U=F$ by the height of intersection of the pencil with the surface we have immediately

$$\tan\sigma_2-\tan\sigma_1=Fh \quad\text{or}\quad \sigma=\sigma_2-\sigma_1=Fh$$

if the angles are small. This gives us the valuable result that the deviation at any zone of a thin lens in prism-dioptres is given by the product of the convergence of the lens in dioptres and the radius of the zone or decentration in cm. The more exact equation is also a valuable introduction to Von Seidel's method of deviations. From the equation $m=\frac{U}{V}$ we have immediately $m=\frac{\tan\sigma_1}{\tan\sigma_2}$ or Helmholtz's expression for the magnification.

Combinations of Thin Lenses in contact.—It is obvious from the summation of the thicknesses that the resultant convergence $F=\Sigma F_1$ for any number of thin lenses in contact, and that in the case of cylindrical or sphero-cylindrical lenses, the powers in any meridian may be added. It has been already shown that the convergence of a cylindrical lens in a meridian making an angle of θ with the axis is

$$F\sin^2\theta=\frac{F}{2}(1-\cos 2\theta).$$

Hence if we have any number of cylindrical lenses in contact of convergences F_1, F_2, &c., and whose axes make angles α_1, α_2, &c., with a reference meridian, we have

$$F=\tfrac{1}{2}\Sigma F_1-\tfrac{1}{2}\Sigma F_1\cos 2(\theta-\alpha_1)$$

in any meridian making an angle θ with the reference meridian. On expanding we have

$$F=\tfrac{1}{2}\Sigma F_1-\tfrac{1}{2}\{\cos 2\theta\Sigma F_1\cos 2\alpha_1-\sin 2\theta\Sigma F_1\sin 2\alpha_1\},$$

and this is a maximum or minimum for

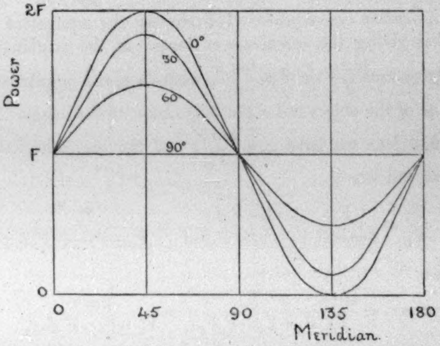
$$\tan 2\theta=\frac{\Sigma F_1\sin 2\alpha_1}{\Sigma F_1\cos 2\alpha_1},$$

giving the direction of the axes of the combination, while the maximum and minimum values are given by the expression

$$\tfrac{1}{2}\Sigma F_1\pm\tfrac{1}{2}\sqrt{(\Sigma F_1\sin 2\alpha_1)^2+(\Sigma F_1\cos 2\alpha_1)^2}.$$

corresponding to the convergences in the two principal meridians. Fig. 5 shows curves exhibiting the relation between the convergence and angle for two cylinders crossed at various angles. The properties of such combinations have already been brought before this Society by Dr. Thompson *, but the writer had come to the same conclusions earlier. It is interesting to note that while the displacement of a prism corresponds to an ordinary vector, the curvature of a cylinder is what is called by Steinmetz a double frequency vector, and this can be well illustrated experimentally. In general the distinction which Prof. Thompson has drawn between unipolar and dipolar quantities has its origin in the distinction between simple and double frequency vectors.

Fig. 5.



Power of combination of two equal cylinders crossed at various angles.

Combinations of Two Thin Lenses.—These may be mentioned here before considering thick lenses, as they do not imply a knowledge of refraction at a single surface. Considering two thin lenses of convergences F_0 and F_2 separated by an interval d_1, parallel light falling on the first emerges from it with a convergence F_0 which becomes $\frac{1}{1/F_0-d}$ or $\frac{F_0}{1-F_0d_1}$

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* S. P. Thompson, Phil. Mag. March 1900.

on reaching the second. To this is added the convergence of the second lens, giving us the emergent convergence as

$$\frac{F_0}{1-F_0d_1}+F_2=\frac{F_0+F_2-F_0F_2d_1}{1-F_0d_1}.$$

This may be termed the "back" or "emergent" convergence if desired, corresponding to the "back focus," and the other emergent convergence will evidently be $\frac{F_0+F_2-F_0F_2d_1}{1-F_2d_1}$.

Equivalent Convergence.—Introducing the equivalent lens as that giving the same size of image as the combination, we have $x=\frac{\tan\alpha}{F}$ or $F=\frac{\tan\alpha}{x}$, where α is the angular magnitude of the object and x the magnitude of the image. For the first lens we have $x_0=\frac{\tan\alpha}{F_0}$, and the magnification by the second lens is

$$\frac{U_2}{V_2}=\frac{F_0}{1-F_0d_1}\,\frac{1-F_0d_1}{F_0+F_2-F_0F_1d_1}=\frac{F_0}{F_0+F_2-F_0F_2d_1}.$$

Hence

$$x=mx_0=\frac{\tan\alpha}{F_0}\,\frac{F_0}{F_0+F_2-F_0F_2d_1}=\frac{\tan\alpha}{F_0+F_2-F_0F_2d_1},$$

and $$F=\frac{\tan\alpha}{x}=F_0+F_2-F_0F_2d_1.$$

Nodal or Principal Points.—The Nodal points being defined as those points through which light passes undeviated, we have, since the deviation at any zone of a lens $= Fh$,

$$F_0h_0+F_2h_2=0,$$

or $$F_0d_0+F_2d_2=0 \quad \text{(see fig. 6)},$$

but $d_0-d_2=d_1$ the distance between the lenses.

Hence $$d_0=\frac{F_2d_1}{F_0+F_2}$$ giving the optical centre,

or $$D_0=\frac{F_0+F_2}{F_2d_1}$$ the convergence of the light focussing at O.

Similarly $D_2 = -\frac{F_0+F_2}{F_0 d_1}$.

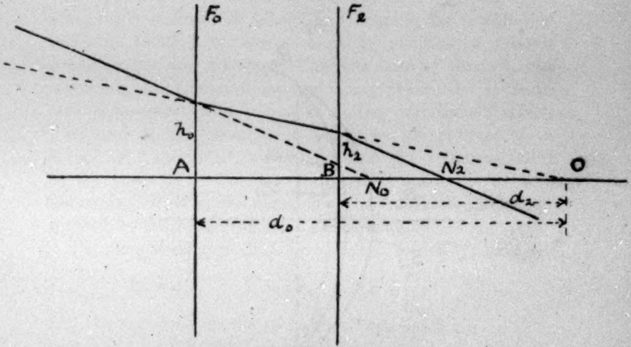
Hence $D_0' = D_0 - F_0 = \frac{F_0+F_2-F_0F_2d_1}{F_2 d_1}$,

the convergence of the light proceeding to N_0,

and $d_0' = \frac{F_2 d_1}{F_0+F_2-F_0F_2d_1}$ the distance AN_0,

while $d_2' = -\frac{F_0 d_1}{F_0+F_2-F_0F_2d_1}$ the distance BN_2.

Fig. 6.



Nodal points of a Two-lens Combination.

We have seen above that the emergent convergence at the second lens $F' = \frac{F_0+F_2-F_0F_2d_1}{1-F_0d_1}$, and now that the convergence at any distance $d_2' = \frac{F'}{1-F'd_2'}$.

But $d_2' = -\frac{F_0d_1}{F_0+F_2-F_0F_2d_1}$; $\therefore\ 1-F'd_2 = \frac{1}{1+F_0d_1}$,

and therefore convergence at Nodal point

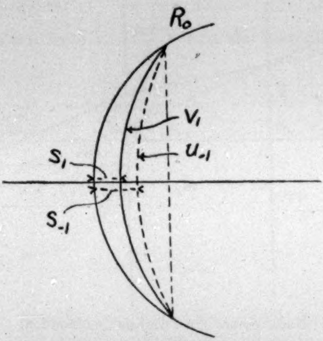
$$= \frac{F_0+F_2-F_0F_2d_1}{1-F_0d_1}(1-F\ d_1) = F_0+F_2-F_0F_2d_1 = F\ ;$$

or the convergence in the plane through the nodal point of emergence is the equivalent convergence; or the distance between the nodal and focal points is the equivalent focal length. This latter might of course have been directly obtained by taking the back focal length and distance of nodal point.

Thick Lenses and Refracting Systems.

Refraction at Single Surface.—Up to the present we have not needed to consider refraction at a single curved surface, but it is necessary to do so before studying thick lenses and lens systems. If light passes through a surface (fig. 7) separating two media of refractive indices μ_{-1} and μ_1, we have always as our fundamental equation $\mu_{-1}s_{-1}=\mu_1 s_1$, where s_{-1} and s_1 are the departures or sags of corresponding parts of the incident and refracted wave-fronts from the surface.

Fig. 7.



Refraction at a Single Spherical Surface.

In the case of a normal homocentric pencil of convergence U and a spherical surface of curvature R_0 this gives us immediately

$$\mu_{-1}(R_0-U_{-1})=\mu_1(R_0-V_1)$$

or

$$\mu_1 V_1-\mu_{-1}U_{-1}=(\mu_1-\mu_{-1})R_0=F_0.$$

In the interspace between two surfaces we have

$$U_1 = \frac{1}{\frac{1}{V_1} - t_1} = \frac{V_1}{1 - V_1 t_1},$$

where U_1 is the convergence on meeting the second surface and t is the thickness.

This equation can, however, be written

$$\mu_1 U_1 = \frac{1}{\frac{1}{\mu_1 V_1} - \frac{t_1}{\mu_1}}.$$

Nomenclature for Thick Lenses and Lens Systems.—We can greatly reduce the labour and simplify the working of problems relating to lens systems by adopting a suitable nomenclature and notation. This was done by Gauss *, who introduced the idea of "absolute" and "reduced" distances and thicknesses. Following this we may introduce the term "reduced convergence" of a pencil as the product of its actual or "absolute" convergence and the refractive index of the material; while the "reduced thickness," as with Gauss, is defined as the negative of the "absolute thickness" divided by the refractive index of the medium.

Denoting these quantities by accented letters, we have

$$U' = \mu U, \qquad V' = \mu V, \qquad t' = -\frac{t}{\mu};$$

and the equations for the surface and interspace are

$$V_1' - U'_{-1} = (\mu_1 - \mu_{-1}) R_0 = F_0,$$

and

$$U_1' = \frac{1}{t_1' + \frac{1}{V_1'}} = \frac{V_1'}{V_1' t_1' + 1}.$$

It is worthy of note that each of the reduced quantities has a simple physical meaning. For in the first equation, if the surface is plane

$$V_1' - U'_{-1} = 0, \quad \text{or} \quad V_1' = U'_{-1}.$$

Consequently the reduced convergence of a pencil does not change in passing through a plane surface separating any two media. But if the second medium is air $\mu_1 = 1$ and

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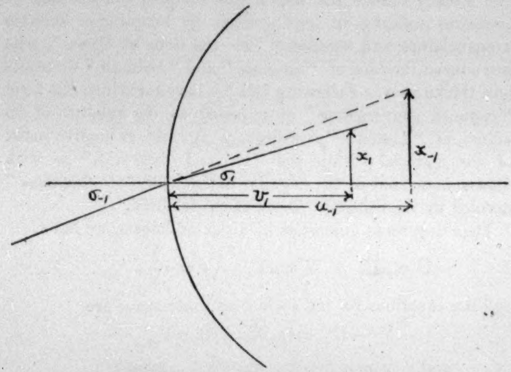
* Pendlebury, 'Lenses and Systems of Lenses.'

$V = U'_{-1}$. Hence the reduced convergence of a wave-front in any medium is the convergence it would have if it emerged from that medium into air through a plane surface. The term "equivalent" instead of reduced convergence may therefore be employed if preferred, but is preferably kept for the equivalent convergence of the combination. Similarly $t_1' = -\frac{t}{\mu}$ is the "apparent thickness" of a parallel plate of the medium when viewed in air.

The magnification produced by a single surface is readily seen by the same device of a pinhole stop at the vertex, fig. 8.

Fig. 8.



Magnification by Single Spherical Surface.

We then have

$$x_{-1} = \mu_1 \tan \sigma_{-1} = \frac{\tan \sigma_{-1}}{U_{-1}}$$

and

$$x_1 = \frac{\tan \sigma_1}{V_1}\,.$$

Hence

$$m = \frac{x_1}{x_{-1}} = \frac{U_{-1}}{V_1}\,\frac{\tan \sigma_1}{\tan \sigma_{-1}}$$

$$= \frac{U_{-1}}{V_1}\,\frac{\sin \sigma_1}{\sin \sigma_{-1}} \text{ approx.} = \frac{U'_{-1}}{V_1'}\,.$$

It will be unnecessary to go here more fully into the properties of the single surface, its two principal convergences of the object and image space, principal and nodal points, &c. We will immediately pass to the case of the thick lens.

Simple Thick Lens.—Denoting the curvatures of the surfaces by R_0 and R_2 respectively, and the refractive indices of the media by μ_{-1}, μ_1, and μ_3, we have :—

At first surface, $V_1' - U_{-1}' = F_0$ or $V_1' = F_0 + U'_{-1}$

Interspace . . . . $U_1' = \dfrac{1}{t_1' + \dfrac{1}{V_1'}} = \dfrac{V_1'}{V_1' t_1' + 1}$

Second surface . . . . . . $V_3' = F_2 + U_1'$.

Combining these equations we have immediately

$$V_3' = F_2 + \frac{F_0 + U'_{-1}}{(F_0 + U'_{-1})t_1' + 1} = \frac{F_0 + F_2 + F_0 F_2 t_1' + (F_2 t_1' + 1)U'_{-1}}{F_0 t_1' + 1 + t_1' U'_{-1}}$$

$$= \frac{A + BU'_{-1}}{C + DU'_{-1}},$$

where $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = -1.$

To find the size of the image produced we have

$$\frac{1}{m} = \frac{1}{m_0}\frac{1}{m_2},$$

where m_0 and m_2 are the successive magnifications at the surfaces.

But $\dfrac{1}{m_0} = \dfrac{V_1'}{U'_{-1}}$ and $\dfrac{1}{m_2} = \dfrac{V_3'}{U_1'} = \dfrac{A + BU'_{-1}}{C + DU'_{-1}} \dfrac{V_1' t_1' + 1}{V_1'}$

and $C + DU'_{-1} = V_1 t_1' + 1$ by the above.

Hence $\dfrac{1}{m_2} = \dfrac{A + BU'_{-1}}{V_1'}$ and $\dfrac{1}{m} = \dfrac{A + BU'_{-1}}{U'_{-1}}$.

This re-establishes the three Gauss relations in the convergence form. It should be noted that the quantity A is the reduced equivalent convergence of the lens.

It will not be necessary to show how the properties of thick lenses may be deduced from these equations as methods similar to those of Gauss can be followed. We may therefore pass on to the consideration of a complete system, but before doing so reference may be made to the deviation or Von Seidel method.

If we take our equation for a single surface

$$\mu_1 V_1 - \mu_{-1} U_{-1} = F_0$$

and multiply it by h_0, the lateral distance of intersection of a pencil, we have

$$\mu_1 \tan \sigma_1 - \mu_{-1} \tan \sigma_{-1} = F_0 h_0.$$

If we now denote the product of the tangent of an angle by the refractive index of the medium, by the term "reduced angle" and represent these reduced angles by accented letters, we have

$$\sigma_1' - \sigma'_{-1} = F_0 h_0.$$

In the interspace between two refracting surfaces we have clearly

$$h_2 - h_0 = -t_1 \tan \sigma_1 = -\frac{t_1}{\mu_1} \mu_1 \tan \sigma_1,$$

or

$$h_2 - h_0 = \sigma_1' t_1'.$$

Hence, for tracing the course of a pencil through any system, we have simply

Surface 0. $\sigma_1' - \sigma'_{-1} = F_0 h_1$ or $\sigma_1' = F_0 h_0 + \sigma'_{-1}$;
Interval 1. $h_2 - h_0 = \sigma_1' t_1'$ or $h_2 = t_1' \sigma_1' + h_0$;
Surface 2. $\sigma_3' - \sigma'_{-1} = F_2 h_2$ or $\sigma_3' = F_2 h_2 + \sigma_1'$;

and this may be extended to any number of surfaces by putting down the convergences and reduced thicknesses, and the corresponding lateral intersections and reduced angles in the order

$$\begin{array}{cccccc} & F_0 & t_1' & F_2 & t_3' & \&c. \\ \sigma'_{-1} & h_0 & \sigma_1' & h_2 & \sigma_3' & \&c. \end{array}$$

each member of the lower series being derived by multiplying the penultimate member by the corresponding value above it, and adding the antepenultimate. This is the extremely valuable method due to Von Seidel *.

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* Dr. S. P. Thompson's translation of Lummer's Photographic Optics, Appendix II.

For a single thick lens we have to take simply the three equations given, and we find

$$h_2 = t_1'\sigma_1' + h_0 = (F_0 t_1' + 1)h_0 + t_1'\sigma'_{-1} = Ch_0 + D\sigma'_{-1}$$

and

$$\sigma_3' = F_2 h_2 + \sigma_1' = (F_0 + F_2 + F_0 F_2 t_1')h_0 + (F_2 t_1' + 1)\sigma'_{-1} = Ah_0 + B\sigma'_{-1}.$$

We may express this either in the ordinary Gauss or in the convergence form by taking

$$v_3' = \frac{h_2}{\sigma_3'}, \quad \text{or} \quad V_3' = \frac{\sigma_3'}{h_2}.$$

In the latter case

$$V_3 = \frac{Ah_0 + B\sigma'_{-1}}{Ch_0 + D\sigma'_{-1}} = \frac{A + BU'_{-1}}{C + DU'_{-1}} \text{ as before.}$$

Extension of Convergence Theory to any System.

Our equations for the convergence method were

$$V_1' - U'_{-1} = F_0\,;$$

$$U_1' = \cfrac{1}{t_1' + \cfrac{1}{V_1'}}\,;$$

$$V_3' - U_1' = F_3, \text{ \&c.}$$

Combining these equations we have for any system

$$V'_{2n+1} = F_{2n} + \frac{1}{t_{2n-1}} + \frac{1}{F_{2n-2}} + \cdots\cdots + \frac{1}{t_1'} + \frac{1}{F_0 + U'_{-1}}$$

corresponding to the ordinary continued fraction of Gauss.

We could derive the properties of complete lens systems by the theory of continued fractions as was done by Gauss; but it is much more simple and satisfactory to have recourse to a proof by induction, as the writer believes was first done by Dr. Sumpner, for the ordinary Gauss method.

Let it be granted that for a given system the equations

$$V'_{2n-1} = \frac{A + BU'_{-1}}{C + DU'_{-1}} \quad \frac{1}{m_{2n-1}} = \frac{A' + BU'_{-1}}{U'} \quad \text{and} \quad \begin{vmatrix} A & C \\ B & D \end{vmatrix} = -1$$

hold.

Now let the light pass through a space t'_{2n+1} and another surface of convergence F_{2n}. We shall have

$$V'_{2n-1}=\frac{A+BU'_{-1}}{C+DU'_{-1}}$$

$$U_{2n-1}=\frac{V'_{2n-1}}{V'_{2n-1}t'_{2n-1}+1};$$

$$V_{2n+1}=U'_{2n-1}+F_{2n}.$$

We then have

$$V_{2n+1}=\frac{A(F_{2n}t'_{2n-1}+1)+CF_{2n}+\{B(F_{2n}t'_{2n-1}+1)+DF_{2n}\}U'_{-1}}{At'_{2n-1}+C+(Bt'_{2n-1}+D)U'_{-1}}$$

$$=\frac{A'+B'U'_{-1}}{C'+D'U'_{-1}},$$

where

$$\begin{vmatrix}A' & C'\\ B' & D'\end{vmatrix}=\begin{vmatrix}A(F_{2n}t'_{2n-1}+1)+CF_{2n} & At'_{2n-1}+C\\ B(F_{2n}t'_{2n-1}+1)+DF_{2n} & Bt'_{2n-1}+D\end{vmatrix}=\begin{vmatrix}A & C\\ B & D\end{vmatrix}=-1$$

by the ordinary properties of determinants.

Also

$$\frac{1}{m_{2n+1}}=\frac{1}{m_{2n}}\frac{1}{m_{2n-1}}$$

$$\frac{1}{m_{2n-1}}=\frac{A+BU'_{-1}}{U'_{-1}};$$

$$\frac{1}{m_{2n}}=\frac{V'_{2n+1}}{U'_{2n-1}}=\frac{A'+B'U'_{-1}}{C'+D'U'_{-1}}\cdot\frac{At'_{2n-1}+C+(Bt'_{2n-1}+D)U'_{-1}}{A+BU'_{-1}}$$

$$=\frac{A'+B'U'_{-1}}{A+BU'_{-1}}.$$

Hence
$$\frac{1}{m_{2n+1}}=\frac{A'+B'U'_{-1}}{U'_{-1}}.$$

Consequently, since these equations were proved to hold for two surfaces, and have now been shown to hold for an additional surface, they must hold generally.

It may be of interest as a conclusion to this section to exhibit the three methods in a comparative form.

Comparative Table of Formulæ for Refracting Systems.

	GAUSS METHOD.	DEVIATION METHOD.	CURVATURE METHOD.
Surface 0 ...	$\frac{1}{v_1'} - \frac{1}{u'_{-1}} = F_0$	$\sigma_1' - \sigma'_{-1} = F_0 h_0$	$V_1' - U'_{-1} = F_0$
Interval 1 ...	$u_1' = v_1' + t_1'$	$h_2 - h_0 = t_1'\sigma_1'$	$U_1' = \frac{V_1'}{t_1 V_1' + 1}$
Surface 2 ...	$\frac{1}{v_3'} - \frac{1}{u_1'} = F_2$	$\sigma_3' - \sigma_1' = F_2 h_2$	$V_3' - U_1' = F_2$
Interval 3 ...	$u_3' = v_3' + t_3'$	$h_4 - h_2 = t_3'\sigma_3'$	$U_3' = \frac{V_3'}{t_3' V_3' + 1}$
&c.	&c.	&c.	&c.
Results referred to first and last surfaces.	$v'_{2n+1} = \frac{Cu + D}{Au + B}$ $\frac{1}{m} = Au + B$ $AD - BC = -1$	$h_{2n} = Ch_0 + D\sigma'_{-1}$ $\sigma_{2n+1} = Ah_0 + B\sigma'_{-1}$ $\frac{1}{m} = \frac{Ah_0 + B\sigma'_{-1}}{\sigma'_{-1}}$ $AD - BC = -1$	$V_{2n+1} = \frac{A + BU'_{-1}}{C + DU'_{-1}}$ $\frac{1}{m} = \frac{A + BU'_{-1}}{U'_{-1}}$ $AD - BC = -1$
Results referred to principal planes.	$f' = \frac{1}{A}$ $\frac{1}{v'} - \frac{1}{u'} = \frac{1}{A}$ $m = \frac{v'}{u'}$	$\sigma'_{2n+1} - \sigma'_{-1} = Ah$* $m = \frac{\sigma'_{-1}}{\sigma'_{2n+1}}$	$F' = A$ $V' - U' = A$ $m = \frac{U'}{V'}$

* h is here the height of intersection with the principal planes.

Aberrations.

The subject of aberrations will have to be treated in a separate paper; but to show how the curvature method lends itself to aberration problems, we will here take the case of oblique refraction in thin lenses. Just as we started the ordinary theory of thin lenses by considering the retardation produced by a plate, we may here commence with oblique refraction by a similar plate.

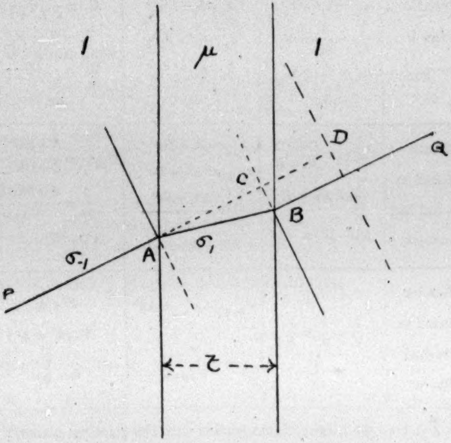
Retardation by Oblique Parallel Plate.—In fig. 9, PABQ is the course of a parallel beam of light, AB being the distance travelled through the plate; we have D as the position the wave-front would have reached in the absence of the plate such that

$$AD=\mu \, . \, AB$$

and retardation

$$s=\mu \, . \, AB-AC.$$

Fig. 9.



Oblique Refraction by a Parallel Plate.

But $$AB=\frac{t}{\cos\sigma_1} \quad \text{and} \quad AC=AB\cos(\sigma_{-1}-\sigma_1).$$

Hence $$s=\frac{\mu-\cos(\sigma_{-1}-\sigma_1)}{\cos\sigma_1}t.$$

Putting $$\sin\sigma_{-1}=\mu\sin\sigma_1$$

and simplifying we have

$$s=\{\sqrt{\mu^2-\sin^2\sigma_{-1}}-\cos\sigma_{-1}\}t$$

accurately ; or to a second approximation

$$s=\left(\mu-1\right)\left(1+\frac{\sigma_{-1}^2}{2\mu}\right)t.$$

Oblique Refraction through Thin Lens.—In the case of a thin lens we may consider the centre of the lens as a parallel plate, and the retardation of the centre of the wave-front is therefore as given above. Hence, if we consider two meridians of the lens, the primary or meridional plane containing the axis of the lens and beam, and the secondary or sagittal plane lying in the axis of the beam perpendicular to the other plane,—

We have in the sagittal plane therefore if ϕ is the obliquity

$$\mathscr{F}_2 = \mathrm{F}\left(1 + \frac{\phi^2}{2\mu}\right).$$

In the meridional plane the retardation of the centre of the beam is the same, but the effective aperture of the lens is reduced in the ratio of $\cos\phi$, and the curvature therefore increased in the ratio $\dfrac{1}{\cos^2\phi} = 1 + \phi^2$ approx., and hence the meridional convergence

$$\mathscr{F}_1 = \mathrm{F}\left(1 + \frac{\phi^2}{2\mu}\right)\left(1 + \phi^2\right).$$

The lens therefore acts as a sphero cylinder of spherical power $\mathrm{F}\left(1 + \dfrac{\phi^2}{2\mu}\right)$ and cylindrical power $\mathrm{F}\phi^2$, or when ϕ is in degrees

$$\text{Spherical power} = \mathrm{F}\left(1 + \frac{\phi^2}{5622\mu}\right).$$

$$\text{Cylindrical}\ \ ,, \ \ = \frac{\mathrm{F}\phi^2}{2861}.$$

The axis of the cylinder is of course in the sagittal plane.

The wave-front also evidently reaches one edge of the lens before the other, giving rise to asymmetrical refraction or coma, but this will not be further considered here.

Primary and Secondary Focal Lines.—We have evidently

$$\text{Primary convergence} = \mathscr{F}_1 = \mathrm{F}\left\{1 + (2\mu + 1)\frac{\phi^2}{2\mu}\right\}$$

$$\text{Mean}\ \ ,, \ \ = \mathscr{F}_0 = \mathrm{F}\left\{1 + (\mu + 1)\frac{\phi^2}{2\mu}\right\}$$

$$\text{Secondary}\ \ ,, \ \ = \mathscr{F}_2 = \mathrm{F}\left\{1 + \frac{\phi^2}{2\mu}\right\}.$$

The mean convergence corresponds to the circle of least confusion. Or in focal lengths we have

Primary focal length $f_1 = f\left\{1-(2\mu+1)\frac{\phi^2}{2\mu}\right\}$

Mean ,, ,, $f_0 = f\left\{1-(\mu+1)\frac{\phi^2}{2\mu}\right\}$

Secondary ,, ,, $f_2 = f\left\{1-\frac{\phi^2}{2\mu}\right\}$

Curvature of Focal Surfaces.—In order that the oblique foci should lie on a plane through the principal focus we have

$$\mathscr{F} = \mathrm{F}\cos\phi = \mathrm{F}\left(1-\frac{\phi^2}{2}\right) \text{ approx.}$$

If $\Delta\mathscr{F}$ is the excess of the actual convergence over that necessary to focus on the plane, we have

$$\text{sag}\, \Delta f = \frac{\Delta\mathscr{F}}{\mathrm{F}^2},$$

and lateral distance of focus from principal focus

$$x = \frac{\tan\phi}{\mathrm{F}} = \frac{\phi}{\mathrm{F}} \text{ approx.}$$

Hence curvature $= \frac{2\Delta f}{x^2} = \frac{2\Delta\mathscr{F}}{\phi^2}$ approx.

Applying this to the above we have :

Curvature of primary focal surface, $\mathrm{R}_1 = \frac{3\mu+1}{\mu}\mathrm{F} = 3\mathrm{F} + \frac{\mathrm{F}}{\mu}$

,, of surface of least confusion, $\mathrm{R}_0 = \frac{2\mu+1}{\mu}\mathrm{F} = 2\mathrm{F} + \frac{\mathrm{F}}{\mu}$

,, of secondary focal surface, $\mathrm{R}_2 = \frac{\mu+1}{\mu}\mathrm{F} = \mathrm{F} + \frac{\mathrm{F}}{\mu}.$

The corresponding radii of curvature can be at once written down and are in agreement with those given by Mr. Dennis Taylor and others*. It is also obvious from the method of proof that the curvature of the image surfaces is independent of the position of the object plane.

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* H. Dennis Taylor, "Design of Telescope Object Glasses," Proceedings Royal Astron. Soc.

Focal Surfaces for Thin Lenses in contact.—Since the curvature is given by $\frac{\Delta \mathscr{F}}{\phi^2}$, and the convergence of any number of thin lenses in contact is the sum of their individual convergences, we immediately have the interesting result that for combinations of thin lenses in contact, without diaphragm separated from the combination, the curvatures of the fields may be added together just as with convergences. We thus have

$$R_1 = 3\mathsf{F} + \Sigma \frac{F}{\mu}$$

$$R_0 = 2\mathsf{F} + \Sigma \frac{F}{\mu}$$

$$R_2 = \mathsf{F} + \Sigma \frac{F}{\mu}$$

where $\mathsf{F} = \Sigma F.$

For such a combination the mean curvature therefore is $2\mathsf{F} + \Sigma \frac{F}{\mu}$, or $2\mathsf{F}$ if the Petzval condition is satisfied, and under these circumstances R_1, R_2, and $R_3 = \mathsf{F}$, $2\mathsf{F}$, and $3\mathsf{F}$ respectively, while the curvature corresponding to the "astigmatic difference" is always $2\mathsf{F}$ independently of the materials of the lenses.

This example will serve to show how easily and directly aberration problems may be solved by physical methods, and the writer proposes to show in another paper that the whole theory of aberrations in refracting systems may be similarly treated with advantage. He ventures to hope, however, that enough has been said in this paper to convince everyone that not only is there no necessity for the abandonment of curvature methods at any stage in optical work, but that there is every advantage in retaining them throughout. It will also be obvious that they lend themselves to combining the study of diffraction and image formation, which should lead to valuable new results.

Discussion.

Prof. S. P. **Thompson** said he had brought before the Society two papers dealing with the curvature method in geometrical optics, one in 1888, and the other more recently when he treated cylindrical lenses. Dr. Drysdale had pointed out that Herschel used the curvature method in 1827, but Prof. Thompson remarked that Herschel then put it forward tentatively and finally abandoned it. Mr. Cheshire had recently shown that in 1857 Porro treated of teaching geometrical optics by curvatures, and it had been stated that the method was known to Galileo. He had, however, sought in vain for any reference to it in his writings. The curvature method was suggested to him in 1876, when Prof. Everett published a paper on C.G.S. units and entered into the question of dimensions as an illustration of physical units. It was in an endeavour to extend the idea of dimensions to optics that he was struck with the fact that nearly every quantity was the reciprocal of a length, and that the ordinary formulæ in connexion with thin lenses and spherical mirrors simply meant that two curvatures could be added together directly to form a third. Thereupon, in 1878, he developed an elementary treatment of lens problems from this point of view, and that treatment he had taught since 1884. He had not gone far into the treatment of aberrations by curvature methods, but he had satisfied himself that it could be carried out. The author of the paper was to be congratulated upon linking the curvature method with the other methods of teaching geometrical optics, particularly that due to Von Seidel. With regard to some of the terms used by Dr. Drysdale, he thought the word "convergence" was unsuitable as it was apt to be confused with the angle between the limiting rays of a converging pencil. This angle was constant, but the convergence altered as the light approached the focus. Dr. Drysdale had used the term dioptrie as a unit of curvature. As a matter of fact the term dioptrie was adopted by an International Congress held in Brussels in 1877 to represent the *power* of a lens of focal length one metre. He thought the term prism dioptrie was an unfortunate one, and suggested prismoptrie in its stead. Prof. Thompson expressed his interest in the fact that the Author proposed to deal with the aberrations of Von. Seidel by the curvature method, and hoped that the treatment of astigmatism would be simplified. One advantage of the curvature method was that it simplified geometry, but its great advantage lay in the fact that it kept as nearly as possible to reality.

Mr. R. J. **Sowter**, in advancing a friendly criticism of the paper, said that although a distinct contribution to the subject, the paper would probably make even more pessimistic those who felt that the curvature method in advanced optics was somewhat of a snare. This was directly opposed to the effect which Dr. Drysdale hoped for. The curvature method was suitable for the solution of many problems, mostly simple; but by its nature, its intimate association with the spherical wave-front, it was so restricted in scope that it had to be discarded when we dealt with the modern problems of photographic optics and optical design. It was not proposed to labour any argument, merely to express an opinion, and in support of this he referred Dr. Drysdale to Dr. Hasting's 'Light,' appendix A, in which book would be found, amongst other things, the curvature method solution of the important Abbe-Helmholtz magnification equation deduced at length by Dr. Drysdale. The aberration problems to be treated in a second paper were awaited with interest. The position of scepticism taken up by Mr. Sowter as to the wholesale superiority of the curvature method was not removed by the paper; and in addition he was supported in such a pessimistic view by Mr. Cheshire, and by the assurance of a leading optical calculator and designer on the continent, that for optical calculations the curvature method was of no use. In Germany, moreover, the method had been discarded.

Mr. T. H. **Blakesley** said he agreed with Mr. Sowter that the curvature method was something of a will-o'-the-wisp. If a ray was looked upon as the orthogonal trajectory of a wave surface, it was just as real as the wave surface; and in many cases it was quite as convenient to use rays as to use curvatures.

Mr. **Chalmers** remarked that from his experience he had come to the conclusion that the curvature method was most effective for getting out approximate solutions to optical problems, but for fine design work it was not so good as the ray method. In dealing with problems from the point of view of least time, he proposed to borrow from the curvature method and its notation in order to simplify the labour in first design work. With regard to the term dioptrie he said it referred to the *power* of a lens. In order to avoid any ambiguity it might be called the *focal power* of a lens. He preferred the term "equivalent thickness" to that of "reduced thickness" used by the Author, and said that it was unfortunate that Dr. Drysdale had followed the notation of Von Seidel and used the suffix -1 to denote the medium before the surface under consideration.

Dr. C. V. **Drysdale**, in reply, said that a good deal of attention had apparently been given to the subject, and that the speakers could be divided into two classes, the optimistic and the pessimistic. He would remain optimistic until he reached a point where the curvature method broke down. He had used the curvature method in dealing with the five aberrations of Von Seidel and also with achromatism, and he hoped to be able to bring the results of his investigations before the Society. A good deal might be said about the notation used in the curvature method, but he thought the method was preferable to others because it was easier to understand the physical processes involved.

XLV. *On Direct Reading Resistance-Thermometers, with a Note on Composite Thermocouples. By* ALBERT CAMPBELL, *B.A.*[*]

THE measurement of high temperatures is a matter of growing importance both in scientific and technical work, and thanks mainly to Prof. Callendar's work on the subject, platinum resistance-thermometers at present hold the foremost place for accuracy in such measurements. In their most usual form, however, resistance-thermometers do not indicate the temperature directly, but it has to be deduced from the reading by means of a formula, table, or curve. In order to get rid of this trouble, I have recently invented two simple arrangements, by either of which the reading of a resistance-box, in connexion with a resistance-thermometer, gives directly the actual temperature to a good degree of accuracy.

I shall designate the two methods :—

(A) The Rectifying Shunt Method,
and (B) The Rectifying Loop Method.

I have named them thus, because the result of using the device is to reduce to a straight line the curve connecting the box-reading with the temperature. I shall first describe (A), which is the simpler but the less accurate of the two.

(A) *The Rectifying Shunt Method.*

In this method the arrangement is extremely simple, and consists merely of the addition of an appropriate shunt to one of the arms of the Wheatstone's bridge used to measure the resistance. Before I show how this shunt is applied, let us consider the connexion between resistance and temperature. For platinum, the formulæ ordinarily used are equivalent to the following

$$\frac{R_t}{R_0} = 1 + \alpha t - \beta t^2,$$

where R_0 and R_t are the resistances at 0° C. and t respectively, and α and β are the constants. As to actual values, α and β

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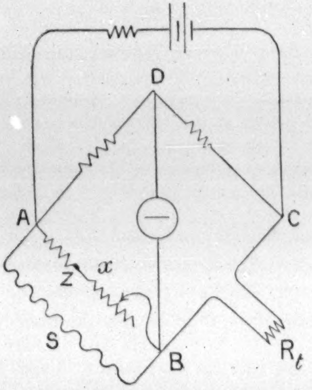
[*] Read March 10, 1905.

vary slightly in different specimens, α being of the order of 0·003, while β is near 0·0000005. This parabolic formula has been found to hold with considerable accuracy over a wide range of temperature (up to 1000° C.) *. For simplicity we may for the present make R_0 equal to 1 ohm.

Then
$$R_t = 1 + \alpha t - \beta t^2.$$

Fig. 1.



Now in fig. 1 let the Wheatstone's bridge shown have, at 0° C., four equal arms, of which BC is the platinum thermometer, and AB consists of a variable resistance x shunted by a fixed resistance s.

If the temperature of R_t is t, and x is altered until a balance is obtained, then

$$\frac{sx}{s+x} = R_t = 1 + \alpha t - \beta t^2.$$

Therefore

$$x = \frac{sR_t}{s - R_t} = \frac{s(1+\alpha t - \beta t^2)}{s - 1 - \alpha t + \beta t^2}$$

$$= \frac{s}{s-1}\left[(1+\alpha t - \beta t^2) \Big/ \left(1 - \frac{\alpha t}{s-1} + \frac{\beta t^2}{s-1}\right)\right].$$

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* See, for example, J. A. Harker, Phil. Trans. vol. cciii. p. 343.

Let $\dfrac{1}{s-1}=q$.

Then

$$x=sq\cdot\frac{1+\alpha t-\beta t^2}{1-\alpha qt+\beta qt^2}$$

$$=sq\left[1+(1+q)\alpha t+(1+q)(q\alpha^2-\beta)t^2\right.$$
$$\left.+\alpha q(1+q)(\alpha^2q-2\beta)t^3+\ldots\right]\quad . \quad (1)$$

Since α^2 and β are small let us neglect for the present the terms beyond t^2.

If we can now make the term in t^2 vanish, x will reduce to the form

$$x_0(1+mt).$$

The condition for this is

$$(1+q)(q\alpha^2-\beta)=0.$$

The factor $(1+q)$, being equal to $\dfrac{s}{s-1}$, cannot vanish since s is not zero, and thus the requisite condition is

$$q\alpha^2-\beta=0.$$

Hence

$$\frac{1}{s-1}=q=\frac{\beta}{\alpha^2};$$

therefore

$$s=1+\frac{\alpha^2}{\beta}.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since α and β are both positive for platinum, $\left(1+\dfrac{\alpha^2}{\beta}\right)$ is positive and therefore s is a real resistance. When s is made to have this value, then

$$x=x_0(1+mt)$$

where x_0 is the reading when $t=0°$ C.

We have

$$x_0=\frac{s}{s-1},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$m=\alpha(1+q)=\frac{\alpha s}{s-1},\quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$t=\frac{x-x_0}{m}.$$

Thus t is directly proportional to the increase of x above the zero-reading x_0.

Clearly this proportionality still holds when the bridge-arms are not all equal, so long as x_0 and s are kept in the proper ratio. The compensating leads, being in the measuring arm, must have resistance in proportion to it. In practice, it is desirable to be able to use for the variable part ZB an ordinary resistance-box with sets of 10-ohm, 1-ohm, and 0·1-ohm coils (with $\frac{1}{100}$ths also, if desired for the highest accuracy); and, in order to do this with a platinum thermometer of any resistance, the arm DC (fig. 1) is made to have the same resistance (at 0° C. say) as the thermometer, whilst the arm AD is made adjustable within a small range so that it can be made equal to AB at the zero-reading. Then if s, m, and x_0 be the values found from α and β as shown above, AZ is set to be $\frac{x_0}{10m}$, and the shunt is made $\frac{s}{10m}$. When these adjustments are right to start with, then each $\frac{1}{10}$ ohm between Z and B represents 1° C.

It will be remembered that in the above investigation we neglected the powers of t above t^2 [equation (1)]. The result of this is that the values of s, x_0, and m calculated as above, give readings slightly in error, particularly at the higher temperatures. It is not difficult, however, by one or two trials to alter the value of s (and hence x_0 and m) so as to give readings whose accuracy is within two or three degrees, from 0° to 1000° C.

I have described this method mainly for two reasons: firstly, that it led up to the discovery of the more exact way (B), and secondly, because it is of some theoretical interest to observe that the resistance of a platinum wire at varying temperature can be very nearly represented by two resistances in parallel, one of them having a constant temperature coefficient and the other being invariable with temperature. Perhaps the first part represents pure metal, while the second (which is about $\frac{1}{30}$ of the first) may be an alloy produced by the impurities present.

If the resistance-thermometer be of iron (at moderate temperatures) or any other metal for which

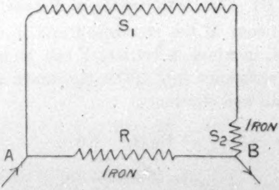
$$R_t = R_0(1 + \alpha t + \beta t^2),$$

the temperature-reading curve can usually be brought very nearly to a straight line by shunting the thermometer-arm itself by a suitable invariable resistance. Neglecting the terms above t^2 as in the preceding investigation, the requisite value of the shunt would be $R_0\left(\frac{\alpha^2}{\beta}-1\right)$. This value is not necessarily always positive, and thus it may happen that the result is not attainable (—it *is* possible for *iron*). As in the case of metals like platinum, a better result can be got by trying various values of the shunt near to $R_t\left(\frac{\alpha^2}{\beta}-1\right)$ in this case.

By introducing a little more complication, as in fig. 2, a nearer approximation to linear variation with temperature can be secured.

Here the main resistance R is shunted by $s_1 + s_2$, of which s_1 is invariable and kept cold, while a small part s_2 is of the same metal as R and at the same temperature. By suitable choice of s_1 and s_2 the resistance from A to B can be made very nearly of the form $R_0(1 + mt)$.

Fig. 2.

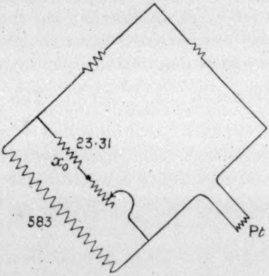


To take an actual example of Method (A),—a certain platinum resistance-thermometer follows approximately the formula

$$\frac{R_t}{R_0} = 1 + 0{\cdot}003946t - 0{\cdot}0000005798t^2. \quad . \quad . \quad (5)$$

Fig. 3 shows the values of the resistances necessary to give direct readings for this instrument.

Fig. 3.



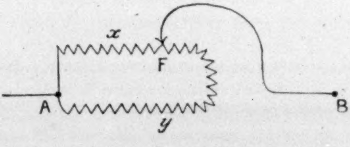
It will be seen that the shunt is here equal to $25x_0$, while the approximate formula gives $27{\cdot}86x_0$.

I need not discuss the first method further, but will pass now to the second, which is much more exact.

(B) *The Rectifying Loop Method.*

The arrangement of the measuring arm in this method is shown in fig. 4, in which a contact F can be moved along a loop of fixed resistance (n). (The resistance of the contact is assumed small and constant.)

Fig. 4.



As before, let us assume the parabolic formula

$$R_t = R_0(1 + \alpha t - \beta t^2),$$

and let us choose $R_0 = 1$,
making

$$R_t = 1 + \alpha t - \beta t^2.$$

In fig. 4 let $x+y=\text{constant}=n$, and let resistance AB be proportional to R_t when a balance (in the bridge) is obtained.

Then
$$kR_t=\frac{xy}{x+y}=\frac{x(n-x)}{n}$$
$$=x-\frac{x^2}{n}.$$

Now suppose that x is altered in proportion to the rise of temperature of R,

$$\text{i. e.} \qquad x=x_0(1+mt).$$

Let $x_0=1$, and therefore $x=1+mt$.

Then
$$k(1+\alpha t-\beta t^2)=kR_t=x-\frac{x^2}{n}=1+mt-\frac{(1+mt)^2}{n};$$
or
$$k(1+\alpha t-\beta t^2)=\left(1-\frac{1}{n}\right)+m\left(1-\frac{2}{n}\right)t-\frac{m^2}{n}t^2.$$

If this is true for all values of t, we have

$$k=1-\frac{1}{n}=\frac{n-1}{n}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\alpha=m\left(\frac{n-2}{n-1}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\beta=\frac{m^2}{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence
$$\alpha=\frac{\beta(n-2)}{m}.$$

Therefore
$$n=\frac{\alpha m+2\beta}{\beta}=\frac{\alpha m}{\beta}+2.$$

Hence from (7) $\quad m^2-\alpha m-\beta=0.$

Therefore
$$m=\frac{\alpha}{2}\left(1\pm\sqrt{1+\frac{4\beta}{\alpha^2}}\right). \quad . \quad . \quad . \quad . \quad (9)$$

Also
$$n=\frac{\alpha m}{\beta}+2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

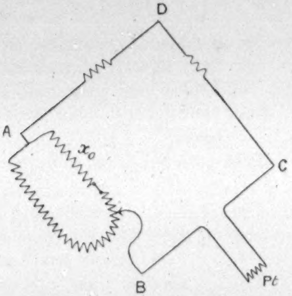
and
$$k=\frac{\alpha m+\beta}{\alpha m+2\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since α and β are both positive (for metals like platinum), real values for m, n, and k can thus always be found by equations (9), (10), and (11). As no terms in t^3 and higher powers have been neglected, *this arrangement gives an exact equivalent of the parabolic formula.* In other words, when m, n, and k have the values found from α and β by these equations, the alteration in x necessary to produce a balance for a given change of temperature will be proportional to that change of temperature.

From equation (9) it will be seen that two values of m satisfy the required conditions. The value got by taking the positive sign in (9) is the convenient one to use, as the other value gives very small change of reading for a given change of temperature.

In order to make the increase of x by 0·1 ohm correspond to a rise of temperature of 1° C., as in Method (A) the resistances (fig. 5) in the loop are raised in suitable proportion *, and the arm AD is set to give a balance at some known temperature (say 0° C.), the compensating leads, increased in like ratio, being between B and the slider.

Fig. 5.



For example, for the thermometer already mentioned [equation (5)] we could make the total resistance of the loop 603·4 ohms, and the resistance x_0 equal to 24·46 ohms.

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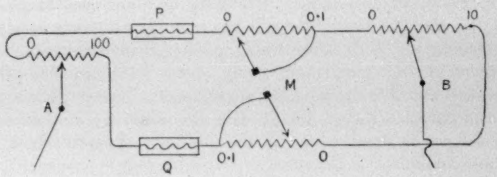
* A similar procedure is required to give equal bridge-arms.

Then each 0·1 ohm moved by the slider will accurately represent 1° C. Without altering these resistances, the combination could be used with fair accuracy with any other thermometer of nearly similar platinum. For the best work, however, the simplest way is to have two small resistance-coils adjusted to correspond to each thermometer, one representing x_0, and the other $n-x_0-100$.

The arrangement shown in fig. 6 is convenient for the Rectifying Loop, the two adjusted coils being at P and Q.

Fig. 6.



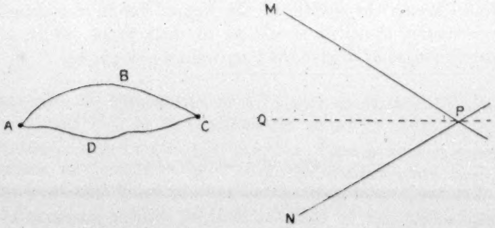
Here A and B, the ends of the bridge-arm, are connected to turning-sliders on the ten 10-ohm and ten 1-ohm coils respectively, while the central turning head is a double one, altering the upper and lower resistances by equal and opposite steps of 0·1 ohm at a time. For clearness I have put this head (M) in the middle of the figure, but in the actual construction the dials should be in their proper order of magnitude, so as to give the temperature at a glance.

APPENDIX.

On Composite Thermocouples.

For the measurement of moderate or small differences of temperature, up to 150° C., sensitive thermo-junctions of several kinds have been successfully used. For example, in the measurement of the small temperature-differences due to the Peltier Effect, I found Iron-German silver thermopiles very convenient (Proc. Roy. Soc. Edin. 1882). The great advantage of this combination lies in the fact that the thermoelectric power is large and almost constant over a good range of temperature, the lines of the iron and German-silver in the thermoelectric diagram being very nearly parallel : thus the voltages given are very nearly proportional to the temperature-differences. Fortunately the Nickel line is also very nearly parallel to that for Iron, and is lower than that for German-silver, making the combination Iron-Nickel considerably more powerful than that of Iron-German silver. Since the introduction of Constantan (or Eureka, a copper-nickel alloy), it has been largely used, particularly in Germany, in conjunction with iron, as a sensitive thermo-junction. The thermoelectric power is likewise very nearly constant and it is considerably greater than either of the others already mentioned, being about 50 microvolts per degree C. The Iron-Nickel combination, however, has for some purposes the advantage over the other two, in that its resistance is much lower for the same size of wires. In all these combinations, the voltage obtained is only approximately proportional to the temperature difference. To make the proportionality much closer I would suggest the use of composite junctions, first employed by Tait in mapping out the thermoelectric diagram.

Fig. 7.



In fig. 7 let ABC and ADC be wires of two different metals joined in parallel, and let MP and NP be their respective lines on the thermoelectric diagram. Then the line for the composite wire AC will pass through P (the Neutral Point of ABC and ADC) in some direction such as QP, and, by altering the ratio of the resistances of ABC and ADC, the direction of QP can be altered continuously so as to lie anywhere desired between MP and NP.

Thus by combining with the iron or the constantan (say) a "modifying" wire of another metal such as copper, the line of the composite wire can be turned into more exact parallelism to the line of the other metal used with it by merely altering the resistance of one of the branches of the composite wire. In the cases already mentioned the "modifying" wire (copper) will have a circuit of relatively high resistance.

In measuring high temperatures with platinum-platinum-rhodium (or PtIr) junctions, a "modifying" wire of less pure platinum, for example, would enable the thermoelectric voltage-curve to be modified at will, and perhaps "normalized" or even rectified.

Teddington, Feb. 23, 1905.

DISCUSSION.

Mr. W. **DUDDELL** expressed his interest in the paper, and asked the Author if he had considered the question of putting a constant resistance in series with the rectifying-loop. By so doing, it would be possible to simplify the mathematical processes involved in the method. With reference to the Author's note on composite thermocouples, he asked if Mr. Campbell could give any information regarding the variation of the thermoelectric properties of constantan with the percentage of nickel and copper present in the alloy.

Prof. **CALLENDAR** said he admired the rectifying-loop method of reducing the direct readings to the Gas-Scale, and thought it would be useful in practical work, especially at high temperatures, but for the most accurate scientific work at moderate temperatures he thought the corrections would be too complicated, and that trouble would be saved by adopting the ordinary method. It was most important in practice to secure accurate compensation of the leads, which could be insured in a satisfactory manner only by making the leads and the ratio arms equal. Instead of making the loop-resistance equal to kR as proposed in the paper, it would be necessary to make it equal to R, and to write $x=c+mt$ or its equivalent. The method suggested by Mr. Duddell of adding to the loop a resistance in series equal to R_0, would simplify the algebraic relations of the coefficients, giving $m=aR_0$, and $n=a^2R_0/b$, but would prevent the thermometer reading on the loop below 0° C. For many purposes this would not matter seriously. Neither the loop method, nor the three-lead arrangement subsequently proposed by the Author, could be used with a bridge-wire, as both involved plug or sliding contacts of negligible resistance. The three-lead arrangement also precluded the possibility of making an insulation-test of the leads at any time.—With regard to composite thermocouples, he did not think the arrangement would be useful except for very moderate ranges of temperature on account of the irrationality of the curves, which could not be satisfactorily represented by parabolic formulæ. He had himself employed a similar arrangement with resistance-thermometers, combining wires of different materials in series, such as platinum and gold, or platinum and iron, so as to eliminate the b term. This was a simple and fairly reliable way of rectifying the curve, since the resistance-variation of most metals followed approximately parabolic formulæ, and the sign of b was positive for many metals but negative for platinum. It was difficult, however, to find a suitable metal capable of standing high temperatures without change.

Dr. **Chree** pointed out that the resistance of the coils and shunt would vary somewhat with temperature unless made of some material with a negligible temperature coefficient. This would presumably necessitate some correction to the reading in the ordinary case. He asked Mr. Campbell whether a change in the total resistance of the wire due to temperature affected in any way the accuracy of the calculations.

Dr. R. T. **Glazebrook** expressed his interest in the paper and in the usefulness of the methods described for ordinary work, but pointed out that for the most accurate work it was advisable to work with the simplest possible form of apparatus and to apply the necessary corrections at the end. With regard to the question of compensation raised by Prof. Callendar, the mathematics given in the paper would embody his suggestion if x_0 were given a suitable value instead of being made equal to unity. The constantan wires referred to by Mr. Campbell had been tested over a large range of temperature, and it had been found that a constantan-iron couple gave a formula which was approximately linear.

Mr. Campbell, in reply to Prof. Callendar, said it would be possible to arrange the Rectifying-Loop with plug-contacts instead of sliders; with 25 ohms in the measuring arm, sliders could not cause serious error. In reply to Mr. Duddell's second question, he did not know if any of the other copper-nickel alloys gave better thermoelectric results (against iron) than constantan. In both of the rectifying methods the temperature-coefficient of the measuring arm caused no more error than in the ordinary method. In further answer to Prof. Callendar's and Mr. Duddell's remarks, he mentioned that for both methods he had indicated clearly in the paper that when once the ratio of the resistances of the zero-reading and the total loop (or the shunt) was found from α and β, then the absolute values of these resistances may be altered in any ratio desired. The bridge can thus have equal or unequal arms at will.

XLVI. *On the Stresses in the Earth's Crust before and after the Sinking of a Bore-hole. By* C. CHREE, *Sc.D., LL.D., F.R.S.*[*]

CONTENTS.

§ 1. Introduction.
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3. Notation.
4. Homogeneous isotropic gravitating sphere.
5. Composite gravitating sphere, crust, and nucleus.
6. Spheroidal rotating "Earth."
7. Solid vertical prism under gravity and pressure.
8. Hollow vertical cylinder under gravity and pressure.
9. Solid prism under pressure.
10. Hollow cylinder under pressure.
11. Solid sphere under pressure.
12. Hollow sphere under pressure.
13. Normal stress on small area of infinite plane.
14-20. Applications to the Earth.

§ 1. IN 'Nature,' October 20, 1904, p. 602, there appeared letters by Mr. G. Martin and the Hon. C. A. Parsons dealing with the size of the stresses in the Earth's crust and speculating as to what would happen if a hole were bored to a depth of twelve miles.

The letters indicate that some interest attaches to the problem, but its true character seems to stand in need of careful consideration. We know at present so little of the nature of the earth's material, even at such small depths as twelve miles, and have such scanty knowledge of the combined effects of high temperature and high pressure, that there are no data for making an *exact* calculation. Rocks as we know them at the earth's surface are not isotropic or even homogeneous solids, and they are not perfectly elastic for any considerable stresses[†]; but the crust of the earth seems to behave as a solid so far as can be inferred from earthquake-waves, and nothing that is positively known forbids the hypothesis that the material a few miles down is elastic for moderate changes of stress. Further, if any calculation is to be made of the internal stresses, there seems no alternative to the application of elastic solid theory. The following results based on the ordinary mathematical theory may thus be worth the consideration of engineers and others interested in the practical problem.

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[*] Read March 10, 1905.

[†] Japanese Earthquake Investigation Committee, Reports No. 17.

I propose first to present the results applicable to a series of mathematical problems, only one or two of which are absolutely novel, and then to consider what their bearing is on the problem now under discussion. Only the general character of the mathematical steps will be indicated.

§ 2. The two best known theories as to rupture, or more strictly as to the limits of application of the elastic solid theory, are:—

(i.) The maximum stress difference theory, according to which $\overline{S}$ the greatest value of S, where S is the *difference* between the algebraically greatest and least of the three principal stresses at a point, must not exceed a certain experimental limit;

(ii.) The greatest strain theory, according to which the limiting value attaches to the largest strain in any part of the structure, assuming that to be an extension.

Theory (i.) seems that least favourable to the permanence of the structure in such problems as those to be considered here, and attention is almost exclusively devoted to it. It has the recommendation that there seems considerable experimental evidence in favour of the view that it is the maximum stress difference on which depends the tendency to flow in solids under severe non-uniform pressure *.

Notation.

§ 3. In isotropic material m and n represent the two elastic constants in Thomson and Tait's notation. These are connected with Young's modulus E, and Poisson's ratio η, by the relations

$$m/1=n/(1-2\eta)=\mathrm{E}\div\{2(1+\eta)(1-2\eta)\}. \quad . \quad . \quad (1)$$

In applications of spherical coordinates, r, θ, ϕ, the displacement along the radius vector is denoted by u; in most of the cases treated here the three principal stresses are the radial $\widehat{rr}$, and the two orthogonal stresses $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$. Also $\widehat{\phi\phi}$ in the cases treated, with the exception of § 6, is equal to $\widehat{\theta\theta}$. The three principal strains are then du/dr, u/r, and u/r.

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* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 247, &c.

In cylindrical coordinates the displacements are u along r, the outward drawn perpendicular from the cylindrical axis, or axis of z, and w parallel to this axis. The principal stresses in the problems treated here are $\widehat{rr}$ and $\widehat{zz}$, parallel respectively to r and z, and $\widehat{\phi\phi}$ perpendicular to these two directions. The corresponding principal strains are du/dr, dw/dz, and u/r.

The dilatation Δ is given in spherical and cylindrical coordinates respectively, in the cases of symmetry here considered, by

$$\Delta = du/dr + 2u/r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\Delta = du/dr + u/r + dw/dz. \quad . \quad . \quad . \quad . \quad (3)$$

Homogeneous isotropic gravitating sphere.

§ 4. If a be the radius, ρ the density, and g 'gravity' at the surface, then, the centre being origin,

$$\left.\begin{aligned} u &= \frac{g\rho r}{10a(m+n)}\left\{r^2 - a^2\frac{5m+n}{3m-n}\right\}, \\ \widehat{rr} &= -g\rho(a^2-r^2)(5m+n) \div \{10a(m+n)\}, \\ \widehat{\theta\theta} &= -g\rho\{a^2(5m+n) - r^2(5m-3n)\} \div \{10a(m+n)\}, \\ \mathrm{S} &= 2g\rho r^2 n \div \{5a(m+n)\}, \\ \bar{\mathrm{S}}\ (\text{at surface}) &= 2g\rho an \div \{5(m+n)\} \equiv \tfrac{1}{5}g\rho a(1-2\eta)/(1-\eta). \end{aligned}\right\} (4)^*$$

At points near the surface for which

$$h \equiv a - r \text{ is small,}$$

we have, retaining only the lowest power of h, as first approximations,

$$\left.\begin{aligned} \widehat{rr} &= -g\rho h(3-\eta) \div \{5(1-\eta)\}, \\ \widehat{\theta\theta} &= -\mathrm{S} = -g\rho a(1-2\eta) \div \{5(1-\eta)\}. \end{aligned}\right\} \quad . \quad . \quad (5)$$

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* Camb. Phil. Soc. Trans. vol. xiv. p. 281.

In the special case when the material is incompressible,

i. e. $n/m = 0, \quad \text{or} \quad \eta = 0{\cdot}5,$

we have throughout the mass of the sphere

$$\left.\begin{aligned} u &= 0, \\ \widehat{rr} &= \widehat{\theta\theta} = -g\rho(a^2-r^2)/2a, \\ \mathrm{S} &= \overline{\mathrm{S}} = 0; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

and close to the surface

$$\widehat{rr} = \widehat{\theta\theta} = -g\rho h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 5. Gravitating spherical "Earth," consisting of a core of radius b, density $\rho+\rho'$ and elastic constants m', n', and of a crust or layer of density ρ, and elastic constants m, n, resting on the core and bounded externally by a spherical surface of radius a.

The expressions for the displacements and stresses are as follows:—

In the core:

$$\left.\begin{aligned} u &= \tfrac{1}{3}r\mathrm{A} + \tfrac{2}{15}\pi\frac{(\rho+\rho')^2r^3}{m'+n'}, \\ \widehat{rr} &= \tfrac{1}{3}(3m'-n')\mathrm{A} + \frac{5m'+n'}{15(m'+n')}2\pi(\rho+\rho')^2r^2, \\ \widehat{\theta\theta} &= \tfrac{1}{3}(3m'-n')\mathrm{A} + \frac{5m'-3n'}{15(m'+n')}2\pi(\rho+\rho')^2r^2. \end{aligned}\right\} \quad . \quad . \quad (8)$$

In the layer:

$$\left.\begin{aligned} u &= \tfrac{1}{3}r\mathrm{B} + r^{-2}\mathrm{C} + \frac{2\pi\rho^2r^3}{15(m+n)} - \frac{2\pi\rho\rho'b^3}{3(m+n)}, \\ \widehat{rr} &= \tfrac{1}{3}(3m-n)\,\mathrm{B} - 4nr^{-3}\mathrm{C} + \frac{5m+n}{15(m+n)}2\pi\rho^2r^2 - \frac{m-n}{m+n}\frac{4}{3}\frac{\pi\rho\rho'b^3}{r}, \\ \widehat{\theta\theta} &= \tfrac{1}{3}(3m-n)\,\mathrm{B} + 2nr^{-3}\mathrm{C} + \frac{5m-3n}{15(m+n)}2\pi\rho^2r^2 - \frac{m}{m+n}\frac{4}{3}\frac{\pi\rho\rho'b^3}{r}. \end{aligned}\right\} \quad (9)$$

Here A, B, C are three arbitrary constants to be determined by the surface conditions, and the gravitational force between two masses μ and μ' at distance R is taken as $(\mu\mu'/\mathrm{R}^2) \times 1$.

We shall suppose the outer surface $r=a$ to be free from force. If uniform normal pressure acts its effects are most easily obtained separately.

Supposing the outer surface $r=a$ free from force, $\widehat{rr}$ must vanish over it, whence

$$\tfrac{1}{3}(3m-n)\mathrm{B}-4na^{-3}\mathrm{C}=-\frac{5m+n}{15(m+n)}2\pi\rho^2a^2+\frac{m-n}{m+n}\frac{4}{3}\pi\rho\rho'\frac{b^3}{a}. \quad (10)$$

Over the common surface $r=b$, the radial displacements must be the same for the core and layer, whence

$$\mathrm{B}+3b^{-3}\mathrm{C}=\mathrm{A}+\frac{2}{5}\frac{\pi(\rho+\rho')^2b^2}{m'+n'}-\frac{2}{5}\frac{\pi\rho^2b^2}{m+n}+\frac{2\pi\rho\rho'b^2}{m+n}. \quad (11)$$

Finally the values of $\widehat{rr}$ in the core and layer must be equal over the common surface, and so

$$\tfrac{1}{3}(3m-n)\mathrm{B}-4nb^{-3}\mathrm{C}=\tfrac{1}{3}(3m'-n')\mathrm{A}+\frac{5m'+n'}{15(m'+n')}2\pi(\rho+\rho')^2b^2$$
$$-\frac{5m+n}{15(m+n)}2\pi\rho^2b^2+\frac{m-n}{m+n}\frac{4}{3}\pi\rho\rho'b^2. \quad (12)$$

These three equations determine A, B, and C without ambiguity, and there is no difficulty, except in the length of the expressions, in obtaining the solution for the general case.

As we shall see, however, physical interest is mainly if not entirely restricted to the case when the material of the core is incompressible or very nearly so, and I shall thus limit myself to the case when n'/m' is negligible, n' being finite.

From (12) we see that when n'/m' is negligible A must vanish. For n' being finite, m' must be infinite, and so the one term $\tfrac{1}{3}(3m'-n')\mathrm{A}$ unless A vanished would become infinite, whilst the other terms in the equation would remain finite. We must thus treat A as negligible, except when multiplied by m', and regard (12) as simply determining the (finite) value of $(3m'-n')\mathrm{A}/3$.

Our surface equations thus reduce to two, viz. (10) and

$$\mathrm{B}+3b^{-3}\mathrm{C}=-\frac{2}{5}\frac{\pi\rho^2b^2}{m+n}+\frac{2\pi\rho\rho'b^2}{m+n}, \quad (11')$$

this being the form taken by (11) when A is negligible and m' is infinite.

The values of B and C may be at once written down from (10) and (11′). Confining our attention to the case where the thickness $a-b=t$ of the crust is so small that $(t/a)^2$ is negligible, we find

$$\left.\begin{aligned} \mathrm{B} &= 2\pi\rho a^2\{-\rho+2\rho'(1-3t/a)\}\div\{3(m+n)\}, \\ \mathrm{C} &= 2\pi\rho a^5\{\tfrac{2}{5}\rho+\rho'(1-3t/a)\}\div\{9(m+n)\}. \end{aligned}\right\} \quad . \qquad (13)$$

Substituting these values of B and C in (9), then putting $r=a-h$, and neglecting $(h/a)^2$, we find after reduction

$$\left.\begin{aligned} \widehat{rr} &= -\tfrac{4}{3}\pi\rho ah\{\rho+\rho'-3(t/a)\rho'\}, \\ \widehat{\theta\theta} &= -\tfrac{4}{3}\pi\rho ah\frac{m-n}{m+n}\left\{\rho+\rho'-3(t/a)\rho'\right\}. \end{aligned}\right\} \quad . \qquad (14)$$

But if g be the acceleration of gravity at the surface

$$\begin{aligned} g &= (4\pi/3)\{\rho a^3+\rho' b^3\}/a^2, \\ &= \tfrac{4}{3}\pi a\{\rho+\rho'-(3t/a)\rho'\}, \end{aligned}$$

when $(t/a)^2$ is neglected.

Thus we have in the crust :

$$\left.\begin{aligned} \widehat{rr} &= -g\rho h, \\ \widehat{\theta\theta} &= -g\rho h(m-n)/(m+n) \equiv -g\rho h\eta/(1-\eta), \\ \mathrm{S} &= g\rho h(1-2\eta)/(1-\eta). \end{aligned}\right\} \quad . \qquad (15)$$

§ 6. Slightly spheroidal, homogeneous, gravitating and rotating earth.

The surface values of the stresses, which are by no means very complicated, will suffice for our immediate object.

Let $2a$, $2c$ represent the equatorial and polar diameters, r the perpendicular from any point on the polar axis, p the perpendicular from the centre on the tangent plane, ρ the density, G the gravitational force between two unit masses at unit distance, and ω the angular velocity.

At the surface the principal stresses are $\widehat{nn}$ along the normal, $\widehat{tt}$ along the tangent in the meridian plane, and $\widehat{\phi\phi}$ perpendicular to the meridian plane.

Neglecting terms of order $(1-c^2/a^2)^2$ we find *

$$\left.\begin{aligned}
\widehat{nn} &= 0,\\
\widehat{tt} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)}\left\{\frac{c^2}{p^2}(1-2\eta)+\frac{a^2-c^2}{a^2}\frac{1-\eta-4\eta^2}{7+5\eta}\right\}\\
&\quad+\frac{\omega^2\rho a^2(c/p)^2}{5(1-\eta)(7+5\eta)^2}\left\{(7+5\eta)(3-6\eta-5\eta^2)+4\frac{a^2-c^2}{a^2}(1+\eta)(6-5\eta-5\eta^2)\right\},\\
\widehat{\psi\psi} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)}\left\{1-2\eta+\frac{a^2-c^2}{a^2}\left(1-3\frac{r^2}{a^2}\right)\frac{1-\eta-4\eta^2}{7+5\eta}\right\}\\
&\quad+\frac{\omega^2\rho a^2}{5(1-\eta)(7+5\eta)^2}\Big[(7+5\eta)(3-6\eta-5\eta^2)+4\frac{a^2-c^2}{a^2}(1+\eta)(6-5\eta-5\eta^2)\\
&\qquad+\frac{r^2}{a^2}\left\{5(7+5\eta)(1-\eta^2)-\frac{a^2-c^2}{a^2}(30-4\eta-82\eta^2-41\eta^3)\right\}\Big].
\end{aligned}\right\}(16)$$

When $c=a$, $4\pi bGa/3=g$ the gravitational acceleration at the surface. Thus in a perfect sphere the contributions from the gravitational terms to $\widehat{tt}$ and $\widehat{\phi\phi}$ are equal, and agree with the value $-g\rho a(1-2\eta)\div\{5(1-\eta)\}$ given in (5) of § 4 for $\widehat{\theta\theta}$; they vanish when the material is incompressible.

The contributions from the "centrifugal force" terms to the surface stresses are in a perfect sphere, λ denoting the latitude,

$$\left.\begin{aligned}
\widehat{tt} &= \theta\theta = \omega^2\rho a^2(3-6\eta-5\eta^2)\div\{5(1-\eta)(7+5\eta)\},\\
\widehat{\phi\phi} &= \frac{\omega^2\rho a^2}{5(1-\eta)(7+5\eta)}\left\{3-6\eta-5\eta^2+5\cos^2\lambda(1-\eta^2)\right\}.
\end{aligned}\right\}(17)$$

The value of $\widehat{\theta\theta}$ is thus constant over the surface. Its sign depends on that of $3-6\eta-5\eta^2$. This expression vanishes when $\eta=\cdot 3798$, or $n/m=\cdot 2404$. It also vanishes when $m/n=\cdot 2404$, but this latter value is physically impossible. Thus $\widehat{\theta\theta}$ is positive or negative, *i.e.* a tension or a pressure, according as η is less or greater than 0·38. $\widehat{\phi\phi}$ is always positive in the equator; at the poles it is equal to $\widehat{\theta\theta}$.

When $\widehat{\phi\phi}$ is positive and $\widehat{\theta\theta}$ negative (which implies η being $>\cdot 3798$) $\widehat{\phi\phi}-\widehat{\theta\theta}$ is the maximum stress difference, and the greatest value occurs in the equator where

$$S=\omega^2\rho a^2(1+\eta)/(7+5\eta). \quad . \quad . \quad . \quad . \quad (18)$$

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* Roy. Soc. Proc. vol. lviii. eqns. (15), p. 43, and eqns. (23) to (26), pp. 44 and 45.

This increases slightly with η, varying from $0{\cdot}155\omega^2\rho a^2$ when $\eta = {\cdot}3798$ to $0{\cdot}158\omega^2\rho a^2$ when $\eta = 0{\cdot}5$.

Supposing the earth an incompressible sphere in which $\rho = 5{\cdot}5$, $a = 3963$ miles, and $\omega^2 a = g/293$, we find for the maximum *surface* value of S

$$\overline{\mathrm{S}} = 12{\cdot}0 \text{ tons weight per square inch.} \quad . \quad . \quad (19)$$

Except for incompressible material, the principal terms depending on ω^2 in $\widehat{tt}$ and $\widehat{\phi\phi}$ are small compared with the principal gravitational terms; and when the material is incompressible the secondary terms depending on ω^2 which contain $a^2 - c^2$ are small compared with the secondary gravitational terms. Thus we may in general neglect the secondary terms in ω^2 for a first approximation. We may also neglect the differences between a, c, and p in all terms depending on ω or containing $a^2 - c^2$ as a factor, and may replace $4\pi \mathrm{G}\rho a/3$ by g, where g is the mean value of gravity over the surface.

Doing so, we find as first approximations to the principal stresses in the surface of an incompressible nearly spherical spheroid

$$\left.\begin{aligned} \widehat{nn} &= 0, \\ \widehat{tt} &= \frac{1}{19} g\rho a\left(\frac{4}{5}\frac{a-c}{a} - \frac{\omega^2 a}{g}\right), \\ \widehat{\phi\phi} &= \frac{1}{19} g\rho a\left(\frac{4}{5}\frac{a-c}{a} - \frac{\omega^2 a}{g}\right)\left(1 - 3\,\frac{r^2}{a^2}\right). \end{aligned}\right\} \quad . \quad (20)$$

Thus $\widehat{tt}$ and $\widehat{\phi\phi}$ would both vanish all over the surface if

$$\frac{4}{5}\frac{a-c}{a} = \frac{\omega^2 a}{g}.$$

In the actual earth

$$\omega^2 a/g = 1/290 \quad \text{approximately.}$$

Bessel's value for $(a-c)/a$ is 1/299, while Clarke's is 1/293·5. Taking the former value, and employing as before λ to denote the latitude, we find in the surface

$$\left.\begin{aligned} \widehat{tt} &= -0{\cdot}90 \quad . \quad . \quad . \quad \text{tons weight per square inch} \\ \widehat{\phi\phi} &= -0{\cdot}90(1 - 3\cos^2\lambda) \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ \overline{\mathrm{S}} &= 2{\cdot}7 \quad . \quad . \quad . \quad . \quad . \quad \text{,,} \quad \text{,,} \quad \text{,,} \end{aligned}\right\} (21)$$

The following particular values of $\widehat{\phi\phi}$ (all in tons weight per square inch) should be noted :—

Equator.	Lat. 35° 16′.	Lat. 54° 44′.	Poles.
+1·8	+0·9	0·0	−0·9

The stress in the meridian plane is always a pressure when the material is incompressible ; but that perpendicular to the meridian is a tension in latitudes below $54\frac{3}{4}°$. The numerical values of these surface stresses would be reduced by assigning to the earth's compression $\overline{a-c}/a$ any value, such as Clarke's, which is larger than Bessel's.

§ 7. Right vertical prism of density ρ, acted on by gravity, the prismatic surface being exposed to normal pressure $p-\mathrm{C}z$, where p and C are constants, and the upper plane surface $z=h$ being free from force.

To give definiteness to the problem we shall suppose the vertical displacement nil at the C.G. of the base. The shape of the cross section does not matter.

The solution is obtainable from one on p. 545 of the Phil. Mag. for November 1901, by suitably altering the notation.

Taking rectangular axes of x, y, z, the axis of z being drawn vertically upwards, and the origin being at the C.G. of the base of the prism, the displacements α, β, γ are given by

$$\left.\begin{aligned}&\alpha/x=\beta/y=(1/\mathrm{E})\{\eta g\rho(h-z)-(1-\eta)(p-\mathrm{C}z)\},\\&\gamma=-(z/\mathrm{E})\{g\rho(h-\tfrac{1}{2}z)-\eta(2p-\mathrm{C}z)\}+(1/2\mathrm{E})(x^2+y^2)\{g\rho\eta-(1-\eta)\mathrm{C}\}.\end{aligned}\right\}\quad(22)$$

The principal stresses and the stress-difference are given by

$$\left.\begin{aligned}\widehat{xx}&=\widehat{yy}=-(p-\mathrm{C}z)=-p_{\mathrm{H}},\\\widehat{zz}&=-g\rho(h-z)=-p_{\mathrm{V}},\\\mathrm{S}&=p_{\mathrm{V}}\sim p_{\mathrm{H}},\end{aligned}\right\}\quad . \; . \; . \quad(23)$$

where p_{V} and p_{H} are the pressures at the depths in question in the vertical and horizontal directions respectively.

Assuming $p-\mathrm{C}z$ everywhere positive, the stresses are all numerically greatest over the base $z=0$; also

$$\mathrm{S}_{z=0}=g\rho h\sim p \quad . \; . \; . \; . \; . \; . \; . \quad (24)$$

§ 8. Hollow, vertical, circular cylinder of density ρ, acted on by gravity, the outer cylindrical surface $r=a$ being exposed to normal pressure $p-\mathrm{C}z$, where p and C are constants, the inner cylindrical surface $r=a'$ and the upper plane surface $z=h$ being free from force.

The solution is obtainable from one on p. 550 of the Phil. Mag. for November 1901 by suitably altering the notation. It is, in cylindrical coordinates,

$$\left.\begin{aligned} u &= (r/\mathrm{E})\left[g\rho\eta(h-z)-(1-\eta)\frac{a^2(p-\mathrm{C}z)}{a^2-a'^2}\right]-\frac{a^2a'^2(p-\mathrm{C}z)}{2nr(a^2-a'^2)}, \\ w &= (z/\mathrm{E})\left[-g\rho h+\frac{2\eta a^2p}{a^2-a'^2}\right]+\tfrac{1}{2}\frac{g\rho}{\mathrm{E}}(z^2+\eta r^2) \\ &\qquad -\frac{\mathrm{C}a^2}{2(a^2-a'^2)}\left\{\frac{(1-\eta)r^2+2\eta z^2}{\mathrm{E}}+\frac{a'^2}{n}\log(r/r_1)\right\}. \end{aligned}\right\} \quad (25)$$

Here r_1 is a constant which can be determined so that w has any assigned value at any given point.

For the stresses we have

$$\left.\begin{aligned} \widehat{rr} &= -\frac{a^2}{a^2-a'^2}\left(1-\frac{a'^2}{r^2}\right)\left(p-\mathrm{C}z\right), \\ \widehat{\phi\phi} &= -\frac{a^2}{a^2-a'^2}\left(1+\frac{a'^2}{r^2}\right)\left(p-\mathrm{C}z\right), \\ \widehat{zz} &= -g\rho(h-z). \end{aligned}\right\} \quad . \quad . \quad (26)$$

Assuming p, C, and $p-\mathrm{C}h$ positive, the stresses are all negative and numerically greatest at the base of the cylinder, where z vanishes; and the greatest value of the maximum stress difference—which is met with over the inner surface—is the greater of the two quantities :—

$$\left.\begin{aligned} \overline{\mathrm{S}}_1 &= 2pa^2/(a^2-a'^2), \\ \overline{\mathrm{S}}_2 &= g\rho h. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

When a'/a is very small, then very approximately

$$\overline{\mathrm{S}}_1 = 2p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

§ 9. Right prism exposed to uniform pressures **P** and p over its flat ends and prismatic surface respectively. The axis of z being along the prismatic axis, the stresses and stress-difference are given by

$$\left.\begin{aligned} \widehat{xx} &= \widehat{yy} = -p, \\ \widehat{zz} &= -\mathrm{P}, \\ \mathrm{S} &= \overline{\mathrm{S}} = \mathrm{P} \sim p. \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (29)$$

§ 10. Hollow right circular cylinder, outer radius a, inner a', exposed to pressures p and P over its external cylindrical surface and flat ends respectively.

The stresses are

$$\left.\begin{aligned} \widehat{rr} &= -pa^2(1-a'^2/r^2)/(a^2-a'^2), \\ \widehat{\phi\phi} &= -pa^2(1+a'^2/r^2)/(a^2-a'^2), \\ \widehat{zz} &= -\mathrm{P}. \end{aligned}\right\} \quad . \; . \; . \quad (30)$$

At a given distance r from the axis the maximum stress difference may according to circumstances be any one of the three following :—

$$\left.\begin{aligned} \mathrm{S}_1 &= \widehat{rr} - \widehat{\phi\phi} = 2pa^2a'^2 \div r^2(a^2-a'^2), \\ \mathrm{S}_2 &= \widehat{rr} - \widehat{zz} = \mathrm{P} - pa'^2(1-a'^2/r^2)/(a^2-a'^2), \\ \mathrm{S}_3 &= \widehat{zz} - \widehat{\phi\phi} = -\mathrm{P} + pa^2(1+a'^2/r^2)/(a^2-a'^2). \end{aligned}\right\}. \quad (31)$$

But for the greatest value of S there are only two options, both referring to the inner surface of the cylinder, viz. :—

$$\left.\begin{aligned} \overline{\mathrm{S}}_1 &= 2pa^2/(a^2-a'^2), \\ \overline{\mathrm{S}}_2 &= \mathrm{P}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (32)$$

When a'/a is very small, a close approximation is

$$\overline{\mathrm{S}}_1 = 2p. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

§ 11. Solid sphere of radius a, exposed to uniform surface-pressure p.

The solution is

$$\left.\begin{aligned} u/r &= du/dr = -p/(3m-n), \\ \widehat{rr} &= \widehat{\theta\theta} = -p, \\ \mathrm{S} &= 0. \end{aligned}\right\} \quad . \; . \; . \quad (34)$$

§ 12. Spherical shell outer radius a, inner a', exposed to uniform pressure p over the outer surface.

The stresses and stress-difference are given by

$$\left.\begin{aligned} \widehat{rr} &= -pa^3(1-a'^3/r^3)/(a^3-a'^3), \\ \widehat{\theta\theta} &= -pa^3(1+a'^3/2r^3)/(a^3-a'^3), \\ \mathrm{S} &= \tfrac{3}{2}\, p(aa')^3 \div \{r^3(a^3-a'^3)\}, \\ \overline{\mathrm{S}} \text{ (at inner surface)} &= \tfrac{3}{2} pa^3/(a^3-a'^3). \end{aligned}\right\} \quad . \quad . \quad (35)$$

When a'/a is very small a closely approximate value is

$$\overline{\mathrm{S}} = 3p/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

§ 13. Material bounded by an infinite plane,—on one side of which it extends to infinity,—acted on by a surface-tension uniformly distributed over a circular area of radius a'.

For clearness suppose the material to be on the lower side of the plane $z=0$, the tension T, per unit area, acting vertically upwards. Let z be measured positively downwards from an origin at the centre of the stressed area, and let R denote the distance between an element $d\sigma$ of the stressed area and a point r, z in the solid, r being the perpendicular on the axis of z.

The solution, as obtained by Boussinesq and Cerruti *, is

$$\left.\begin{aligned} u &= (\mathrm{T}/4\pi n)\frac{d}{dr}\left\{\frac{d}{dz}\iint \mathrm{R}\,d\sigma + (1-2\eta)\iint \log(z+\mathrm{R})\,d\sigma\right\}, \\ w &= (\mathrm{T}/4\pi n)\left\{\frac{d^2}{dz^2}\iint \mathrm{R}\,d\sigma - (3-2\eta)\iint \frac{1}{\mathrm{R}}\,d\sigma\right\}, \\ \Delta &= -(\mathrm{T}/2\pi n)(1-2\eta)\frac{d}{dz}\iint \frac{1}{\mathrm{R}}\,d\sigma. \end{aligned}\right\} . \quad (37)$$

The integrals extend over the whole of the stressed area, *i. e.* over the area enclosed by the circle $r=a'$.

The above integrals in their general form are somewhat unmanageable. At a point whose distance from the stressed area is such that a'/R is always small, closely approximate values for the displacements are

$$\left.\begin{aligned} u &= -\frac{\mathrm{T}a'^2 r}{4n\mathrm{R}}\left\{\frac{z}{\mathrm{R}^2} - \frac{(1-2\eta)}{\mathrm{R}+z}\right\}, \\ w &= -\frac{\mathrm{T}a'^2}{4n\mathrm{R}}\left\{\left(\frac{z}{\mathrm{R}}\right)^2 + 2(1-\eta)\right\} \end{aligned}\right\}; \quad . \quad . \quad (38)$$

where R now denotes $\sqrt{r^2+z^2}$.

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* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 1492, equations (ix.).

Along a given radius the above displacements vary inversely as the distance from the origin, *i. e.* from the centre of the stressed area; consequently the strains and stresses vary inversely as the square of this distance. For instance, at a point at a distance z vertically below the centre of the stressed area, a'/z being small,

$$\left.\begin{aligned} \widehat{rr} = \widehat{\phi\phi} &= -(1-2\eta)\mathrm{T}a'^2/(4z^2), \\ \widehat{zz} &= 3\mathrm{T}a'^2/(2z^2) \end{aligned}\right\}; \quad . \quad . \quad . \quad (39)$$

while in the plane of the loaded area, but outside it, a'/r being small,

$$\left.\begin{aligned} -\widehat{rr} = \widehat{\phi\phi} &= (1-2\eta)\mathrm{T}a'^2/2r^2, \\ \widehat{zz} &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (40)$$

Thus the effects diminish very rapidly as the distance from the loaded area increases, and so far as rupture is concerned interest centres in the material close to the loaded area. The determination of what happens close to the loaded area is in general rather a delicate operation, especially at points situated close to its boundary.

At any point on the axis of z

$$\mathrm{R}^2 = z^2 + r'^2,$$

where r' is the distance of the element $d\sigma$ from the centre of the loaded area, and so

$$d\mathrm{R}/dz = z/\mathrm{R}.$$

Thus from the last of equations (37)

$$\begin{aligned} \frac{2\pi n\Delta}{(1-2\eta)\mathrm{T}} = \iint \frac{z}{\mathrm{R}^3}\, d\sigma = \int_0^{a'} \frac{z \,.\, 2\pi r' dr'}{(z^2+r'^2)^{\frac{3}{2}}} \\ = 2\pi\{1 - z/(a'^2+z^2)^{\frac{1}{2}}\}. \end{aligned}$$

Thus in the limit when $z=0$, *i. e.* at the centre of the loaded area,

$$\Delta = \mathrm{T}(1-2\eta)/n. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Now by symmetry it is clear that $\widehat{rr}$, $\widehat{\phi\phi}$, $\widehat{zz}$ are the principal stresses at the centre, and also that

$$\widehat{rr} = \widehat{\phi\phi}.$$

Thus

$$\begin{aligned} 2\widehat{rr} + \widehat{zz} &= \widehat{rr} + \widehat{\phi\phi} + \widehat{zz} \\ &= (3m-n)\Delta = 2(1+\eta)\mathrm{T}. \end{aligned}$$

But by the surface conditions

$$\widehat{zz}=\mathrm{T}.$$

Hence and

$$\left.\begin{aligned} \widehat{rr}&=(1+2\eta)\mathrm{T}/2, \\ \mathrm{S}=\widehat{zz}-\widehat{rr}&=(1-2\eta)\mathrm{T}/2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

As this method of obtaining the stresses is very artificial, I may add that I have deduced directly from the second of equations (37) for points in the axis of z

$$w=(\mathrm{T}/2n)\left[(1-2\eta)z-2(1-\eta)(a'^2+z^2)^{\frac{1}{2}}+z^2(a'^2+z^2)^{-\frac{1}{2}}\right].$$

Thus in the limit when $z=0$, *i. e.* at the centre of the loaded area,

$$dw/dz=\mathrm{T}(1-2\eta)/2n.$$

I have also succeeded in deducing from the first of equations (37) as limiting values when $z=0$

$$\alpha/x=\beta/y=u/r=du/dr=\mathrm{T}(1-2\eta)/4n.$$

These values for the strains are in harmony with the values given above for the stresses.

In the case of tension or pressure over a very small area it is practically immaterial whether the bounding surface is a plane or a sphere of large radius. Thus in practice we can regard (15) and (37) as applicable to an elastic "Earth" so long as h is a very small fraction of the radius. Superposing the two stress systems, and replacing T by $g\,\rho\,h$, we have a small area $\pi a'^2$ at a depth h from the surface free from pressure, whilst other elements at the same depth experience a pressure $g\rho h$. The solution thus answers closely to the conditions existing below a bore-hole of depth h. At the base of the hole, at its centre, we find combining (15) and (42), using the notation of (15),

$$\left.\begin{aligned} \widehat{rr}&=0, \\ \mathrm{S}=\widehat{\theta\theta}&=g\rho h\frac{(1-2\eta)}{1-\eta}\,\frac{1+\eta}{2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

This value of S it will be noticed bears to that in (15) the ratio

$$1+\eta : 2.$$

Applications to the Earth.

§ 14. The first question to consider is the state of matters prior to the existence of the bore-hole. The Earth is spheroidal and rotates about its axis of figure, circumstances which introduce variety into the conditions in different latitudes. Results such as (17), (18), and (19) show that an elastic sphere of the Earth's size rotating in a day must suffer stresses from the "centrifugal forces" which are very considerable even for material such as steel. The results (20) and (21) show, however, that the stresses due to centrifugal force in a homogeneous Earth are at least in large measure neutralized by the compensating action of the spheroidal form on the gravitational stresses. In our ignorance of the distribution of density and elasticity throughout the Earth, there is necessarily uncertainty as to the extent of this compensating action. The compensation may be less exact than according to (21), but it is at least as likely to be more exact. Further, whilst the pure centrifugal stresses are very considerable at the surface, their rate of variation with the depth is small. There is thus every reason to believe that whatever may be the combined effect of rotation and ellipticity, it will not produce throughout a bore-hole only ten or twelve miles deep effects differing in any essential respect from those observed in ordinary mines. Owing to centrifugal forces there may be a slightly greater tendency for a bore-hole to collapse in a north-south than in an east-west direction. Rotation may slightly facilitate collapse ; but everything points to the conclusion that its effects, whether direct or indirect, are only of secondary importance.

§ 15. A more serious source of uncertainty arises in connexion with the value (5) of the horizontal stress $\widehat{\theta\theta}$ in a homogeneous gravitating sphere. If, for example, we suppose η equal 0·25, which is approximately true of glass or iron, the value of $-\widehat{\theta\theta}$ is $(2/15)g\rho a$; this means a pressure answering to the weight of a column of density ρ some 500 miles high. Near the Earth's surface this enormous horizontal pressure would exist in company with only a relatively small vertical pressure ; and under such conditions no known material would continue an elastic solid. There is further the observed fact that near the Earth's surface large horizontal stresses are normally at least non-existent. If they occurred, mines could not be constructed. No theory, in short, of the Earth's condition can well be entertained which is incompatible with horizontal stresses remaining small at moderate depths.

The most obvious way out of the difficulty is to assume the homogeneous "Earth" incompressible. We then, as appears from (6) and (7), have the horizontal stresses vanishing at the surface equally with the vertical stresses. To this hypothesis there is, however, the objection that no known material is incompressible, and that the materials of which the Earth's crust is composed do not, so far as is known, show any close approach to such a condition.

The hypothesis § 5, which limits the incompressibility to the deep-seated material, is less open to criticism. At great depths, according to any view yet suggested, the material must be exposed to very severe pressure; and it is difficult to imagine material exposed to pressure of hundreds of tons weight on the square inch being anything but nearly incompressible under variations of that pressure. The hypothesis of an incompressible nucleus and compressible crust is of course at best only an approximation to the truth. A theory which treated the material as varying continuously with the depth, or as consisting of a large number of homogeneous layers, would presumably be more exact. Its complication, however, would be great; and in the absence of data the simplest consistent hypothesis seems the best.

§ 16. The above considerations indicate that in the absence of a bore-hole the stresses at depths of a few miles in reasonably homogeneous strata are probably not widely different from those appearing in (15). According to this equation the pressure on a horizontal plane equals the weight of the material between it and the surface. This pressure, in short, is the same as if the superincumbent material were liquid. The result implies that the superincumbent material is so much dead weight, and does not act to an appreciable extent as a protecting arch. In reality, some slight action of this kind is likely to exist. In not a few cases, the bending visible in rocks is suggestive of large horizontal thrusts. The condition, however, of the material when the bending occurred may have been plastic.

An idea of the probable diminution in the vertical pressure due to the horizontal thrust in superincumbent material may be derived from (5). If in this case we suppose η equal 1/4 the vertical pressure, $-\widehat{rr}$, is only 11/15 of its value when the material is incompressible; but the horizontal or arching pressures to which this reduction is due (*i. e.* the value of $-\widehat{\theta\theta}$ in (5)) if we suppose the density to be 5·5, amounts at the surface to nearly 3000 tons weight on the square inch. The natural inference is that whilst $g\rho h$ is a maximum estimate for the vertical pressure on a horizontal plane, it is unlikely to be many per cent. in error.

As regards horizontal pressure, the largest value given by (15) for $-\widehat{\theta\theta}$ is $g\rho h$, answering to $\eta=0{\cdot}5$. It follows from our previous considerations that if shallow borings show no considerable horizontal pressure, the value of this pressure at depth h is unlikely to be much in excess of $g\rho h$. On the other hand, according to (15), $-\widehat{\theta\theta}$ vanishes when $\eta=0$, and is only $g\rho h/3$ when $\eta=\frac{1}{4}$. It is thus by no means improbable that the horizontal pressure may be very considerably less than $g\rho h$, and so less than the vertical pressure.

Noticing that $\widehat{rr}$ in (15) is really a vertical stress, it is obvious that the stress systems (15) and (23) are of the same type. The system (23) is the more general, in so far as it does not assume any relationship between the magnitudes of the vertical and horizontal pressures, and does not make the latter necessarily vanish at the upper surface. The material in the Earth's crust thus presents the same elastic conditions as a vertical prism of similar density acted on by gravity, provided the latter be exposed to suitable horizontal pressures whose intensity depends only on the level.

§ 17. As a numerical example, let us take Mr. Martin's case, where h is twelve miles and ρ thrice the density of water. Employing these values in (15) or (23), we find for the vertical pressure at this depth

$$-\widehat{zz}=36{\cdot}8 \text{ tons weight per square inch.}$$

Mr. Martin himself speaks of 440 tons, which suggests the omission of 12 in some divisor. Mr. Parsons gives 40 tons as a rough approximation, which agrees substantially with the above estimate. If the prism existed as an isolated pillar, it would, neglecting atmospheric pressures, be free from horizontal pressure, and the maximum stress-difference at its base would be simply the above value of $-\widehat{zz}$, or 36·8 tons weight on the square inch. But when the prism forms part of the Earth's crust, horizontal pressures exist; and so long as they are less than $g\rho h$, the larger they are the smaller is S. If, for example, we supposed the horizontal pressure to be given by $-\widehat{\theta\theta}$ in (15) with $\eta=1/4$ we should find S reduced to 24·4 tons weight per square inch.

Comparing (15) and (23) with (29), we see that our reasoning points to the conclusion that the material at any given depth in the crust, like the material in a gravitating prism, is under the same condition as a short non-gravitating prism over whose flat ends and curved surface there act *uniform* pressures, equal respectively to the vertical and horizontal pressures actually existent at the point in the crust. Thus experiments on a solid prism might supply definite information as to the elastic state of the Earth's crust if rocks were available of the same composition as the more deep-seated material, and the temperature at which the experiments were made was sufficiently high.

Effects of a Bore-hole.

§ 18. The existence of a bore-hole may profoundly influence the conditions in its immediate neighbourhood; but clearly it will not produce an appreciable effect at distances from the hole which are a large multiple of its diameter. Over the cylindrical surface of the hole the normal stress must vanish, *i. e.* the horizontal pressure must be *nil*, supposing the hole vertical. Thus if we suppose an imaginary cylindrical surface coaxial with the bore-hole, but of relatively large diameter, the material within it will be practically in the state of the hollow cylinder of § 8 when we suppose a'/a very small. From this conclusion we must exclude the material immediately surrounding the extreme foot of the hole, whose condition will be somewhat modified. Excluding this very limited volume, we infer from (27) and (28) that the maximum stress-difference S at depth h, which was $g\rho h \sim p$ before the hole was bored, is, subsequent to the boring, the greater of the two quantities $2p$ and $g\rho h$. Here p denotes the horizontal pressure at depth h at a distance from the hole.

If p vanishes, *i. e.* if the material is in the condition of an isolated pillar, the presence of the bore-hole makes no difference.

On the hypothesis of a compressible crust on an incompressible nucleus, after the hole is formed S equals $g\rho h$ so long as η does not exceed 1/3; thereafter it increases with η to a maximum of $2g\rho h$ when η equals 0·5. Particulars on this hypothesis are as follows:—

Value of $S/g\rho h$.

$\eta =$	0	0·2	·25	·3	$\cdot\dot{3}$	·4	·45	·5
Before hole made	1	0·75	$0{\cdot}\dot{6}$	0·57	0·5	$0{\cdot}\dot{3}$	0·18	·00
After „ „	1	1	1	1	1	$1{\cdot}\dot{3}$	1·64	2·00

When a bore-hole is formed the engineer may thus anticipate over its surface a stress-difference varying from p to $2p$ according to circumstances, where p represents the hydrostatic pressure at the same depth in a liquid of the same mean density as the superincumbent material.

From the identity in type of the results in §§ 8 and 10, the probable behaviour of the tube-walls may be deduced from experiments in which a circular cylinder of the material with a coaxial hole of small bore is exposed to the combined effects of uniform pressures over its flat ends and curved surface. The former should equal the pressure in a liquid, whose density equals that of the crust, at a depth equal to that of the projected boring; the latter may be as large as the former or have any smaller value. Under the conditions supposed by Mr. Martin the pressures, in tons weight on the square inch, would be

36·8 on the flat ends,
from 0 to 36·8 on the curved surface.

§ 19. Near the bottom of the hole the conditions are complicated. An approximate idea of the conditions immediately below the hole may be derived from § 13. At the centre of the bottom of the hole the tendency to rupture according to (43) is actually less than before the hole was bored, unless the material be incompressible when it vanishes in either case. This diminution is almost certainly confined to the central part of the base of the hole. When a core is hollowed out of a cylinder exposed to uniform pressure p the stress-difference—as shown in § 10—rises to $2p$. Under the same circumstances, when a core is taken out of a sphere the stress-difference—as shown in (36)—becomes $3p/2$. The conditions at the edge of the base of the hole seem more likely to approach the conditions in the sphere than those in the cylinder. Thus while uncertainty prevails as to the conditions in the base—a good deal depending in practice on the borer—this seems on the whole not so weak a spot as the walls of the boring a few diameters of the bore above the base.

The above considerations unite in indicating that the largest value to be anticipated for the maximum stress-difference anywhere over the surface of the bore-hole is $2p$, where p is the hydrostatic pressure that would exist at the bottom of the hole if filled with a liquid whose density equals the mean density of the core extracted.

§ 20. There are, however, two contingencies to be borne in mind. Whilst large horizontal pressures of the order of hundreds of tons weight on the square inch may be regarded as practically impossible near the surface, it is quite conceivable that smaller but still large horizontal pressures may exist in strata which differ notably from adjacent material. If a bore-hole should pierce such a stratum a nipping action may well ensue. Again, whilst no one is likely to select the immediate neighbourhood of an active volcano as a site for a boring, it might be well in any case to be prepared for the possibility of piercing material in so unstable a condition that sudden relief of pressure may lead to semi-volcanic action.

XLVII. *On the Lateral Vibration of Bars of Uniform and Varying Sectional Area.* *By* JOHN MORROW, *M.Sc.* (*Vict.*); *M.Eng.* (*Liverpool*); *Lecturer in Engineering, University College, Bristol* [*].

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I. *Introduction.*

IN the theory of vibrating rods expressions for the frequency and the form of the displaced bar are obtained from the equation

$$\frac{d^4y}{dx^4} = \mu^4 y,$$

in which y is the displacement at a distance x from the origin.

The solution is

$$y = \mathrm{A}\sin\mu x + \mathrm{B}\cos\mu x + \mathrm{C}\sinh\mu x + \mathrm{D}\cosh\mu x,$$

A, B, C, D being arbitrary constants to be determined by the end conditions.

In the case of a uniform bar supported at each end the numerical solution presents no difficulty. In other elementary cases the results are deduced from the values of the roots of equations which would be troublesome if not previously tabulated.

Lord Rayleigh [†] has shown that the natural period of a bar may be obtained approximately by simpler means. For example, he supposes that the curve assumed by a clamped-free bar whilst vibrating is the same as that which it would take up if displaced from its position of equilibrium by a lateral force acting on the bar at some point in its length. He shows that the period is given with considerable accuracy if the point of application of the force is one quarter of the length from the free end.

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[*] Read March 10, 1905.

[†] See Rayleigh's 'Sound,' vol. i. pp. 233–235.

More recently, Garrett [*] has investigated this subject and, from instantaneous photographs of the vibrating bar, has concluded that a better approximation to the centre-line of the bar is obtained if the lateral force is assumed to act at a point one-fifth of the length from the free end.

Rayleigh's method is, of course, applicable to rods under other conditions of support. It rests on the fact [†] that a close approach to the true period can be obtained by assuming a type of vibration which is admissible as an initial configuration, as a considerable departure from the true type leads to only a small error in the estimate of the frequency.

Garrett's method and results have been further investigated and compared with Lord Rayleigh's by Dr. Chree [‡], and it is shown that, whilst Rayleigh's depends on a recognized dynamical principle, Garrett's apparently has no such basis.

These simple solutions may be of considerable importance in acoustics and mechanics. They have been employed in problems of the "whirling" and vibration of rotating shafts by Dunkerley [§] and later by Chree [||].

It is the object of the present paper to show that by assuming an equation which completely satisfies the end conditions a better approximation is obtained, and that, by a process of continuous approximation, the vibration curve and the period of the fundamental are given to any required degree of accuracy.

The treatment is capable of very general application, and will be chiefly useful in problems of which the solution has not previously been obtained. The first three cases dealt with will serve as illustrations under various types of terminal conditions. It is a noteworthy fact that, whilst Rayleigh's approximations are necessarily overestimates, the method described below gives initially too small a value for the frequency.

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[*] Phil. Mag. Nov. 1904.

[†] Rayleigh's 'Sound,' vol. i. §§ 88, 89, 182.

[‡] Phil. Mag. Jan. 1905, p. 134.

[§] Phil. Trans. A, 1894, p. 279.

[||] Phil. Mag. May 1904, p. 504.

When the density and flexural rigidity of a bar are variable from point to point in its length, its treatment by the ordinary method of analysis presents difficulties which have not yet been completely overcome. Many of the solutions are obtained easily by the method here given, and several special cases are considered.

2. *Approximate Method of Solution.*

If a bar is vibrating so that every point in it has the same period, the ratio of the acceleration to the displacement is constant for all points.

If y_z be the displacement at a distance z from the point chosen as origin, so that $\frac{d^2y_z}{dt^2}$ is the acceleration, and if y_1 be the displacement of any chosen point (say that at which the displacement is a maximum), then the above statement may be expressed by the equation

$$\frac{d^2y_z}{dt^2}\Big/ y_z = \frac{d^2y_1}{dt^2}\Big/ y_1$$

or

$$\ddot{y}_z = \frac{\ddot{y}_1}{y_1} y_z. \qquad \ldots\ldots\ldots \quad (1)$$

This is equivalent to the usual relation in frequency problems denoted by

$$y = \cos kt \,.\, f(x).$$

Now, if we consider the reversed effective forces due to the acceleration of an increment of the length of the bar, we have, by the ordinary Euler-Bernoulli theory, that the couple at any section in the direction tending to increase $\frac{dy}{dx}$ is equal to the product of the curvature and the flexural rigidity at that section. That is, in view of the approximate straightness of the rod,

$$\frac{d^2y}{dx^2} = \frac{\mathrm{M}}{\mathrm{EI}}, \qquad \ldots\ldots\ldots \quad (2)$$

where M is the couple due to the reversed effective forces,
E = Young's Modulus,
and I = Moment of Inertia of the cross-section about the neutral axis.

Since M is a function of the mass per unit length, the end forces, and the accelerations, we can express it in terms of $\frac{\ddot{y}_1}{y_1}$ y_z and the density and cross-sectional area of the bar. And assuming for y_z a hypothetical type of vibration satisfying the end conditions, we can insert the value of M thus obtained in the equation (2). Solving for y we have a second approximation to the centre-line of the bar, and can calculate the period by

$$N = \frac{1}{2\pi}\sqrt{-\frac{\ddot{y}_1}{y_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The next approximation consists in using this solution for y to find a more accurate expression for M, inserting this in equation (2) and again solving.

The process can be repeated to any extent, but in common with other methods it neglects the rotatory inertia of the rod.

In the following work the bars will be considered to be of uniform sectional area, density, and flexural rigidity except when otherwise stated.

3. *Clamped-Free Bar.*

Taking first the case of a rod "clamped" or "built-in" at one end and free at the other, let the origin, O, be at the free end.

If ρ = density of material of bar,
ω = cross-sectional area,

then the Moment of the Inertia forces about X (distant x from O)

$$= \int_0^x \rho\omega\, \ddot{y}_z (x - z)\, dz,$$

and therefore the differential equation of the centre line of the beam is, by (1) and (2),

$$-\frac{d^2y}{dx^2} = \frac{\rho\omega}{\mathrm{EI}} \frac{\ddot{y}_1}{y_1} \int_0^x y_z (x - z)\, dz, \quad . \quad . \quad . \quad . \quad (4)$$

$\rho\omega$ being taken from under the integral sign on account of the uniformity of the density and sectional area throughout the length.

The object is now to find the solution of this equation in ascending powers of x, and hence to obtain the period of vibration.

Assume a value of y_z which satisfies the end conditions. Thus let

$$y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4.$$

The end conditions at the free end, $x=0$, are

$$\frac{d^2y}{dx^2} = 0, \quad \therefore \quad \mathrm{C} = 0,$$

$$\frac{d^3y}{dx^3} = 0, \quad \therefore \quad \mathrm{D} = 0,$$

$$y = y_1, \quad \therefore \quad \mathrm{A} = y_1,$$

and at the fixed end, $x = l, \qquad y = 0, \qquad \frac{dy}{dx} = 0,$

$$\mathrm{B} = -\frac{4}{3}\frac{y_1}{l}, \qquad \mathrm{E} = \frac{y_1}{3l^4},$$

and the equation is

$$y = y_1\left(1 - \frac{4}{3}\frac{x}{l} + \frac{1}{3}\frac{x^4}{l^4}\right). \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for y_z in equation (4)

$$-\frac{d^2y}{dx^2} = \frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\int_0^x\left(1 - \frac{4}{3}\frac{z}{l} + \frac{1}{3}\frac{z^4}{l^4}\right)(x-z)\,dz$$

$$= \frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\left(\cdot 5x^2 - \cdot\dot{2}\frac{x^3}{l} + \cdot 0\dot{1}\frac{x^6}{l^4}\right),$$

and performing the integrations, evaluating the constants by the end conditions,

$$-y = \frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\Big(\cdot 0819\dot{4}\,l^4 - \cdot 112{,}698{,}41\,l^3x + \cdot 041\dot{6}\,x^4$$

$$- \cdot 0\dot{1}\frac{x^5}{l} + \cdot 000{,}198{,}41\frac{x^8}{l^4}\Big), \quad . \quad . \quad (6)$$

which is approximately the curve of vibration of the bar and gives

$$-y_1 = \cdot 0819\dot{4}\frac{\rho\omega l^4}{\mathrm{EI}}\ddot{y}_1,$$

or from (3)

$$N = \frac{3{\cdot}493}{2\pi} \frac{kU}{l^2},$$

where k = radius of gyration of cross-section,

$$U = \sqrt{\frac{E}{\rho}}$$ = velocity of transmission of longitudinal vibrations in the rod.

The expression given by Rayleigh may be put in the form

$$N = \frac{3{\cdot}5160}{2\pi} \frac{kU}{l^2},$$

showing that the ratio between the two values of N is 0·9935.

The next approximation gives a still closer value and a ratio of 0·9998. It is obtained by inserting y_2 from equation (6) in (4), and using the value just obtained for y_1, whence

$$-\frac{d^2y}{dx^2} = \frac{12{\cdot}204\rho\omega}{EI\,l^4}\ddot{y}_1 \int_0^x \Big({\cdot}0819\dot{4}\, l^4 - {\cdot}112{,}698{,}41\, l^3 z + {\cdot}041\dot{6} z^4$$

$$- {\cdot}0\dot{1} \frac{z^5}{l} + {\cdot}000{,}198{,}41 \frac{z^8}{l^4} \Big)(x - z)dz$$

$$= \frac{12{\cdot}204\rho\omega}{EI\,l^4}\ddot{y}_1 \Big({\cdot}04097\dot{2}\, l^4 x^2 - {\cdot}018{,}783{,}07\, l^3 x^3 + {\cdot}0013\dot{8}\, x^6$$

$$- {\cdot}000{,}264{,}55 \frac{x^7}{l} + {\cdot}000{,}002{,}21 \frac{x^{10}}{l^4} \Big),$$

$$\therefore\ -y = \frac{{\cdot}12204\rho\omega\ddot{y}_1}{EI\,l^4} \Big({\cdot}663084\, l^8 - {\cdot}912718\, l^7 x + {\cdot}341435\, l^4 x^4$$

$$- {\cdot}093915\, l^3 x^5 + {\cdot}002480\, x^8$$

$$- {\cdot}000367 \frac{x^9}{l} + {\cdot}000002 \frac{x^{12}}{l^4} \Big).$$

This is a more accurate expression for the vibration-curve than is (6), but it can be seen that the difference is not great.

$$-y_1 = {\cdot}0809227 \frac{\rho\omega\, l^4}{EI}\ddot{y}_1,$$

and

$$N = \frac{3{\cdot}5153}{2\pi} \frac{kU}{l^2},$$

which is correct to 0·02 per cent. of the value given by exact methods.

4. *Bar "Supported" at Both Ends.*

This is the case in which the directions at the ends are free, but the ends themselves are constrained to remain at rest by "supporting" forces of the required magnitude.

For a uniform bar these forces must each equal one half of the total force due to the inertia of the bar, that is, taking O at one end,

$$\tfrac{1}{2}\rho\omega\frac{\ddot{y}_1}{y_1}\int_0^l y\,dx.$$

Equation (2) then becomes

$$\mathrm{EI}\frac{d^2y}{dx^2}=\tfrac{1}{2}\rho\omega\frac{\ddot{y}_1}{y_1}(l-x)\int_0^l y\,dx-\rho\omega\frac{\ddot{y}_1}{y_1}\int_x^l y_z(z-x)dz. \quad . \quad (7)$$

Taking $y=\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+\mathrm{E}x^4+\mathrm{F}x^5$,

$x=0,\quad y=\frac{d^2y}{dx^2}=0,\qquad \therefore\ \mathrm{A}=\mathrm{C}=0$

$x=\frac{l}{2}\quad \frac{dy}{dx}=0,\qquad \therefore\ 0=\mathrm{B}+\tfrac{3}{4}\mathrm{D}l^2+\tfrac{1}{2}\mathrm{E}l^3+\tfrac{5}{16}\mathrm{F}l^4$

$x=\frac{l}{2}\quad y=y_1,\qquad \therefore\ y_1=\mathrm{B}\frac{l}{2}+\mathrm{D}\frac{l^3}{8}+\mathrm{E}\frac{l^4}{16}+\mathrm{F}\frac{l^5}{32}$

$x=l\quad y=0,\qquad \therefore\ 0=\mathrm{B}l+\mathrm{D}l^3+\mathrm{E}l^4+\mathrm{F}l^5$

$x=l\quad \frac{d^2y}{dx^2}=0,\qquad \therefore\ 0=6\mathrm{D}l+12\mathrm{E}l^2+20\mathrm{F}l^3;$

and the last four of these give

$$\mathrm{B}=\frac{16}{5}\frac{y_1}{l},\quad \mathrm{D}=-\frac{32}{5}\frac{y_1}{l^3},\quad \mathrm{E}=\frac{16}{5}\frac{y_1}{l^4},\quad \mathrm{F}=0.$$

Hence the type to be assumed as a first approximation is

$$y=\frac{16}{5}\frac{y_1}{l^4}(l^3x-2lx^3+x^4),$$

and (7) becomes

$$\frac{d^2y}{dx^2}=\frac{\rho\omega\ddot{y}_1}{\mathrm{EI}}\left\{\frac{l-x}{2}\int_0^l\frac{16}{5l^4}(l^3x-2lx^3+x^4)dx\right.$$
$$\left.-\frac{16}{5l^4}\int_x^l(l^3z-2lz^3+z^4)(z-x)dz\right\}$$
$$=\frac{16}{5}\frac{\rho\omega\ddot{y}_1}{\mathrm{EI}l^4}\left(\frac{l^5x}{10}-\frac{l^3x^3}{6}+\frac{lx^5}{10}-\frac{x^6}{30}\right).$$

Integrating and keeping the end conditions in view,

$$y=\frac{16}{5}\,\frac{\rho\omega\ddot{y}_1}{\mathrm{E}\mathrm{I}l^4}(\cdot 01\dot{6}\,l^5x^3-\cdot 008\dot{3}\,l^3x^5+\cdot 002{,}380{,}95\,lx^7$$
$$-\cdot 000{,}595{,}24\,x^8-\cdot 010{,}119{,}05\,l^7x), \quad . \quad . \quad (8)$$

which is the approximate vibration-curve, and

$$-\frac{\ddot{y}_1}{y_1}=97\cdot 1098\,\frac{\mathrm{E}\mathrm{I}}{\rho\omega l^4}.$$

For the second approximation, using (8) in (7) and simplifying as before,

$$-\frac{d^2y}{dx^2}=\cdot 310{,}751\,\frac{\rho\omega\ddot{y}_1}{\mathrm{E}\mathrm{I}l^8}(-1\cdot 02513\,l^9x+1\cdot 68650\,l^7x^3$$
$$-\cdot 8\dot{3}\,l^5x^5+\cdot 19841\,l^3x^7-\cdot 03307\,lx^9+\cdot 00662x^{10})$$
$$\therefore\ -y=\cdot 310751\,\frac{\rho\omega\ddot{y}_1}{\mathrm{E}\mathrm{I}l^8}(\cdot 10386\,l^{11}x-\cdot 17086\,l^9x^3$$
$$+\cdot 0843\dot{3}\,l^7x^5-\cdot 01984\,l^5x^7$$
$$+\cdot 00276\,l^3x^9-\cdot 00030\,lx^{11}+\cdot 00005\,x^{12}).$$

This is a close approximation to the equation of the centre-line, and gives

$$-y_1=\cdot 0102725\,\frac{\rho\omega l^4}{\mathrm{E}\mathrm{I}}\,\ddot{y}_1$$
$$\therefore\ \mathrm{N}=\frac{9\cdot 866}{2\pi}\,\frac{k\mathrm{U}}{l^2}.$$

The correct value of the numerical constant in the numerator is π^2, so that the ratio of the two values is 0·9996.

The last expression for the displacement may be conveniently compared with the exact type. Thus it may be written

$$y=\mathrm{A}x\left\{1-\frac{\cdot 17086}{\cdot 10386}\frac{x^2}{l^2}+\frac{\cdot 08433}{\cdot 10386}\frac{x^4}{l^4}\cdot\cdot\cdot\cdot\cdot-\frac{\cdot 00030}{\cdot 10386}\frac{x^{10}}{l^{10}}+\frac{\cdot 00005}{\cdot 10386}\frac{x^{11}}{l^{11}}\right\},$$

whilst the method indicated at the commencement of this paper gives, by the insertion of the end conditions,

$$y\propto\sin\frac{\pi x}{l},$$

or

$$y=\mathrm{A}'x\left\{1-\frac{\pi^2(x/l)^2}{\underline{|3}}+\frac{\pi^4(x/l)^4}{\underline{|5}}\cdot\cdot\cdot\cdot\cdot-\frac{\pi^{10}(x/l)^{10}}{\underline{|11}}\right\}.$$

Comparing the coefficients of x^2, x^4, &c. inside the bracket shows how the accuracy alters in passing from the lower to the higher powers. For example,

$$\frac{\cdot 17086}{\cdot 10386} = 1\cdot 645. \qquad \frac{\pi^2}{6} = 1\cdot 645.$$

$$\frac{\cdot 08433}{\cdot 10386} = \cdot 8120. \qquad \frac{\pi^4}{\lfloor 5} = \cdot 8117.$$

$$\frac{\cdot 00030}{\cdot 10386} = \cdot 000289. \qquad \frac{\pi^{10}}{\lfloor 11} = \cdot 000235.$$

5. *Free-Free Bar.*

In the case in which both ends are free and no external forces act on the bar, the conditions at each end are

$$y = y_1, \qquad \frac{d^2y}{dx^2} = 0, \qquad \frac{d^3y}{dx^3} = 0.$$

Assuming

$$y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4 + \mathrm{F}x^5 + \mathrm{G}x^6,$$

the conditions at $x=0$ give

$$\mathrm{A} = y_1, \qquad \mathrm{C} = \mathrm{D} = 0,$$

and those at $x=l$ give

$$\mathrm{B} = -\tfrac{1}{2}\,\mathrm{G}l^5, \qquad \mathrm{E} = \tfrac{5}{2}\,\mathrm{G}l^2, \qquad \mathrm{F} = -3\,\mathrm{G}l.$$

To determine G we have that, since the total force on the bar vanishes,

$$\rho\omega\frac{\ddot{y}_1}{y_1}\int_0^l y\,dx = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is

$$\int_0^l (y_1 - \tfrac{1}{2}\,\mathrm{G}l^5x + \tfrac{5}{2}\,\mathrm{G}l^2x^4 - 3\,\mathrm{G}lx^5 + \mathrm{G}x^6)dx = 0$$

$$\therefore\ \mathrm{G} = \frac{28\,y_1}{3l^6},$$

and hence the equation to be assumed is

$$y = y_1\left(1 - \frac{14}{3}\frac{x}{l} + \frac{70}{3}\frac{x^4}{l^4} - 28\frac{x^5}{l^5} + \frac{28}{3}\frac{x^6}{l^6}\right).$$

Equation (2) becomes for this case

$$-\frac{d^2y}{dx^2}=\frac{\rho\omega}{\mathrm{EI}}\frac{\ddot{y}_1}{y_1}\int_0^x y_z(x-z)\,dz,$$

and substituting for y_z

$$-\frac{d^2y}{dx^2}=\frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\int_0^x\left(1-\frac{14}{3}\frac{z}{l}+\frac{70}{3}\frac{z^4}{l^4}-28\frac{z^5}{l^5}+\frac{28}{3}\frac{z^6}{l^6}\right)(x-z)\,dz$$

$$=\frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\left(\cdot 5x^2-\cdot\dot{7}\frac{x^3}{l}+\cdot\dot{7}\frac{x^6}{l^4}-\cdot\dot{6}\frac{x^7}{l^5}+\cdot 1\dot{6}\frac{x^8}{l^6}\right).$$

Integrating twice and evaluating the second constant by virtue of (9)

$$-y=\frac{\rho\omega}{\mathrm{EI}}\ddot{y}_1\left(\cdot 001{,}993{,}14\,l^4-\cdot 00\dot{9}2\dot{5}\,l^3x+\cdot 041\dot{6}\,x^4-\cdot 03\dot{8}\frac{x^5}{l}\right.$$
$$\left.+\cdot 013\dot{8}\frac{x^8}{l^4}-\cdot 00\dot{9}2\dot{5}\frac{x^9}{l^5}+\cdot 00\dot{1}8\dot{5}\frac{x^{10}}{l^6}\right).$$

and

$$\mathrm{N}=\frac{22\cdot 399}{2\pi}\frac{k\mathrm{U}}{l^2}.$$

The more correct value is 22·373, so that in this case the first approximation gives a ratio of the two values as near as 0·9988.

6. *Clamped-Free Bar of Varying Breadth.* ($b=\mathrm{A}x$.)

Let b be the breadth of the bar, and d its depth in the plane in which the vibration occurs.

Let $d=$constant, and $b=\mathrm{A}x$. Equation (4) then becomes

$$-\frac{d^2y}{dx^2}=\frac{1}{\mathrm{EI}}\int_0^x \rho\omega_z\ddot{y}_z\,(x-z)\,dz$$

$$=\frac{2\rho\ddot{y}_1}{\mathrm{E}d^2}\left(x^2-\frac{2}{3}\frac{x^3}{l}+\frac{1}{21}\frac{x^6}{l^4}\right),$$

by virtue of (1) and (5).

$$\therefore\ -y=\frac{2\rho\ddot{y}_1}{\mathrm{E}d^2}\left(\cdot 122{,}619{,}04\,l^4-\cdot 173{,}469{,}38\,l^3x+\cdot 08\dot{3}\,x^4\right.$$
$$\left.-\cdot 0\dot{3}\frac{x^5}{l}+\cdot 000{,}850{,}34\frac{x^8}{l^4}\right)$$

$$-\frac{\ddot{y}_1}{y_1}=4\cdot 07767\frac{\mathrm{E}d^2}{\rho l^4}.$$

In the next approximation the differential equation reduces to

$$-\frac{d^2y}{dx^2}=\frac{97\cdot 86408\rho\ddot{y}_1}{Ed^2l^4}\Big(\cdot 020{,}436{,}51\,l^4x^2-\cdot 014{,}455{,}78\,l^3x^3$$

$$+\cdot 001{,}984{,}13\,x^6-\cdot 000{,}59523\frac{x^7}{l}+\cdot 000{,}007{,}73\frac{x^{10}}{l^4}\Big);$$

whence

$$-y=\frac{\cdot 9786408\rho\ddot{y}_1}{Ed^2l^4}\Big(\cdot 240053\,l^8-\cdot 340800\,l^7x+\cdot 170304\,l^4x^4$$

$$-\cdot 072279\,l^3x^5+\cdot 003543\,x^8-\cdot 000827\frac{x^9}{l}+\cdot 000006\frac{x^{12}}{l^4}\Big)$$

$$-\frac{\ddot{y}_1}{y_1}=4\cdot 25\dot{6}\,\frac{Ed^2}{\rho l^4},$$

and $$N=\frac{2\cdot 063}{2\pi}\,\frac{dU}{l^2}.$$

All clamped-free bars in which the depth is constant and the breadth proportional to the distance from the free end will vibrate with a period which is independent of the breadth at the fixed end (that is of the value of A), as given by the above formula.

7. *Clamped-Free Bar of Varying Breadth.* ($b=Ax^2$.)

Here $$-\frac{d^2y}{dx^2}=\frac{12\rho}{Ed^2x^2}\,\frac{\ddot{y}_1}{y_1}\int_0^x z^2(x-z)y_z\,dz$$

$$=\frac{\rho\ddot{y}_1}{Ed^2}\Big(x^2-\cdot 8\frac{x^3}{l}+\cdot 07142857\frac{x^6}{l^4}\Big),$$

$$\therefore\ -y=\frac{\rho\ddot{y}_1}{Ed^2}\Big(\cdot 098{,}928{,}57\,l^4-\cdot 143{,}537{,}41\,l^3x+08\dot{3}\,x^4$$

$$-\cdot 04\frac{x^5}{l}+\cdot 001{,}275{,}51\frac{x^8}{l^4}\Big),$$

and

$$-\frac{\ddot{y}_1}{y_1}=10\cdot 1083\frac{Ed^2}{\rho l^4}.$$

For the second approximation

$$-\frac{d^2y}{dx^2}=\frac{121\cdot 3\rho\ddot{y}_1}{Ed^2l^4x^2}\int_0^x z^2(x-z)\Big(\cdot 09892857\,l^4-\cdot 14353741\,l^3z$$

$$+\cdot 08\dot{3}\,z^4-\cdot 04\frac{z^5}{l}+\cdot 001{,}275{,}51\frac{x^8}{l^4}\Big)dz$$

$$=\frac{\cdot 1213\rho\ddot{y}_1}{Ed^2l^4}\Big(8\cdot 24405\,l^4x^2-7\cdot 17687\,l^3x^3+1\cdot 48809\,x^6$$

$$-\cdot\dot{5}\frac{x^7}{l}+\cdot 00966\frac{x^{10}}{l^4}\Big),$$

$$\therefore\ -y=\frac{\cdot 1213\rho\ddot{y}_1}{\mathrm{E}d^2l^4}\Big(\cdot 75074\,l^8-1\cdot 09782\,l^7x+\cdot 68700\,l^4x^4$$

$$-\cdot 35884\,l^3x^5+\cdot 02657\,x^8-\cdot 00772\frac{x^9}{l}+\cdot 00007\frac{x^{12}}{l^4}\Big)$$

$$-\frac{\ddot{y}_1}{y_1}=10\cdot 9812\frac{\mathrm{E}d^2}{\rho l^4}.$$

This gives $$\mathrm{N}=\frac{3\cdot 313}{2\pi}\frac{d\mathrm{U}}{l^2}.$$

For a third approximation the differential equation reduces to

$$-\frac{d^2y}{dx^2}=\frac{15\cdot 984\rho\ddot{y}_1}{\mathrm{E}d^2l^8}\Big(\cdot 062562\,l^8x^2-\cdot 054891\,l^7x^3+\cdot 012268\,l^4x^6$$

$$-\cdot 004984\,l^3x^7+\cdot 000201\,x^{10}-\cdot 000049\frac{x^{11}}{l}\Big),$$

$$\therefore\ -y=\frac{\cdot 15984\rho\ddot{y}_1}{\mathrm{E}d^2l^8}\Big(\cdot 5655\,l^{12}-\cdot 8275\,l^{11}x+\cdot 52135\,l^8x^4$$

$$-\cdot 27446\,l^7x^5+\cdot 02191\,l^4x^8-\cdot 00692\,l^3x^9$$

$$+\cdot 00015\,x^{12}-\cdot 00003\frac{x^{13}}{l}\Big).$$

Whence $$-\frac{\ddot{y}_1}{y_1}=11\cdot 063\frac{\mathrm{E}d^2}{\rho l^4},$$

and $$\mathrm{N}=\frac{3\cdot 326}{2\pi}\frac{d\mathrm{U}}{l^2}.$$

This is again independent of the breadth of the bar.

8. *Clamped-Free Bar of Varying Depth.* ($d=\mathrm{A}x$.)

The exact form of the equation first assumed for the type of vibration is immaterial to the final result, but, by choosing a suitable equation, the labour involved in obtaining this result to any required degree of accuracy is reduced.

In the present case (as, indeed, in the last two) $\frac{d^2y}{dx^2}$ is not necessarily zero when $x=0$, for there the area of the section of the bar also vanishes.

Hence we may start with an equation

$$y=y_1\Big(1-\frac{2x}{l}+\frac{x^2}{l^2}\Big), \quad . \quad . \quad . \quad . \quad (10)$$

which, inserted in

$$-\frac{d^2y}{dx^2}=\frac{12\rho}{\mathrm{E}b\mathrm{A}^3x^3}\frac{\ddot{y}_1}{y_1}\int_0^x b\mathrm{A}z(x-z)y_z\,.\,dz$$

gives
$$-\frac{d^2y}{dx^2}=\frac{\rho\ddot{y}_1}{\mathrm{EA}^2}\left(2-\frac{2x}{l}+\cdot 6\frac{x^2}{l^2}\right);$$

$$\therefore\quad -y=\frac{\rho\ddot{y}_1}{\mathrm{EA}^2}\left(\cdot 48\dot{3}\,l^2-1\cdot 2\,lx+x^2-\cdot\dot{3}\frac{x^3}{l}+\cdot 05\frac{x^4}{l^2}\right)$$

and
$$-\frac{\ddot{y}_1}{y_1}=\frac{60}{29}\frac{\mathrm{EA}^2}{\rho l^2}.$$

In the second approximation

$$-\frac{d^2y}{dx^2}=\frac{720}{29}\frac{\rho\ddot{y}_1}{\mathrm{EA}^2l^2}\left(\cdot 080\dot{5}l^2-\cdot 1lx+\cdot 05x^2-\cdot 01\frac{x^3}{l}\right.$$
$$\left.+\cdot 001{,}190{,}48\frac{x^4}{l^5}\right),$$

$$\therefore\quad -y=\frac{72}{29}\frac{\rho\ddot{y}_1}{\mathrm{EA}^2l^2}\left(\cdot 1742064\,l^4-\cdot 4468254\,l^3x+\cdot 402\dot{7}\,l^2x^2\right.$$
$$\left.-\cdot 1\dot{6}\,lx^3+\cdot 041\dot{6}\,x^4-\cdot 00\dot{5}\frac{x^5}{l}+\cdot 0003968\frac{x^6}{l^2}\right),$$

$$\therefore\quad -\frac{\ddot{y}_1}{y_1}=2\cdot 312\frac{\mathrm{EA}^2}{\rho l^2};$$

and in the third approximation

$$-\frac{d^2y}{dx^2}=\frac{199\cdot 763\,\rho\ddot{y}_1}{29\,\mathrm{EA}^2\,l^4}\left(\cdot 29034\,l^4-\cdot 37235\,l\,x+\cdot 2014\,l^2x^2\right.$$
$$\left.-\cdot 0\dot{5}\,lx^3+\cdot 0099\,x^4-\cdot 0001\frac{x^5}{l}\right).$$

$$\therefore\quad -y=\frac{199\cdot 763}{29}\frac{\rho\ddot{y}_1}{\mathrm{EA}^2l^4}(\cdot 06192\,l^6-\cdot 15936\,l^5x+\cdot 14517\,l^4x^2$$
$$-\cdot 06206\,l^3x^3+\cdot 01678\,l^2x^4-\cdot 002\dot{7}\,lx^5+\cdot 000\dot{3}\,x^6)$$

$$-\frac{\ddot{y}_1}{y_1}=2\cdot 3445\frac{\mathrm{EA}^2}{\rho l^2}$$

and
$$\mathrm{N}=\frac{1\cdot 5312}{2\pi}\frac{\mathrm{AU}}{l}.\quad\ldots\ldots\ldots\ldots\quad(11)$$

In this case the period is independent of the breadth of the bar, and the influence of the depth is exhibited by means of the constant A.

9. *Kirchhoff's Investigation.*

The solution for some cases of non-uniform cross-section was considered by Kirchhoff in 1879 [*]. He remarked that the general equation

$$\frac{d^2}{dx^2}\left(\mathrm{EI}\frac{d^2y}{dx^2}\right)+\rho\omega\frac{d^2y}{dt^2}=0\ \dagger,$$

can be solved in general terms when the depth and breadth of the cross-section are of the form

$$d=x^m f_1(\chi), \qquad b=x^n f_2(\chi),$$

χ being any variable quantity.

The only cases he investigates fully are

(*a*) $m=1$, $n=0$, $\chi=$constant.
(*b*) $m=1$, $n=1$, $\chi=$constant.

The former corresponds to that in the last section and the numerical result obtained by Kirchhoff may be written

$$\mathrm{N}=\frac{1{\cdot}534}{2\pi}\frac{\mathrm{AU}}{l}.$$

Agreeing well with equation (11) above.

University College, Bristol,
January 1905.

Discussion.

Dr. Chree pointed out that the Author's formulæ for the frequency contained factors of the type $\sqrt{\ddot{y}/y}$. y varied, however, as $\cos kt$, and so $\ddot{y}/y$ was negative. The explanation was that Mr. Morrow's "inertia forces" ought really to be taken with the reversed sign and ought to be the "reversed effective forces" ordinarily employed in dynamical problems. He remarked that the results to which the method led were very close approximations indeed, but that the successive approximations entailed laborious algebra. The value of the method would be much enhanced if it could be shown that the phenomenon exhibited by all the cases treated by Mr. Morrow of the calculated frequency being too low was universally true. If that were the case, one would know that the correct value was intermediate between that given by Mr. Morrow's first approximation and by Lord Rayleigh's method. Dr. Chree also pointed out that Kirchhoff had obtained exact solutions for a variety of forms of varying section, and that one of his cases, discussed in Todhunter and Pearson's 'History of Elasticity,' was the same as the last of the Author's cases. In this instance the frequency obtained by Mr. Morrow's third approximation came very near Kirchhoff's exact result.

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[*] *Berliner Monatsberichte*, 1879, p. 815.

[†] Rayleigh's 'Theory of Sound,' vol. i. art. 187.

Mr. MORROW, in reply, said that by his method the first value for the frequency was sometimes an appreciable amount out, but that the true value was gradually approached by successive approximations. By making two approximations, therefore, it was possible in any case to determine whether the initial value of the frequency was too high or too low. The method employed by Mr. Garrett gave one approximation only, and there was no continuous approach to the true frequency.

XLVIII. *The Application of the Cymometer to the Determination of the Coefficient of Coupling of Oscillation Transformers.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London**.

[Plate XIX.]

The author has already described one form of direct-reading cymometer for the determination of wave-lengths and frequencies in connexion with electric-wave telegraphy†. Since that publication another more compact form of instrument has been designed by him, adapted for measuring a larger range of oscillation constant.

The principle on which the above-mentioned appliance works is as follows:—If there be any circuit, open or closed, in which electrical oscillations are taking place, we can set up oscillations of identical frequency in another inductively coupled and adjustable circuit, provided that the mutual inductance of the two circuits is very small.

If this last circuit can have its inductance (L) and capacity (C) varied, and if the values of these quantities are known, the maximum resonance will be excited when

$$p=2\pi n=1/\sqrt{\mathrm{CL}},$$

where n is the frequency in the primary or tested circuit. This simple relation holds good, however, only when the mutual inductance of the two circuits is sufficiently small.

For the sake of brevity the author calls the quantity $\sqrt{\mathrm{CL}}$ the *oscillation constant* of the circuit. Let us assume then that we have a circuit consisting of an inductance-coil and a condenser in series with each other, the circuit being closed on itself. Let part of this circuit consist of a straight wire or rod which can be placed near to any circuit in which oscillations are taking place. Then these last induce others in the first-named circuit. Suppose that means exist for varying together both the inductance (L) and the capacity (C) of this closed circuit; we can bring it into a condition in which its oscillation constant, viz. $\sqrt{CL}$, has such a relation to the frequency of the oscillations in the primary circuit that $n=1/2\pi\sqrt{CL}$, and then the maximum potential difference will exist between the plates of the condenser forming the closed circuit. This critical condition may be ascertained by the employment of a vacuum-tube of the spectroscopic type preferably containing rarefied Neon, which is connected to the plates of the condenser.

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* Read March 24, 1905.

† See J. A. Fleming, "On an Instrument for the Measurement of the Length of Long Electric Waves, and also Small Inductances and Capacities," Proc. Roy. Soc. Lond. vol. lxxiv. p. 488 (1905).

Such an arrangement constitutes the instrument called by the author a cymometer*.

In its practical and most recent form it is constructed as follows:—On a mahogany baseboard about 136 cms. long and 30 cms. wide is mounted an inductance-coil H, H (see Plate XIX.). This consists of an ebonite tube 108 centims. long and 3 cms. diameter, having a helical groove cut on its surface, with 4 turns to the centimetre (10 to the inch exactly). In this groove is wound a bare copper wire 1·22 mm. in diameter, the ends of which are secured to brass collars clamped on the ebonite tube. The inductance L of such a spiral can be calculated very approximately by the formula

$$L=(\pi DN)^2 l,$$

where D is the mean diameter of one circular turn of the helix and N the number of turns of wire per centimetre or otherwise, πDN represents the length of wire which is wound on one unit of length (1 cm.) of the helix, and l is the length of the helix.

This helix is supported by wood brackets, *a*, *b*, on the base-board.

Parallel to it is fixed a sliding tubular condenser. This consists of a brass tube I, having a solid pin fixed in one end by means of which it is clamped in ebonite clips fixed to a wooden block *c*. This tube has a length of 104 centims. and an outside diameter of 2·49 centims. Over this brass tube is fixed tightly a thin ebonite tube E, the sides of which are only 1·6 mm. thick, and this extends for a length of 5 centims. beyond the inner brass tube. Over this ebonite tube slides easily another brass tube O, 100 centims. long and 3·05 centims. outside diameter. This tube has a heavy collar, *k*, at one end, to which is attached an ebonite handle, *h*, by which to displace the tube, and also a projecting pin, *l*, which carries a semicircular collar, K, resting on the inductance spiral HH. Also this pin or rod carries an index P, which moves over a divided scale SS.

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* From κῦμα, a wave. The author is indebted to his colleague Prof. Platt, M.A., for the correct form of this word.

The instrument is completed by a copper bar, L_1 L_2 L_3, of square section, 6 mm. in side, which joins one end of the inductance spiral to the inner tube of the condenser. The outer tube of the condenser moves through an insulated metal ring, *g*, and from this ring and from a terminal T at the end of the inner condenser-tube stout wires are brought to the ends of a vacuum-tube V.

It will then be seen that if the handle is moved along, the outer condenser-tube slides off the inner one, thus reducing the capacity of the condenser by an amount which is almost exactly proportional to the displacement of the handle.

Experiment shows that the measured capacities of the tubular condenser with the handle in various positions plot out into a nearly straight line in terms of the displacement of the outer tube. Again, the same movement of the handle reduces proportionately the amount of inductance included in the closed circuit, because the inductance of that portion of the spiral included between the collar at the end *b* and the semicircular clip K is almost exactly proportional to the length of the spiral included. Therefore the oscillation constant of the circuit or the quantity $\sqrt{CL}$ for various positions of the handle is proportional to the displacement, and can be marked on the scale.

In the instrument here described the scale covers a range of oscillation constant from 0 to 12, and it is almost exactly equidivisional.

The scale values have been determined experimentally by measuring the capacity of the sliding condenser and the inductance of the spiral for various positions of the handle, by methods already described to the Physical Society*.

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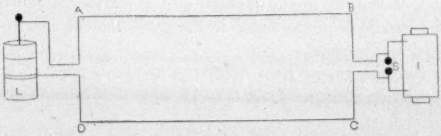
* See J. A. Fleming and W. C. Clinton, "On the Measurement of Small Inductances and Capacities," Phil. Mag. ser. 6, vol. v. p. 493 (1903); also Proc. Phys. Soc. Lond. vol. xviii. p. 386 (1903).

If, then, we have any circuit, open or closed, in which oscillations are taking place, we can determine their frequency if we place the copper bar forming part of the cymometer circuit near and parallel to the first-named circuit, and move the handle until the vacuum-tube shines most brilliantly. The scale-reading will then show us the oscillation constant of the cymometer circuit in that position. Since the frequency corresponding to that reading is related to the constant as the abscissæ and ordinates of a rectangular hyperbola, the scale of the cymometer can also be graduated to show frequencies as well. Again, if the circuit tested is an open circuit which is radiating electric waves, there is a definite relation between the length of the radiated waves and their frequency expressed by $V = n\lambda$, where V is the velocity of electromagnetic radiation. Hence the scale of the cymometer can also be graduated to show wave-lengths directly, and it becomes a direct-reading electric-wave meter for measuring wave-lengths as used in wireless telegraphy.

The instrument has also uses in the laboratory such as determining capacities, inductances, and coefficients of coupling of induction-coils.

For this purpose, the box in which the instrument is packed has a rectangular-shaped circuit of insulated wire ABCD (No. 16 S.W.G.) fixed to the lid. This rectangular-shaped circuit is interrupted in two places, and tails of wire left as in fig. 1.

Fig. 1.



This circuit is given a form such that its inductance for high frequency currents can be calculated. It is intended to be used as follows:—Suppose we desire to determine the capacity of a small leyden-jar for high frequency currents. The jar is connected by outside and inside to one pair of ends of the rectangle, and the other pair of ends are connected to the secondary spark-balls of an induction-coil. We then place the cymometer-bar near and parallel to one long side of the rectangle, and set up oscillations in the leyden-jar circuit with the induction-coil. Next, the cymometer handle is moved until the vacuum-tube (preferably a Neon tube) glows most brightly, and the value of the oscillation constant read off on the scale. We then know that this also must be the oscilllation constant of the jar circuit.

The value of the high frequency inductance of the rectangular circuit is given with the instrument, and hence if we call this value L_1 centimetres and the capacity of the jar C_1 microfarads, and the scale-reading of the oscillation constant of the cymometer corresponding therewith O_1, we have

$$C_1 = \frac{O_1^2}{L_1}.$$

The calculated value of the inductance of the rectangular circuit is obtained by the following formula.

We consider in the first place two equal rectangular circuits the sides of which are respectively S and S′ in length, placed with planes parallel and at a distance b apart. If these circuits are filamentary circuits, then, starting with the Neumann formula for the mutual potential energy M of such circuits, viz.,

$$M = \iint \frac{ds\, ds'}{r} \cos\theta,$$

where r is the distance of any pair of elements ds and ds' of the two circuits and θ their inclination, we extend the integration over every possible pair of elements in two rectangular circuits. It is not difficult to arrive at the following result for the mutual potential energy of the two rectangular circuits conveying unit currents, viz.*

$$M = 4\Big[S \log \frac{(S+\sqrt{S^2+b^2})\sqrt{S'^2+b^2}}{(S+\sqrt{S^2+S'^2+b^2})b} + S' \log \frac{(S'+\sqrt{S'^2+b^2})\sqrt{S^2+b^2}}{S'+\sqrt{S'^2+S^2+b^2})b}$$
$$+2\sqrt{S^2+S'^2+b^2} - 2\sqrt{S'^2+b^2} - 2\sqrt{S^2+b^2} + 2b\Big].$$

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[*] There is no need to give the various steps of the integration as substantially the above formula is given in Mascart and Joubert's Treatise on Electricity and Magnetism, Atkinson's English Translation, vol. ii. p. 477.

To obtain the inductance of a rectangular-shaped circuit, we have then to obtain the mutual potential energy of two rectangular circuits conveying unit currents placed at a distance apart equal to the *geometric mean distance* of all the filamentary currents into which we can suppose the real current in the real circuit divided. If we consider that circuit to be a circular-sectioned wire and the currents to be high frequency, then, as Maxwell shows, the geometric mean distance of all points lying on the circumference of a circle is equal to its radius*.

Hence, in the above formula, if we make b small compared with S or S′, and substitute for b the value $d/2$, where d is the diameter of the round wire, we shall have an expression for the inductance L_1 of a rectangular-shaped circuit of such wire for currents of infinite frequency. The expression is as follows:—

$$L_1 = 4\Big[(S+S')\log\frac{4SS'}{d} - S\log(S+\sqrt{S^2+S'^2})$$

$$-S'\log(S'+\sqrt{S^2+S'^2}+2\sqrt{S^2+S'^2}-2(S+S')\Big] \quad (1)$$

The above formula is a strictly accurate one for infinite frequency, and can easily be applied to any case of a real rectangular circuit. The logarithms are of course Napierian.

We can therefore construct a rectangular circuit of wire attached to the lid of the box of the cymometer, which has a known predetermined inductance of say 5000 centims. Strictly speaking there is a small correction for the tails of parallel wire which connect the rectangle to the jar at one end and to the coil at the other. If considered necessary, this may be taken into account by employing a reduced case of the above formula for the inductance of the rectangle. If there be a pair of round wires of diameter d placed at a distance D apart, the inductance for a length l of the parallel wires is given by the formula

$$L' = 4l\left(\log\frac{2D}{d}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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* See Clerk Maxwell's Treatise on Electricity and Magnetism, 2nd ed. vol. ii. p. 298.

Hence, if the length of the tails of wire at each end of the rectangle is the same and equal to l, the inductance of the whole circuit is equal to L_1+2L', where L_1 and L' have the values given by the formulæ above.

If we wish to determine the high frequency inductance of a short length of wire, say a loop of copper wire of one or two turns, we proceed as follows:—We insert this loop in series with the rectangular circuit and with a condenser of which the capacity has been determined, and employ the cymometer as above described to determine the oscillation constant of the circuit. Then let L_2 be the inductance of the loop of wire, and L_1 that of the rectangle, and O_1 the observed oscillation constant when L_1 is used alone, and O_2 when L_1 and L_2 are in series. Then we have

$$O_1^2=CL_1$$

$$O_2^2=C(L_1+L_2);$$

whence

$$L_2=\frac{O_2^2-O_1^2}{C}.$$

We can always check the result by using as a loop some form of circuit, of which the inductance can be calculated.

Thus if we bend a bare round-sectioned copper wire into a square with the ends brought quite near together, we can predetermine its inductance.

We have here a reduced case of the general formula for a rectangular circuit. In equation (1) above put $S=S'$ and put $4S=l$, then the formula reduces to

$$L=2l\left(\log_e\frac{4l}{d}-2{\cdot}853\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Strictly speaking we should add to the value of the expressions (1) and (3) for the inductance of a rectangle and a square a term equal to $R'/2\pi n$, where R' is the high frequency resistance corresponding to a frequency n. The formulæ (1) and (3) as they stand give the inductance for infinite frequency. The value of $R'/2\pi n$ is, however, generally negligible compared with the other term, and the expressions given may be taken to be the inductances for any frequency of the order of 10^6.

In the next place, we may employ the same instrument to determine the coefficients of coupling of the circuits of an air core-transformer such as an oscillation transformer used in wireless telegraphy. Suppose the inductance of the primary circuit to be denoted by L, that of the secondary by N, and the mutual inductance by M. Then $M/\sqrt{LN}$ is called the *coefficient of coupling*, and is a quantity of importance in the theory of high-frequency transformers.

We may join the two circuits of the oscillation transformer into one circuit so that they assist or oppose each other in creating co-linked flux. In one case the effective inductance is equal to $L+2M+N$, and in the other case it is $L-2M+N$.

Hence if we treat the oscillation transformer so joined up in the two ways, and measure as above its effective inductances, and call them L_1 and L_2, we have

$$L_1=L+2M+N,$$

$$L_2=L-2M+N.$$

Hence

$$M=\frac{L_1-L_2}{4},$$

and

$$L+N=\frac{L_1+L_2}{2}.$$

We can then determine directly and independently the larger of the two inductances L or N, and hence we can calculate the value of $M/\sqrt{LN}$ or the coefficient of coupling of the circuits. As an instance of such a determination we may give the measurements made with a form of oscillation transformer used in wireless telegraphy. The primary circuit consisted of one single turn composed of 8 turns of 7/22 insulated copper wire in parallel wound round a square wooden frame. The secondary circuit consisted of turns of the same stranded wire wound over the primary circuit. The resultant inductances were measured by the cymometer with the circuits joined up to add and oppose each other. The measured values were as follows :—

$$L_1=L+2M+N=62576 \text{ centims.}$$

$$L_2=L-2M+N=49621 \quad ,,$$

$$N=55445 \quad ,,$$

Whence we have $M = 3239$ centims.
and $L+N = 56098$ „
therefore $L = 653$ „
and $N = 55445$ „
Therefore $M/\sqrt{LN} = 0{\cdot}54$

The coupling would therefore be called "close," as it is usual to call the coupling "close" or "tight" when the coefficient exceeds 0·5, and "loose" when it is smaller.

The theory of the instrument is involved in that of oscillation transformers generally. The case of two inductively coupled circuits, each consisting of an inductance of negligible resistance placed in series with a condenser has been fully treated by A. Oberbeck *. It has also been discussed by G. Seibt. If we assume, as we may do, that all the currents and electromotive forces vary in simple sinoidal manner, we may employ complex quantities to represent the vectors with which we are concerned. Suppose then that V represents the potential-difference of the condenser-plates, I the current in the circuit, whilst C, L, and M denote the capacity, inductance, and mutual inductance. The symbols stand for the maximum values of the periodic quantities, and suffixes 1 or 2 have reference to the primary or secondary circuits. Then neglecting the energy dissipation by resistance we can write the vector equations connecting these variables and constants for the two circuits in the form

$$V_1 + jpL_1I_1 + jpMI_2 = 0,$$
$$V_2 + jpL_2I_2 + jpMI_1 = 0,$$
$$I_1 = jpC_1V_1,$$
$$I_2 = jpC_2V_2,$$

where j has the usual significance of the sign of perpendicularity, and $p = 2\pi$ times the frequency.

Then, eliminating I_1, I_2, V_1, V_2, we have

$$p^4 - p^2\frac{C_1L_1 + C_2L_2}{C_1C_2(L_1L_2 - M_2)} + \frac{1}{C_1C_2(L_1L_2 - M_2)} = 0.$$

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* See A. Oberbeck, Wied. *Ann. der Physik*, vol. lv. p. 627 (1895).

Hence

$$p^2 = \frac{1}{2C_1C_2(L_1L_2 - M^2)}\{(C_1L_1 + C_2L_2 \mp \sqrt{(C_1L_1 - C_2L_2)^2 + 4C_1C_2M^2}\}$$

Suppose now that the two circuits have independently and separately the same oscillation constant or same natural time-period T_0. Then $C_1L_1 = C_2L_2 = CL$, and if we call k the coefficient of coupling so that $M^2 = k^2L_1L_2$, we have for the value of p^2 the expression

$$p^2 = \frac{1}{CL} \cdot \frac{1 \pm k}{1 - k^2}.$$

Hence if k has a value different from zero, there will be two oscillations of different frequency induced which have frequencies given by the equations

$$n_1 = n_0 \frac{1}{\sqrt{1-k}},$$

$$n_2 = n_0 \frac{1}{\sqrt{1+k}},$$

where $n_0 = 1/2\pi\sqrt{CL}$ is the natural frequency of each circuit separately. Consequently

$$n_1^2(1-k) = n_0^2, \quad n_2^2(1+k) = n_0^2$$

are two equations which determine the relation of the component frequencies of the complex oscillation set up when two circuits of equal natural time-period are inductively connected. It is evident therefore, that if a circuit has in it oscillations of a certain frequency n_0, and we couple it inductively with another circuit which can be adjusted to have the same oscillation constant $(\sqrt{CL})$; in order that oscillations of only one single frequency equal to n_0 should be induced in this adjustable circuit, it is essential that the coefficient of coupling k of the two circuits should be very small. Otherwise two oscillations of different frequency are excited, the frequency of one being greater and that of the other being less than that of the free independent original frequency n_0 it is desired to determine.

In the form of cymometer here described, this necessary result is obtained by making the mutual inductance between the cymometer and the circuit being tested very small. We have then to employ a sensitive detector for the condition of resonance, viz., a Neon vacuum-tube. This condition is not fulfilled if the whole inductance of the cymometer takes part in creating mutual inductance. The necessary condition cannot be easily satisfied if the detecting arrangement consists of any form of hot-wire ammeter. In using a cymometer for measuring the frequency of the oscillation in any circuit, we have to be on our guard against disturbing the very quantity we wish to measure, or setting up in the cymometer circuit some oscillation of a different frequency. It is an obvious deduction from the above investigation, that in using the cymometer we should place the bar of the cymometer as far away as possible from the circuit being tested. Suppose, however, that we have two circuits of the same time-period when separated and we couple them together inductively. Then, if we investigate with the cymometer the oscillations set up in the secondary circuit, we find it to be a complex oscillation resolvable into components of different periods. The cymometer therefore acts just like an electrical spectroscope. It resolves the complex vibrations in a circuit into their simple components and shows us what they are. This effect is very marked in the case of inductively coupled aerials in wireless telegraphy. If we have a nearly closed condenser circuit with spark-gap in which oscillations are set up, which is inductively coupled to an aerial or antenna, then, even if the two circuits are in common language "tuned" to each other so that they have the same independent time-period, yet when coupled, if coupled at all tightly, there are two oscillations set up in the aerial of different frequencies, and two waves radiated of different wave-length, which may differ in length by 15 or 20 per cent.

The cymometer in the above described form can be used to measure not merely the length of the outgoing wave from a sending aerial, but also the length of the wave being received. To do this a coil of a few turns is wrapped around the long helical inductance-coil, and this is connected between the earth and the receiving aerial. The vacuum-tube is then replaced by any form of electric-wave detector such as a coherer or Marconi magnetic detector, and the cymometer circuit adjusted by moving the handle until the maximum effect is produced, or until the receiver indicates when this is the case, the scale-reading of the cymometer will indicate the length of the arriving waves.

DISCUSSION.

Dr. W. **WATSON** expressed his interest in the paper, and said that the cymometer, on account of its simplicity, would be useful for lecture purposes. He asked Dr. Fleming if he thought that any of the anomalous results which had been obtained in determining specific inductive capacity by methods involving the mutual induction between two circuits might be due to the fact that oscillations of two different frequencies were induced in the secondary circuit. How did a change in the coefficient of coupling affect the energy transmitted to a secondary circuit?

Prof. A. W. **REINOLD** said that the accuracy of the experiments depended upon the use of Neon tubes, and he expressed the hope that it would soon be possible to obtain them easily. Referring to the fact that oscillations in a circuit containing capacity and inductance induced in a neighbouring circuit oscillations of two distinct periods, he asked Dr. Fleming what it was that determined the energies of the two oscillations. He also asked how, when measuring the wave-lengths of waves incident on an aerial, the fact that two oscillations were necessarily induced in the cymometer affected the utility of the instrument.

Mr. A. **CAMPBELL** pointed out that ebonite was used as the dielectric in the sliding condenser, and asked if there was any variation in the capacity with change of frequency.

Dr. **FLEMING**, in reply to Dr. Watson's question, said that undoubtedly errors might result in employing methods involving the mutual induction of two circuits for the determination of specific inductive capacities, if the observer were not acquainted with the peculiarities of coupled circuits and high frequency oscillations. The general theorem involved is that if there be two circuits each having inductance and capacity and oscillations are set up in one circuit by disruptive discharge in the other, then we have a compound oscillation set up in the secondary circuit which may be resolved into two frequencies which are different from the free natural frequency of each circuit separately. The theory has been fully worked out by A. Oberbeck (Wied. *Ann. der Physik*, 1895). Hence if it were assumed that the resulting oscillation in the secondary circuit, even if it had the same natural period as the primary, was a single oscillation of the same period, an error would undoubtedly be made. Oberbeck has not given expressions for the respective energies of the two oscillations, but he has shown that the two oscillations have different damping. The coefficient of coupling determines the respective energies of the two oscillations, but Dr. Fleming said he had not yet given sufficient attention to the theoretical side of the subject to be able to furnish expressions for the energies of the two oscillations. In reply to Prof. Reinold, he said that if the mutual inductance between the aerial wire and the cymometer were small, as it was in the case of the instrument exhibited, then the cymometer only detected the oscillations which existed in the aerial. There was no creation of duplicate oscillations in and by the cymometer itself. In reply to Mr. Campbell, Dr. Fleming said that it had been shown by Prof. J. J. Thomson and others that the variation in dielectric constant with frequency was very small or even absent in the case of ebonite. This was the reason why ebonite was chosen for the dielectric tube. In the case of some kinds of glass there is a large change in dielectric constant with frequency. As regards the commercial production of Neon vacuum-tubes, that subject was engaging his attention, and he hoped that they might soon be placed on the market. The beautiful process devised by Sir James Dewar of absorbing the common constituents of air by cocoanut charcoal at low temperatures enabled Neon and Helium to be prepared from the air without much difficulty.

XLIX. *On the Flux of Light from the Electric Arc with Varying Power-Supply.* *By* G. B. DYKE, *B.Sc.*[*]

[Plate XX.]

IN 1896 a paper was communicated to the Physical Society by Dr. J. A. Fleming, F.R.S., and Mr. J. E. Petavel[†], recording the results of numerous observations on the Electric Arc, dealing, amongst other matters, with the question of the relation between the flux of light, or mean spherical candle-power, and the watts expended in the arc.

The time and labour necessary to obtain the mean spherical candle-power of any source of light from observations made on the point-by-point method then employed, prevented any very extended study of this relation.

The construction of the integrating photometer, described by the author in his paper "On the Practical Determination of the Mean Spherical Candle Power of Incandescent and Arc Lamps"[‡], having, to a very large extent, removed these difficulties, it was thought that a more extended series of observations in this direction might be undertaken, and facilities for this purpose were kindly granted in the Pender Electrical Laboratory of University College, London, by Prof. J. A. Fleming.

The objects in view may be thus stated :—

(i.) To obtain a series of curves for both continuous and alternating current arcs, showing the relation between the mean spherical candle-power and the watts taken up in the arc.

(ii.) To compare the "efficiencies" of the continuous and alternating arcs as illuminants when supplied at different powers, the "efficiency" being expressed in mean spherical candles per watt.

As the field covered by these statements was too broad to be covered in its entirety in the time at the author's disposal, it was decided to impose the following restrictions:—

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[*] Read March 24, 1905.

[†] "An Analytical Study of the Alternating-Current Arc," by J. A. Fleming, M.A., D.Sc., F.R.S., and G. E. Petavel, Phil. Mag. April 1896, p. 315.

[‡] "On the Practical Determination of the Mean Spherical Candle-Power of Incandescent and Arc Lamps," by G. B. Dyke, B.Sc., Phil. Mag. Jan. 1905, p. 136, and Proc. Phys. Soc. vol. xix.

The make and diameters of the carbons used to remain unaltered throughout the experiments.

Alternating current of one frequency only to be employed, this frequency to be 80~

In continuous-current experiments the top carbon to be invariably positive.

Arc-lengths not to exceed 0·5 inch.

Power to be supplied at rates not greater than 1500 watts.

The carbons used may be specified thus:—

Upper Carbon—

Maker	C. Conradty, Nuernberg.
Description	Cored. Marke W.C.
Diameter	12 mm.
Diameter of core	2 "
Weight per cub. centim.	1·41 gramme,
Weight per c.c. of outer carbon	1·43 "
Weight per c.c. of core	0·93 "

Lower Carbon—

Maker	C. Conradty, Nuernberg.
Description	Solid. Marke C.
Diameter	10 mm.
Weight per cub. centim.	1·46 gramme.

The mean spherical candle-power was measured directly by means of an integrating photometer.

This instrument was described in detail in the paper referred to above, and it will be sufficient to say that it consists of a system of pairs of mirrors arranged in a semi-circle, about 6 feet in diameter, in such a manner that light emitted from a source placed at the centre of the system shall be incident upon a photometer-screen, also situated at the centre, at the same angle with the vertical as that which it made on emission from the source.

The intensity of illumination of the screen is then proportional to the mean spherical candle-power. As previously arranged for testing incandescent lamps, the photometer measured directly the ratio of the mean spherical candle-power to the mean horizontal candle-power. When dealing with arc-lamps, however, this ratio is not of much practical value, so a small alteration was made to adapt the instrument to read the absolute value of the mean spherical candle-power. This was effected by replacing the horizontal sliding pair of mirrors by a standard incandescent lamp on a sliding support.

This alteration will of course alter the constant of the instrument.

The formula given before was

$$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}} = \frac{5{\cdot}09}{d^2},$$

where d is the distance in feet from source to screen by way of the sliding pair of mirrors.

Now the sliding pair of mirrors reflected 70·5 per cent. of the light incident on them; hence it is obvious that if these mirrors are replaced by a lamp of known candle-power C.P. we have

$$\text{M.S.C.P.} = \frac{5{\cdot}09}{{\cdot}705} \times \frac{\text{C.P.}}{d^2}.$$

Hence the formula becomes

$$\text{M.S.C.P.} = \frac{7{\cdot}22}{d^2} \times \text{C.P. of balancing-lamp},$$

where d is the distance of balancing-lamp from photometer-screen in feet.

As this result is obtained somewhat indirectly the arc was replaced by an incandescent lamp, and the polar diagram constructed by opening the slides in front of the mirrors one at a time and making a photometric measurement for each.

The Rousseau diagram was then drawn, and the mean spherical candle-power deduced and found to be 22·49. All the mirror-slides were then opened and a balance obtained at 15·95 inches, the candle-power of the balancing-lamp being 5·51.

Hence, applying the formula given above,

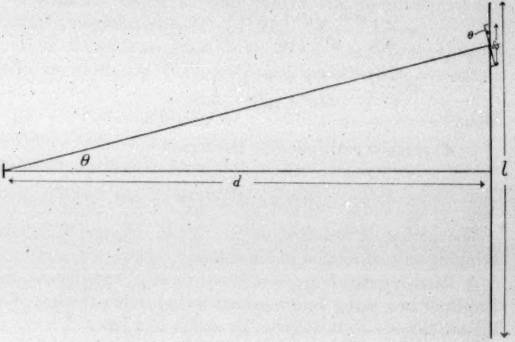
$$\text{constant of instrument} = \frac{22{\cdot}49}{5{\cdot}51} \times \left(\frac{15{\cdot}95}{12}\right)^2$$
$$= 7{\cdot}21.$$

This check-experiment being considered satisfactory, the value 7·22 was taken as the constant of the instrument.

When dealing with arcs of some 2000 mean spherical candle-power, it was necessary to bring the balancing-lamp very close to the photometer.

In this case it was necessary to apply a correction to the scale-reading of the distance *d*, due to the angle subtended by the filament of the balancing-lamp becoming appreciable.

Fig. 1.



If *l* is the length of the filament, and *i* the brightness per unit-length.

Then, considering a flat-loop lamp at a distance *d* from the photometer (see fig. 1), we have :—

Intensity of illumination due to 2 legs of filament

$$= 2\left\{2i\int_0^{\frac{l}{2}}(\delta s\cos\theta)\cos\theta\Big/\left(\frac{d}{\cos\theta}\right)^2\right\}$$

$$= 4i\int_0^{\tan^{-1}\frac{l}{2}/d}\left(\frac{d}{\cos\theta}\times\delta\theta\right)\cos\theta\times\left(\frac{\cos\theta}{d}\right)^2$$

$$= \frac{4i}{d}\int_0^{\tan^{-1}\frac{l}{2}/d}\cos^2\theta\,\delta\theta$$

$$= \frac{2i}{d}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\tan^{-1}\frac{l}{2}/d}$$

$$= \frac{2i}{d}\left[\tan^{-1}\tfrac{l}{2}/d + \frac{\frac{1}{2}ld}{d^2+(\frac{1}{2}l)^2}\right]$$

or approximately

$$=\frac{2i}{d}\left[\frac{\frac{1}{2}d}{\sqrt{d^2+(\frac{1}{2}l)^2}}+\frac{\frac{1}{2}ld}{d^2+(\frac{1}{2}l^2)^2}\right]$$

$$=\frac{il}{d}\left[\frac{d+\sqrt{d^2+(\frac{1}{2}l)^2}}{d^2+(\frac{1}{2}l)^2}\right]$$

$$=\frac{2il}{d^2}\left(\frac{d^2+d\sqrt{d^2+(\frac{1}{2}l)^2}}{2(d^2+(\frac{1}{2}l)^2}\right)$$

Hence

d^2 must be multiplied by the factor

$$\frac{2\{d^2+(\frac{1}{2}l)^2\}}{d^2+d\sqrt{d^2+(\frac{1}{2}l)^2}}$$

The author is indebted to Mr. W. C. Clinton, B.Sc., for bringing this correction to his notice.

A hand-regulated arc was used in the experiments, the length of arc being kept constant by observing the image of the arc thrown on to a screen by means of a lens.

During each series of experiments the arc was maintained at a constant length, whilst the power supplied was varied by means of an adjustable resistance.

In order to obtain a mean result ten photometric readings were taken at each setting.

Altogether eight lengths of arc were employed, viz.

$\frac{1}{16}$ in., $\frac{3}{32}$ in., $\frac{1}{8}$ in., and hence by 16ths to $\frac{7}{16}$ in.

The table which follows gives the mean results of these experiments, involving nearly four thousand observations.

If now these results are plotted, taking watts as abscissæ and mean spherical candle-power as ordinates, we obtain the series of curves shown in Pl. XX., the full-line curves referring to continuous-current arcs and the dotted lines to alternating-current arcs.

It will be seen that, within the limits of experimental error, the relation between mean spherical candle-power and watts follows a straight-line law. This straight line, however, does not pass through the origin, but for zero candle-power there is still an outstanding amount of power amounting to some 200 to 400 watts.

The amount of this outstanding effect increases, in general, with the arc-length, and is probably due to the energy dissipated as heat by radiation, and as chemical energy in evaporation of the carbon at the crater.

The author hopes, at some future date, to investigate this matter experimentally.

It is found from these curves that for each arc-length the point in which the efficiency curve cuts the axis of watts is

Arc length.	Continuous Current Arc.				Alternating Current Arc.			
Inches.	Volts.	Amperes.	Watts.	M.S.C.P.	Volts.	Amperes.	Watts.	M.S.C.P.
$\frac{1}{16}$	50·0	6·4	318	207	39·8	5·6	221	155
	46·3	7·7	355	267	37·8	7·1	268	236
	44·7	9·3	417	370	37·4	9·2	344	359
	45·1	12·1	544	542	(violent hissing if current increased)			
	45·4	16·3	741	777				
	44·2	20·1	890	1078				
$\frac{3}{32}$	50·5	6·7	336	257	47·5	6·7	317	156
	48·9	7·9	384	354	40·5	9·3	378	293
	48·0	9·6	459	490	39·7	11·2	442	445
	47·0	12·4	583	738	39·8	13·4	533	624
	46·4	16·9	786	1119	40·1	16·0	643	855
	45·3	20·9	947	1476	40·7	19·8	807	1104
$\frac{1}{8}$	53·5	6·0	323	290	43·8	6·8	299	124
	51·0	7·5	382	390	42·1	8·6	362	241
	52·2	9·2	481	609	41·6	10·3	429	297
	49·5	12·3	609	901	41·7	12·5	520	477
	47·5	16·8	796	1295	42·1	14·6	616	579
	47·0	20·4	961	1761	42·2	18·8	793	846
					40·3	20·1	811	977
$\frac{3}{16}$	58·0	5·7	333	237	51·0	4·9	250	80
	54·6	7·1	386	353	48·2	6·3	304	132
	52·1	8·9	462	524	45·4	8·5	384	244
	51·2	11·7	599	767	43·5	12·6	547	543
	48·8	16·3	796	1219	42·1	15·8	655	712
	48·8	20·2	985	1669	42·2	19·9	837	990
$\frac{1}{4}$	64·6	6·8	441	410	56·1	4·6	259	87
	60·9	8·7	527	683	53·9	5·4	291	154
	59·4	12·2	727	1161	50·6	7·1	360	196
	55·9	15·4	863	1532	46·0	10·7	490	432
	57·0	19·4	1104	2210	45·6	15·9	727	863
$\frac{5}{16}$	63·0	8·7	548	549	57·9	8·3	478	270
	60·8	10·4	632	727	56·3	10·5	617	382
	59·0	12·5	728	921	53·6	13·7	735	596
	58·5	14·9	871	1284	49·2	17·5	861	797
	61·1	18·8	1150	1900	48·9	21·7	1062	1024
$\frac{3}{8}$	67·2	6·8	459	350	57·7	8·1	470	238
	62·5	9·0	565	557	53·2	11·3	602	446
	62·1	12·2	760	1130	54·1	14·1	763	720
$\frac{7}{16}$	65·8	10·0	655	621	56·8	19·5	593	407
	65·9	11·6	762	810	56·6	13·8	783	660
	64·9	14·3	927	1202	55·6	18·1	1003	1070
	62·4	18·9	1182	1703				

the same whether the arc is supplied with continuous or alternating currents, pointing to the fact that the power expended in forms of energy other than light, *e. g.* volatilization of carbon, radiation of heat, sound, &c., is exactly the same for similar alternating and continuous arcs.

For lengths of arc greater than $\frac{1}{8}$ in. the ratio of the efficiencies of the continuous and alternating arcs is about 3 to 2, agreeing very well with that obtained by Dr. Fleming and Mr. Petavel.

If, however, the arc be made shorter than $\frac{1}{8}$ in. the efficiency line for the alternating-current arc will be found to gradually approach the line for the continuous-current arc until for an arc-length of $\frac{3}{32}$ in. they are practically coincident (Pl. XX.).

The arc is quite stable at this point, and even in the alternating-current case may be supplied at more than 600 watts without causing hissing.

Any further decrease in the arc-length will result in the alternating-current efficiency line rising above the continuous-current line until at an arc-length of $\frac{1}{16}$ in. they have the position shown in the first curve in Pl. XX.

With so small an arc-length, however, the alternating-current arc begins to hiss badly when suppled at a rate greater than 350 watts, although the continuous-current arc is still silent up to 900 watts.

As far as the author is aware, this is the first case in which an alternating current arc of efficiency greater than the corresponding continuous-current arc has been realized in practice, although Dr. Fleming and Mr. Petavel recognized the possibility of such a phenomenon in their paper referred to above.

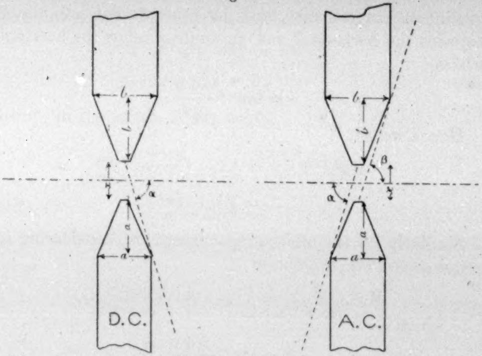
Why, then, does the alternating-current arc give the same efficiency as the continuous-current arc at this arc-length?

This can perhaps be most clearly seen by making some simple approximations with regard to the shape of the carbons, and calculating the length of arc necessary to give the same candle-power with the same power supply.

Let the carbons be supposed to assume the shape indicated in fig. 2, truncated cores whose heights are equal to their bases; this being approximately the shape found by experiment.

Also let all the light in the case of the continuous-current arc be supposed to come from the positive crater.

Fig. 2.



Now, the expression for the mean spherical candle-power of a source symmetrical about an axis is

$$\text{M.S.C.P.} = \tfrac{1}{2}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \mathrm{I}_\theta \cos\theta \, d\theta,$$

where I_θ is the luminous intensity in a direction making an angle θ with the horizontal.

Now let I be the luminous intensity of the crater when viewed normally.

Then, as a rough approximation, assuming the crater to be a flat circular disk, small compared with the arc length,

$$\mathrm{I}_\theta = \mathrm{I} \sin\theta.$$

Hence for the arc

$$\text{M.S.C.P.} = \tfrac{1}{2}\mathrm{I}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta$$

$$= \tfrac{1}{4}\mathrm{I}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$= \tfrac{1}{8}\mathrm{I}\Big[-\cos 2\theta\Big]_{\frac{\pi}{2}}^{-\frac{\pi}{2}} .$$

Let a be the diameter of the lower carbon (−), and b the diameter of the upper carbon (+).

Then, making the above assumptions, all the light, in the continuous-current case, will be concentrated into the zone between the horizontal and an angle α below the horizontal: where

$$\alpha = \tan^{-1}\frac{a+x}{\frac{1}{2}a}.$$

Hence we get

$$\text{M.S.C.P.}_{\text{C.C.}} = \tfrac{1}{8}\,\text{I}_{\text{C.C.}}\left[-\cos 2\theta\right]_0^{\alpha}.$$

$$= \tfrac{1}{4}\,\text{I}_{\text{C.C.}}\sin^2\alpha.$$

Similarly for the alternating-current arc, considering each crater separately, we obtain

$$\text{M.S.C.P.}_{\text{A.C.}} = \tfrac{1}{4}\,\text{I}_{\text{A.C.}}(\sin^2\alpha+\sin^2\beta),$$

where

$$\beta = \tan^{-1}\frac{b+x}{\frac{1}{2}b}.$$

Hence

$$\frac{\text{M.S.C.P.}_{\text{C.C.}}}{\text{M.S.C.P.}_{\text{A.C.}}} = \frac{\text{I}_{\text{C.C.}}}{\text{I}_{\text{A.C.}}}\,\frac{\sin^2\alpha}{\sin^2\alpha+\sin^2\beta}.$$

Some experiments mentioned by Dr. Fleming and Mr. Petavel give:—

for a continuous-current arc

$$\text{I}_\theta = 910 \quad \text{when} \quad \theta = 40{\cdot}5^\circ.$$

Hence

$$\text{I}_{\text{C.C.}} = \frac{910}{\sin 40{\cdot}5^\circ} = 1400;$$

and for a similar alternating-current arc

$$\text{I}_\theta = 300 \text{ when } \theta = 60^\circ \text{ for top crater,}$$

and

$$\text{I}_\theta = 370 \text{ when } \theta = 63^\circ \text{ for bottom crater.}$$

Hence as a mean

$$\text{I}_{\text{A.C.}} = 707.$$

So we get

$$\frac{\text{I}_{\text{C.C.}}}{\text{I}_{\text{A.C.}}} = \frac{1400}{707} = 1{\cdot}99.$$

Therefore

$$\frac{\text{M.S.C.P.}_{\text{C.C.}}}{\text{M.S.C.P.}_{\text{A.C.}}} = 1{\cdot}99\,\frac{\sin^2\alpha}{\sin^2\alpha+\sin^2\beta}.$$

Hence for equal efficiency we have

$$1{\cdot}99\sin^2\alpha = \sin^2\alpha+\sin^2\beta.$$

$$\therefore \qquad {\cdot}99\sin^2\alpha = \sin^2\beta\,;$$

or putting in the values of the angles

$$\frac{{\cdot}99(a+x)^2}{(a+x)^2+{\cdot}25a^2} = \frac{(b+x)^2}{(b+x)^2+{\cdot}25b^2}.$$

Now

$$a = 10 \text{ mm.}$$

$$b = 12 \text{ mm.}$$

Hence

$$\frac{{\cdot}99(10+x)^2}{(10+x)^2+25} = \frac{(12+x)^2}{(12+x)^2+36}.$$

Writing out and arranging

$$x^4+44x^3-340x^2-6000x+18000=0.$$

Solving this equation graphically we obtain as the positive finite value of x

$$x = 2{\cdot}7 \text{ mm.},$$

which agrees very closely with that found by experiment.

This consideration, however, by no means gives a complete solution of the problem of the relative efficiencies of the continuous and alternating current arc, but is only intended to account to some extent for the phenomenon of continuous and alternating current arcs of equal efficiency.

A complete analysis of the subject could only be undertaken on the basis of the evidence of a much more extended series of experiments, of which the time at the author's disposal would not allow.

In conclusion the author wishes to thank Dr. J. A. Fleming, F.R.S., for his many suggestions and constant advice, and also for his kindness in placing the resources of the Pender Laboratory at his disposal, and to express his indebtedness to Messrs. J. S. Westerdale and H. C. Bullman for the very efficient assistance they have rendered him, at great personal inconvenience, in making some thousands of electrical measurements, without which the investigation would have been impossible.

DISCUSSION.

Dr. J. A. FLEMING congratulated the Author, and referred to the practical value of the experiments. The ordinary method of comparing lamps by their M.H.C.P. was unscientific. What was wanted was the total flux of light per watt, and this important figure was obtained at once by the integrating photometer used by the Author. Prof. Fleming instanced the case of a Nernst lamp and an ordinary carbon-filament lamp, pointing out that the ordinary tests indicated that the Nernst lamp was twice as efficient as the ordinary lamp, whereas a comparison of the total flux per watt showed an increased efficiency in the Nernst lamp of only 25 per cent.

Mr. PATERSON expressed his interest in the paper and pointed out that the variation of the light from an electric arc necessitated a great number of readings in order to get a satisfactory mean value: the photometer used by the Author integrated the light in one plane, but it would be still more satisfactory to integrate in two planes at right angles to one another when photometering lights which changed their candle-powers so rapidly. The correcting formula for the length of the filament in a standard lamp, as given by the Author, was unnecessarily long. By a more complete expansion of the first expression given, the formula obtained for the correcting factor for the candle-power of the standard was $K = 1 - \frac{1}{6}\frac{l^2}{d^2}$, which has very nearly the same numerical value but is simpler than the expression given in the paper.

L. *Note on the Voltage Ratios of an Inverted Rotary Converter. By* W. C. Clinton, *B.Sc., A.M.Inst.E.E.; Demonstrator in the Pender Electrical Laboratory, University College, London**.

If a separately excited rotary converter be driven at a constant speed with the rings and brushes on open circuit, the ratios of the maximum alternating E.M.F. between the rings to the E.M.F. over the brushes is given by a well-known expression on the assumption of a sinusoid distribution of the air-gap magnetic flux over the surface of the armature.

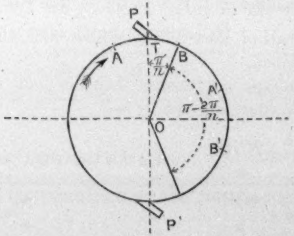
Putting E_a and E_c for the alternating and continuous E.M.F.'s respectively, and n for the number of rings, then

$$\frac{E_a}{E_c} = \sin\frac{\pi}{n}$$

where $\frac{2\pi}{n}$ is the angular width in degrees of phase between the contacts for two successive rings.

If, however, the converter be driven by current supplied to either side, the other side being on open circuit, then a correction on the above value becomes necessary owing to ohmic drop in the armature conductors.

Let direct current be supplied to the commutator, the rings being on open circuit. Let the resistance of a single path



through the armature from brush to brush be R, and let the current flowing in that path be C. Then the ohmic drop from brush to brush is CR.

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* Read March 24, 1905.

Let AB be points of connexion of a pair of rings to the winding.

The P.D. between the rings due to drop is zero at AB, increases uniformly till A is under the brush at P, and then remains constant until B comes under the brush P′. It decreases to zero again with A 180 phase degrees from its first position.

Let $\frac{2\pi}{n}$ = angular distance in phase degrees from A to B, and let θ = phase displacement of the centre T of the section AB from the line OP.

As θ changes from O to $\frac{\pi}{n}$ the P.D. between A and B due to ohmic drop increases from O to $\frac{2CR}{n}$ as AB is the $\frac{2}{n}$th part of a single path between the brushes.

The value of this P.D. for an intermediate value of θ may be written as

$$\frac{2CR}{n} \cdot \frac{\theta}{\frac{\pi}{n}} = \frac{2CR\theta}{\pi}.$$

For further increase of θ up to $\pi - \frac{\pi}{n}$ this drop is constant and equal to $\frac{2CR}{n}$.

From $\pi - \frac{\pi}{n}$ to π the ohmic drop diminishes to zero again, in the same manner as it rose in the section 0 to $\frac{\pi}{n}$

The other half of the cycle is similar with the polarity of the rings reversed.

Let V = voltage on brushes, then the total instantaneous voltage on the rings is given by :—

$$(V-CR)\sin\frac{\pi}{n}\sin\theta + \frac{2CR\theta}{\pi} \quad \text{when } \theta \text{ is between O and } \frac{\pi}{n} \quad . \quad . \quad (1)$$

$$(V-CR)\sin\frac{\pi}{n}\sin\theta + \frac{2CR}{n} \quad ,, \quad ,, \quad \frac{\pi}{n} \text{ and } \pi - \frac{\pi}{n} \quad . \quad (2)$$

$$(V-CR)\sin\frac{\pi}{n}\sin\theta + \frac{2CR(\pi-\theta)}{\pi} \quad ,, \quad ,, \quad \pi - \frac{\pi}{n} \text{ and } \pi \quad . \quad (3)$$

The R.M.S. value of the voltage on the rings can be obtained by proceeding in the usual way with the above expressions.

$$\int_0^{\pi/n} \left\{ (\mathrm{V}-\mathrm{CR}) \sin\frac{\pi}{n} \sin\theta + \frac{2\mathrm{CR}\theta}{\pi} \right\}^2 d\theta$$

$$= \left\{ (\mathrm{V}-\mathrm{CR})^2 \sin^2\frac{\pi}{n} \left(\frac{\pi}{2n} - \frac{\sin\frac{2\pi}{n}}{4} \right) + \frac{4\mathrm{CR}(\mathrm{V}-\mathrm{CR}) \sin\frac{\pi}{n}}{\pi} \left(\sin\frac{\pi}{n} - \frac{\pi}{n} \cos\frac{\pi}{n} \right) + \frac{4\mathrm{C}^2\mathrm{R}^2\pi^2}{n^3} \right\}. \quad (4)$$

$$\int_{\pi n}^{\pi-\pi/n} \left\{ (\mathrm{V}-\mathrm{CR}) \sin\frac{\pi}{n} \sin\theta + \frac{2\mathrm{CR}}{n} \right\}^2 d\theta$$

$$= (\mathrm{V}-\mathrm{CR})^2 \sin^2\frac{\pi}{n} \left\{ \frac{\pi}{2} - \frac{\pi}{n} + \frac{\sin\frac{2\pi}{n}}{2} \right\} + \frac{8\mathrm{CR}(\mathrm{V}-\mathrm{CR}) \sin\frac{\pi}{n} \cos\frac{\pi}{n}}{n} + \frac{4\mathrm{C}^2\mathrm{R}^2}{n^2} \left(\pi - \frac{2\pi}{n} \right). \quad (5)$$

The sum of the squares for the period 0 to π is equal to twice expression (4) + expression (5).

The value of this is

$$(\mathrm{V}-\mathrm{CR})^2 \sin^2\frac{\pi}{n} \cdot \frac{\pi}{2} + \frac{4\mathrm{CR}(\mathrm{V}-\mathrm{CR})}{n} \sin\frac{\pi}{n} \left\{ 2\sin\frac{\pi}{n} - \frac{2\pi}{n} \cos\frac{\pi}{n} + 2\cos\frac{\pi}{n} \right\} + \frac{8\mathrm{C}^2\mathrm{R}^2\pi^2 + 4\mathrm{C}^2\mathrm{R}^2(n-2)\pi}{n^3}.$$

Dividing through by π, taking the square root, and dividing this again by V we get as the value of the ratio of the alternating P.D. to the direct current P.D.

$$\frac{1}{\mathrm{V}} \sqrt{ \frac{(\mathrm{V}-\mathrm{CR})^2 \sin^2\frac{\pi}{n}}{2} + \frac{8\mathrm{CR}(\mathrm{V}-\mathrm{CR}) \sin\frac{\pi}{n}}{n\pi} \left\{ \sin\frac{\pi}{n} + \cos\frac{\pi}{n} - \frac{\pi}{n} \cos\frac{\pi}{n} \right\} + \frac{4\mathrm{C}^2\mathrm{R}^2}{n^3} (2\pi + n - 2) }$$

Let $\mathrm{CR} = \frac{\mathrm{V}}{m}$. Then the expression for the ratio becomes :—

$$\sqrt{ \frac{(m-1)^2 \sin^2\frac{\pi}{n}}{2m^2} + \frac{(m-1) \sin\frac{\pi}{n}}{m^2 n\pi} \left\{ \sin\frac{\pi}{n} + \cos\frac{\pi}{n} - \frac{\pi}{n} \cos\frac{\pi}{n} \right\} + \frac{4}{m^2 n^3} (2\pi + n - 2) }.$$

The corrected ratios of conversion are compared with the uncorrected values for different values of m in the following table :—

No. of Phases.	Corrected Ratios (as percentages).						Uncorrected Ratio as percentages.
	$m=10$.	$m=20$.	$m=50$.	$m=100$.	$m=200$.	$m=500$.	
Single phase ...	67·13	68·32	69·61	70·14	70·42	70·58	70·72
Three phase ...	56·70	58·71	60·18	60·69	60·95	61·10	61·24
Four phase ...	46·04	47·87	49·11	49·55	49·75	49·905	50·00
Six phase	32·46	33·82	34·73	35·03	35·19	35·29	35·35
Twelve phase...	15·60	17·50	17·97	18·13	18·21	18·26	18·30

It is of interest to construct a table showing the differences between the above ratios and the corresponding uncorrected ratio, as a percentage of the latter :—

No. of Phases.	Percentage difference from uncorrected ratio.					
	$m=10$.	$m=20$.	$m=50$.	$m=100$.	$m=200$.	$m=500$.
Single phase	5·076	3·39	1·57	·82	·42	·20
Three phase	7·413	4·13	1·73	·90	·47	·23
Four phase	7·920	4·26	1·78	·90	·50	·19
Six phase	8·176	4·33	1·75	·83	·45	·17
Twelve phase ...	14·75	4·37	1·80	·93	·49	·22

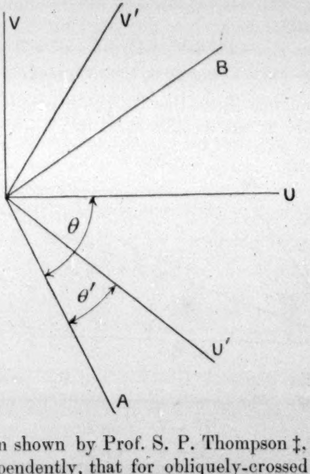
This table brings out the fact that the percentage correction due to voltage drop on the unloaded machine is practically independent of the number of phases, if the value of m is of the order usually found in good design.

LI. *On Ellipsoidal Lenses.*

By R. J. Sowter, *B.Sc., A.R.C.Sc.*[*]

THIS note extends the treatment of thin ellipsoidal or astigmatic lenses by the author's method[†], and gives a simple solution for complex problems of the following types:—"To determine the astigmatic pencil, after refraction of an astigmatic pencil by an ellipsoidal lens." And "to find the ellipsoidal lens equivalent to two cylindrical lenses placed a definite distance apart, with their axes inclined at any angle." The method of treatment can be applied to crossed ellipsoidal lenses, in contact, or separated, and is applicable in general to astigmatic pencils.

Fig. 1.



It has been shown by Prof. S. P. Thompson[‡], and by the author independently, that for obliquely-crossed cylindrical lenses an important double-angled parallelogram of powers can be constructed. This parallelogram is associated with the resolution of a lens-power P into the two powers P $\cos^2\theta$, P $\sin^2\theta$, at right angles. It is here "generalized." If an astigmatic pencil having two focal lines at distances $u\,v$ from an ellipsoidal lens of focal powers A, B is refracted by the lens, which is further defined in position with respect to the focal lines by θ, then by resolving along two directions at right angles and adopting the curvature notation, we get:—

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[*] Read April 14, 1905.

[†] "On Astigmatic Lenses," Proc. Phys. Soc. vol. xvii.

[‡] Proc. Phys. Soc. vol. xvi., and Phil. Mag. March 1900.

$$-(U\cos^2\theta+V\sin^2\theta)+U'\cos^2\theta'+V'\sin^2\theta'=A\ , \qquad \text{(i.)}$$

$$-(U\sin^2\theta+V\cos^2\theta)+U'\sin^2\theta'+V'\cos^2\theta'=B\ . \qquad \text{(ii.)}$$

where

$$U=\frac{1}{u},\quad V=\frac{1}{v},$$

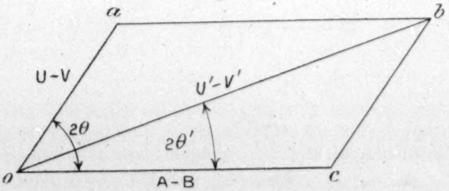
and U', V', θ' define the focal lines in the refracted pencil. Fig. 1 shows the orientation of the powers, or principal curvatures, with respect to each other. Equations (i.) and (ii.) give:—

$$-(U+V)+U'+V'=A+B\quad .\ .\ .\quad \text{(I.)}$$

$$-(U-V)\cos 2\theta+(U'-V')\cos 2\theta'=A-B\ .\quad \text{(II.)}$$

This last equation shows that the power parallelogram $o\,a\,b\,c$, shown in fig. 2, has its sides equal to $(U-V)$, $(A-B)$, its

Fig. 2.



angle 2θ, the diagonal $(U'-V')$, and the inclination of the diagonal $2\theta'$. The parallelogram affords the equation .—

$$\frac{U'-V'}{U-V}=\frac{\sin 2\theta}{\sin 2\theta'}\ .\quad .\ .\ .\ .\quad \text{(III.)}$$

Equations I., II., III., or the parallelogram $o\,a\,b\,c$ together with the simple equation (I.), give a complete solution. The signs to be adopted are those that are usual, and such as to make

$$-(U+V)+U'+V'=A+B.$$

The usual *lengthy* solution by the use of the characteristic function resolves itself into that given by Herman* employing Malus' theorem.

If a parallel pencil fall upon the first lens of a crossed and separated cylindrical combination, the equations (i.) (ii.) are

$$-\mathrm{U}\cos^2\theta+\mathrm{U}'\cos^2\theta'+\mathrm{V}'\sin^2\theta'=\mathrm{A},$$
$$-\mathrm{U}\sin^2\theta+\mathrm{U}'\sin^2\theta'+\mathrm{V}'\cos^2\theta'=0,$$

where U is the equatorial curvature of the cylindrical wave at the second lens, power A, θ is the angle of crossing of the lenses, and $\mathrm{U}'\ \mathrm{V}'\ \theta'$ define the refracted pencil, and consequently the equivalent ellipsoidal lens.

The sides of the parallelogram are A and U.

The following experiments were performed :—

Experiment I. Light from a small aperture having vertical and horizontal cross-wires was brought to a parallel beam by a convex lens, and the beam fell on a convex cylindrical lens of focal length 18·8 cms. supported with its axis vertical. At 8 cms. from this lens another convex cylindrical lens of focal length 25·3 cms. was supported with its axis inclined at 60° to the horizontal. The focal lines were at 8·1 cms., and 140 cms. from the second lens, and the near line was inclined at 21° to the axis of the inclined lens. Equation (I.) gives :—

$$\mathrm{U}'+\mathrm{V}'=-{\cdot}132,$$

and equations II. and III., or the parallelogram, give :—

$$(\mathrm{U}'-\mathrm{V}')\cos 2\theta'=\mathrm{A}+\mathrm{U}\cos 2\theta=-{\cdot}0858$$
$$(\mathrm{U}'-\mathrm{V}')\sin 2\theta'=\mathrm{U}\sin 2\theta \qquad =-{\cdot}0802$$

from which
$$\mathrm{U}'-\mathrm{V}'=-{\cdot}118$$
$$\tan 2\theta' = {\cdot}935$$

that is,
$$\mathrm{U}'=-{\cdot}125 \qquad (u'=\ -8\ \text{cms.})$$
$$\mathrm{V}'=-{\cdot}007 \qquad (v'=-143\ \text{cms.})$$
$$\theta'=21\tfrac{1}{2}^\circ,$$

values in agreement with those observed.

Experiment II. A beam diverging from a circular aperture having vertical and horizontal cross-wires was made parallel, and fell upon a concave cylindrical lens ($f = 18{\cdot}8$ cms.) supported with its axis vertical. At 10 cms. from this lens, two convex cylindrical lenses were placed in contact and crossed at right angles. Their axes were at 60° and 30° respectively to the horizontal, and their focal lengths were 18·8 cms. and 25·3 cms. The focal lines were at 23 cms. and 70 cms. from the crossed cylinders, and the inclination of one focal line to one of the cylinder axes was 41°.

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* Geom. Optics, § 174.

Equations (i.) and (ii.) give at once :—

$$-\mathrm{U}\cos^2\theta + \mathrm{U}'\cos^2\theta' + \mathrm{V}'\sin^2\theta' = \mathrm{A}$$
$$-\mathrm{U}\sin^2\theta + \mathrm{U}'\sin^2\theta' + \mathrm{V}'\cos^2\theta' = \mathrm{B},$$

and the parallelogram, the sides being A − B and U, gives at once :—

$$(\mathrm{U}' - \mathrm{V}')\sin 2\theta' = \mathrm{U}\sin 2\theta,$$

from which we get :—

$$\begin{aligned} \mathrm{U}' + \mathrm{V}' &= -{\cdot}058 \\ \mathrm{U}' - \mathrm{V}' &= {\cdot}0302 \\ \tan 2\theta' &= 8{\cdot}11 \end{aligned}$$

or

$$\begin{aligned} \mathrm{U}' &= -{\cdot}0139 \qquad (u' = -72 \text{ cms.}) \\ \mathrm{V}' &= -{\cdot}0441 \qquad (v' = -23 \text{ cms.}) \\ \theta' &= 41\tfrac{1}{2}^\circ. \end{aligned}$$

This method of power resolutions, as also the associated "generalized" parallelogram, can only be applied to astigmatic problems in which the wave-fronts after refraction are paraboloidal, or are ellipsoidal and can be assumed paraboloidal in the neighbourhood of some point.

Sturm appears to have been the first to show that all the normals to a paraboloid in the neighbourhood of a point converge to or diverge from two lines at right angles to one another.

DISCUSSION.

Mr. W. BENNETT congratulated the Author on his interesting solution of an important problem. The method of resolving the convergence or divergence of the incident pencil, and the power of the lens, in two directions at right angles should prove of great utility, particularly in arithmetical computation. Referring to the statements at the end of the paper, he asked if it was not a fact that any surface could be regarded as paraboloidal in the neighbourhood of a point. If we consider the paraboloid of curvature to be adjusted to the surface at the point, and neglect quantities of the first order, the normals of a small element of the paraboloid pass through two lines, the difference between the element of the surface and the element of the paraboloid depending only on quantities of the second order. The matter may be looked at in another way: the normals to any wave-front touch a caustic surface of two sheets, the normals to any finite region of the wave-front mapping out a finite region on each sheet of the caustic. As the element of the wave-front is made small the corresponding elements of the caustic become smaller, but do not in general become linear. Their projections, however, on the orthogonal surface become approximately linear, and hence the normals pass approximately through two straight lines and any thin pencil may be regarded as a standard astigmatic pencil. The standard astigmatic pencil of finite aperture, however, cannot exist. Maxwell has shown that the only form of wave-front for which the two sheets of the caustic surface degenerate into lines is the cyclide of Bonnet. These focal lines are not straight, but are two conics in planes at right angles to one another, the foci of either conic lying on the vertices of the other. In the general case these conics are an ellipse and a hyperbola respectively. In one particular case the conics are parabolas and in another they become a circle and a straight line through the centre of the circle. The wave-front in the latter case is an anchor-ring.

LII. *The Determination of the Moment of Inertia of the Magnets used in the Measurement of the Horizontal Component of the Earth's Field.* *By* W. WATSON, *A.R.C.S., D.Sc., F.R.S., Assistant Professor of Physics at the Royal College of Science, London* *.

ONE of the constants required when determining the horizontal component of the earth's magnetic field by the ordinary method, is the moment of inertia of the magnet which is used in the vibration experiment. Nearly all the magnetometers which are used in English-speaking lands are tested at the Kew Observatory; and the custom there has been, I believe, to determine the moment of inertia of the cylindrical brass bar supplied with each instrument by calculation from its dimensions, then by measuring the period of the magnet alone and when loaded with this bar to calculate the moment of inertia of the magnet. This method presupposes that the density of the inertia-bar is uniform throughout. Now it is by no means easy to secure a bar of which the density is uniform throughout, and further it is difficult to test whether such uniformity has been secured.

This question as to the uniformity of the inertia-bars supplied with magnetometers has been brought into some prominence lately; for the differences at first obtained when comparing the magnetometers intended for the Indian Magnetic Survey with the Kew standards were finally traced to want of uniformity in the inertia-bars.

It seems to me that more reliable and uniform results would be obtained if a somewhat different procedure were adopted. Namely, to determine once for all, with very great care, the moment of inertia of a standard bar, and then to determine the moment of inertia of the bars supplied with the different magnetometers sent to be tested by comparing them experimentally with the standard bar. Even if the standard adopted were slightly wrong, so that the values given for the bars belonging to the different instruments were uniformly either too high or too low, the confusion caused would be much less than where, as at present, the values obtained may, owing to want of uniformity in the bars, be sometimes too high and sometimes too low.

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* Read April 14, 1905.

In the following paper is described an instrument suitable for comparing the moment of inertia of bars, together with some experiments made with a view to determining the moment of inertia of a standard bar, and an investigation on the influence of the air on the period.

§ 1. *Determination of the Moment of Inertia of a Standard Bar.*

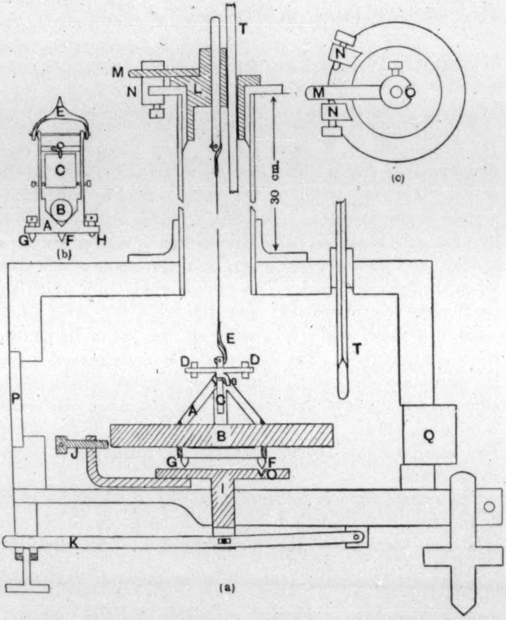
The instrument is shown in fig. 1. It consists of a cradle, A, suspended by a stout quartz fibre (length 30 cms., diameter 0·37 mm.). The bar, B, to be tested rests on this cradle while the position of the cradle is read by reflexion of a scale in the mirror C. A table, I, which can be raised or lowered by a lever, K, has a hole-slot-and-plane on its upper surface, and the three feet, F, G, H, of the cradle rest in these when the table is raised. The conical hole, O, in which the leg F rests is so placed that the cradle when supported on the table lies about half a millimetre to the left of its position when hanging free. The object of this arrangement is that the position of the bar B, with reference to the cradle, can be adjusted by means of the screw J, and yet, when the cradle is released, the bar may swing clear of the point of this screw. Two small weights D serve to adjust the balance and period of the cradle.

The quartz fibre is soldered to a clip E which hooks on to the cradle and another clip which hooks on to the end of a rod carried by the torsion-head L. An arm, M, attached to the torsion-head can be moved between two adjustable stops N. This arm serves to give the torsion-head a small to-and-fro motion of adjustable and fixed amount, and thus allows of the bar being set swinging.

The bar, &c., is enclosed in a wooden box the sides of which are removable. A window P allows of the mirror being seen; while a hole Q, which can be closed by a cork, serves for the removal of the bar from the cradle. The torsion-head is supported by a brass tube 30 cms. long attached to the top of the box. This brass tube is wound round with a layer of felt to reduce the temperature changes.

The method adopted for adjusting the instrument is as follows. The base of the instrument having been levelled, the cradle is supported on the table with the bar in position,

Fig. 1.



and a striding level is placed on the bar. By adjusting the screws G and H the bar is brought into a horizontal position, and a telescope and scale are set up at a distance of about 70 cms. from the instrument, and so adjusted that the horizontal cross wire coincides with the ends of the division-lines of the scale as seen reflected in the mirror C. The bar is then removed and the cradle released. The weights D are adjusted till the horizontal wire of the telescope again coincides with the images of the ends of the division-lines of the scale. When this adjustment is complete the line joining the V's of the cradle will be horizontal. The bar having been placed in position, the screw J is adjusted till on releasing the cradle the horizontal wire of the telescope again coincides with the ends of the division-lines of the scale, and therefore the axis of the bar is horizontal.

The vertical cross wire having been brought into coincidence with some well-marked division of the scale, the arm M is moved from its position against one of the stops N up to the other stop and back again, thus starting the bar swinging. The two thermometers T are read, and the period is determined by means of a chronometer in the usual manner. The following is an example of such a determination of period, though in some of the observations, particularly those in which the air effect was being investigated, the vibrations were allowed to continue for over 30 minutes.

Amplitude { at commencement 4°
at end............... 1° 30′

Temperature { 14°·9 14°·9
14°·7 14°·7

Times of Transit. Scale moving to right.		Difference.	Times of Transit. Scale moving to left.		Difference.
h m s	h m s	m s	h m s	h m s	m s
11 46 13·6	12 6 15·1	20 1·5	11 46 34·2	12 6 35·7	20 1·5
47 13·7	7 15·2	1·5	47 34·4	7 36·0	1·6
48 13·8	8 15·3	1·5	48 34·5	8 36·0	1·5
49 14·0	9 15·4	1·4	49 34·6	9 36·0	1·4
50 14·0	10 15·6	1·6	50 34·6	10 36·1	1·5
4 0·4	Mean ...	20 1·50	4 0·4	Mean ...	20 1·50
16 1·6			16 1·6		
12 6 15·6			12 6 36·2		

Approximate period determined with stop-watch = 3·75. Hence in 60 secs. there are 16 vibrations, and in 4 m. 0·4 s. there are 64. Thus in 16 m. 1·6 s. there are 256. Hence the number of vibrations corresponding to each of the above differences is 320.

Mean time for 320 vibrations . . 20 m. 1·50 s.
„ „ one vibration . . . 3·7547 sec.

The period of the cradle alone has also to be determined, and it will be found convenient to adjust the period by the weights D to nearly one second, so that the method of coincidence may be conveniently employed.

The following table gives particulars of the different inertia-bars which have been compared.

Distinguishing number.	Material, &c.
1.	Rolled brass.
2.	Cast silver.
3.	Rolled copper.
4.	Copper with 0·5 per cent. zinc. Cast.
5.	Copper with 0·5 per cent. phosphide of tin. Cast.
6.	Gun-metal. Cast.
7.	Rolled copper.
8.	Rolled copper.
9.	Rolled copper.
10.	A rolled brass bar with slightly rounded corners and gilt.

Note.—Bars 2, 4, and 5 were made from castings prepared with very great care at the Royal Mint under the direction of Dr. Rose; and I am much indebted to him for the trouble he has taken in the matter.

The bars were turned and then ground true, while special care was taken to have the ends plane and perpendicular to the axis of the cylinder. In the case of bars numbers 7, 8, and 9 the ends were ground and polished to an optical surface, which was tested by obtaining Newton's rings between the metal surface and a piece of plane glass. The ends were slightly rounded; but as this did not amount to more than about five half wave-lengths of sodium light, and this only near the edge, the accuracy of the measurements did not necessitate any correction being applied *.

To measure the lengths of the bars two cylindrical palpers having the same diameter as the bars were prepared, one end of each being turned to a spherical surface. A fiducial line was scratched on one cylinder. These cylinders rested in a

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* This question has also been tested at the National Physical Laboratory, where they measured the lengths of the generators of two of these cylinders across two diameters, and found that the lengths were constant except at the very edge, and that there the "rounding" did not amount to more than ·0006 cm.

V-groove in the iron bed of a comparator fitted with micrometer-microscopes, one palper bedding against a stop. The two palpers being placed in contact, by means of a force of 70 grams weight, the right microscope was adjusted to the fiducial mark on the left-hand palper. The bar to be measured having been introduced between the palpers, which were kept pressed together by the same force as before, the left-hand microscope was adjusted to the fiducial mark on the left-hand palper. The micrometers having been read, the bar was turned through 90° and the adjustment repeated. In this way four measurements were taken, each measurement consisting of four readings of the micrometers, the bar being turned through a right angle between each measurement. To obtain the distance between the micrometers a nickel steel (Invar) metre which had been calibrated at the International Bureau at Sèvres was substituted for the bars, and the micrometer-readings taken for two divisions 10 centimetres apart, care being taken not to displace either micrometer more than necessary from the position it occupied when setting on the bars.

In the case of the bars with optically worked ends, the length was also measured by setting the micrometers halfway between the point of a needle and its image as seen reflected in the end surface of the bar. At first this method gave values which were consistently 0·001 centimetre lower than those obtained by the contact method. This difference was found to be due to the objectives of the microscopes not being aplanatic, for in this method only half of the objective of the microscope is used. On replacing the objectives with others which were optically much better, and taking care to focus accurately, the measurements obtained by the two methods were in exact agreement.

The diameters were measured at three places along each bar by means of a Brown and Sharp micrometer-screw gauge which had been tested against the standard metre, measurements being taken along different diameters at each place.

The masses were obtained by means of a Bunge balance, which reads easily to a tenth of a milligram. The method of double weighing was employed, the weights used having been calibrated and compared with a standard 100 gram weight the error of which has been determined at Sèvres. Except in the case of the silver bar no correction has been applied for buoyancy, as all the weights, except the fractions of a gram, were of brass.

In the following table are given the dimensions of the bars and the calculated values of the moment of inertia :—

No. of Bar.	Length at 15°. cm.	Diameter at 15°. cm.	Mass. grams.	Logarithm of the Moment of Inertia at 15°.
1	9·9755	·9909	65·230	2·73633
2	9·9811.	·9974	81·814	2·83525
3	9·9998	·9975	69·204	2·76416
4	10·0044	·9979	69·437	2·76602
5	10·0017	·9985	69·391	2·76549
6	9·9878	·9972	68·516	2·75878
7	9·9964	·9994	68·745	2·76099
8	9·9837	·9973	68·414	2·75779
9	9·9856	·9976	68·481	2·75837

All the observations of period used in the comparisons of the moments of inertia of the bars were made at temperatures not greatly differing from 15° C. In order to be able to allow for the small departures from this temperature, a series of observations were made with the whole instrument heated in a well-stirred air-bath to a temperature of 30°. The change of period with change of temperature is very small, owing to the fact that the rigidity of quartz increases with rise of temperature, and thus produces the opposite effect to that due to the increase in the dimensions of the bars owing to expansion. In the case of the copper bars the coefficient of increase of period with temperature is 0·00004, and in the case of brass 0·000035. By means of the coefficients the observations of period recorded in the following table have been reduced to what they would have been at 15°, the actual temperatures at which they were made being included between 14° and 18°.

In the case of bar 1, since the observations with this bar range over a very considerable time, the dates of the various measurements of period are given to indicate the constancy which can be obtained with such a quartz-fibre suspension.

No. 1.

Date.	Period.
13. x. 03.	3·6615
14. x. 03.	11
26. x. 03.	10
27. x. 03.	11
19. xi. 03.	09
25. xi. 03.	12
26. xi. 03.	13
2. xii. 03.	14
21. i. 04.	11
5. ii. 04.	11
5. ii. 04.	11
Mean...	3·6612

No. 2.

	4·0543
	43
Mean...	4·0543

No. 3.

	3·7672
	70
	69
Mean...	3·7670

No. 4.

	3·7751
	48
	48
Mean...	3·7749

No. 5.

	3·7726
	23
	22
Mean...	3·7724

No. 6.

	3·7467
	71
Mean...	3·7469

No. 7.

	3·7545
	47
	45
	49
Mean...	3·7547

No. 8.

	3·7421
	22
	24
	24
Mean ...	3·7423

No. 9.

	3·7446
	47
	49
	48
Mean...	3·7448

No. 10.

	3·6682
	81
	83
	86
	84
	84
	81
	81
Mean...	3·6683

Cradle only.

	1·2427
	9
	6
	5
	2
	2
	5
	6
Mean...	1·2425

If t_1 is the period with a bar of moment of inertia K_1 in the cradle, while t_0 is the period of the cradle alone, K_0 being its moment of inertia, we have

$$t_1 = 2\pi\sqrt{\frac{K_1 + K_0}{c}},$$

$$t_0 = 2\pi\sqrt{\frac{K_0}{c}},$$

where c is a constant depending on the suspension fibre. Hence

$$\frac{K_1}{t_1^2 - t_0^2} = \frac{c}{4\pi^2}.$$

Thus if the bars are uniform in density and diameter we have, subject to variations due to errors of experiment, that the quotient $K_1/(t_1^2 - t_0^2)$ ought to be the same for all the bars. The following table gives the value of the quotient for the different bars :—

Number of Bar.	$\frac{K_1}{t_1^2 - t_0^2}$	Difference from mean.
1	45·944	+·008
2	47	+·011
3	40	+·004
4	21	−·015
5	31	−·005
6	23	−·013
7	41	+·005
8	44	+·008
9	37	+·001
	Mean 45·936$_4$	±·0078

The mean value for $\frac{K_1}{t_1^2 - t_0^2}$ is 45·936$_4$. I do not, however, consider that this mean is the best value to take, as owing to the extra care taken in making the three bars numbers 7, 8, and 9, I consider it advisable to give the results with these bars treble weight. Doing this the mean becomes

45·938.

By means of this mean value we may calculate the moment of inertia of the standard bar No. 10. The value obtained in this way at 15° C. is

547·24.

Owing to the fact that the rigidity of fused silica increases as the temperature rises, the period changes very little with temperature. Thus for the cradle alone the percentage increase of the period for a fall of temperature of one degree centigrade is ·0036. With a brass bar of the dimensions used in these experiments, the percentage increase of period for a fall of temperature of one degree is ·0035. As long, therefore, as care is taken to have the temperature uniform throughout the instrument, the correction for changes of temperature can be made without difficulty.

In order to obtain some idea as to the accuracy with which the moments of inertia of bars may be *compared* by the method employed, we may take the differences from the respective means of the determinations of period given on page 643, taken irrespective of sign. The number obtained is 0·00014. This corresponds to a mean difference from the mean of 0·0037 per cent. This value, if we neglect the error produced by an erroneous value obtained for the period of the cradle, which error can easily be made negligible, corresponds to a mean error of ·018 per cent. in the ratio of the moments of inertia.

To test the question as to whether the vibrations were strictly isochronous, that is whether the period was independent of the amplitude, a series of measurements were made with different initial amplitudes, and the results are shown in the following table :—

Amplitude.		Period.
At commencement.	At end.	
14° 24′	7° 29′	3·6616
7 52	4 15	3·6613
4 55	2 25	3·6611
2 44	1 20	3·6611

It would thus appear that with an initial amplitude less than 5° the period is not, within the limits of the accuracy of the observations, affected by the variation in the amplitude. This point was also brought out because in many of the observations of period starting with an initial amplitude of between 4° and 5° one set of period observations were made, and then, without altering the amplitude of the vibrations, another set was made. The values obtained for the periods were, however, as often larger with the smaller amplitude as smaller. As a further precaution the initial amplitude was in all cases taken as about the same.

When obtaining the expression for $\frac{K_1}{t_1^2 - t_0^2}$ above it has been assumed that the torsional rigidity of the suspension fibre is the same whether the cradle is empty or loaded with one of the bars. To test whether this assumption is allowable, a set of measurements were made in which the period of the cradle alone was measured. Then its period when loaded with each of two bars separately, and finally when loaded with the two bars*. The results obtained in the case of two fibres are shown in the following table :—

	Weight of Suspended System.	Period.	Value of C.
Cradle only	27 grms.	1·2426	
Cradle+No. 1	92 ,,	3·6612	1813·5
Cradle+No. 4	96 ,,	3·7749	1813·6
Cradle+No. 1+No. 4	161 ,,	5·1094	1813·9
Cradle only	27 ,,	0·8713	
Cradle+No. 10	92 ,,	2·5713	3691·3
Cradle+No. 7	95 ,,	2·6318	3691·5
Cradle+No. 7+No. 10	160 ,,	3·5744	3692·2

Assuming in each case that the change in the rigidity of the suspending fibre is proportional to the load, we obtain the following values for the ratio of the change in c for an increase in load from 27 grams to 95 grams to the value of c, that is the value of $\frac{\delta c}{c}$.

From the observations with the first fibre . .	·00017
From observations with the second fibre . .	·00022

The effect of this increase in the rigidity of the suspension fibre with increase in load on the ratio deduced for the moment of inertia of the bars, will increase as the difference in the moment of inertia (and mass) of the bars increases. To obtain an idea of the magnitude of the effect we may calculate the correction in the case of the bars numbers 1 and 2, which are the lightest and heaviest bars respectively. The numbers obtained are :—

Ratio of moments of inertia obtained when c is assumed constant	125574
Ratio of the moments of inertia obtained when corrected for the change in c produced by change in load	125568

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* The cradle used differed slightly from that shown in fig. 1, in that a second pair of V's were provided to allow of two bars being suspended simultaneously.

It will be seen that the only effect of applying the correction is to alter the value of the ratio by one part in 20000, so that it may be neglected.

§ 2. *The Effect of the Inertia of the Air on the Period of a Vibrating Magnet.*

In order to test the influence of the air carried along by the bars when swinging on their period, a vacuum-chamber was prepared which was capable of enclosing the whole instrument. This chamber consisted of a wooden box, the sides having a thickness of two inches, which was coated outside with thin tinplate, the joints being soldered up. Two windows were fitted, one of considerable size and easily removable, through which the bars could be changed, and the other for the observation of the mirror. An arrangement was also fitted by means of which the fibre could be removed and the top of the torsion-head turned so as to start the oscillations. A side tube was connected to a manometer and a Fleuss pump, which was driven by an electric motor and was capable of maintaining a pressure of about 5 centimetres of mercury. No attempt was made to stop whatever small leaks there were, which prevented a lower pressure being attained, as the range of pressure obtained was amply sufficient for the purpose. The thermometer, which projects into the box of the instrument, was read through the large window. A special experiment showed that the change of pressure from 76 cm. to 5 cm. did not affect the reading of this thermometer by 0°·05 C.

The procedure adopted was to observe the period at atmospheric pressure, then exhaust and observe the period, and finally, the air having been readmitted, to again observe at atmospheric pressure. In every case at least an hour was allowed after the exhaustion or admission of the air for the temperature to become uniform.

The quartz fibre used in the comparison of the inertia-bars having been broken when fitting up the vacuum-chamber a new fibre had to be fitted. The following table gives the observed periods.

Cradle and Bar No. 1.

Period at 15°.	Pressure.	Period at 15°.	Pressure.
2·5651	75·3 cm.	2·5646	4·0 cm.
2·5650	75·3	2·5645	4·0
2·5648	76·8	2·5644	4·0
2·5647	76·8	2·5641	4·6
2·5647	72·0	2·5641	4·5
2·5647	72·0	2·5641	4·3
		2·5641	4·3
		2·5641	4·2
		2·5641	4·2
Means ... 2·5648	74·7	2·5642	4·2
Cradle only.			
·87049	76·0	·87001	4·7
·87046	75·5	·87003	4·7
Means ... ·87047	75·7	·87002	4·7

If t_1 and t_2 are the periods at the lower and higher pressures respectively, and K is the moment of inertia of the suspended system, while k_1 and k_2 are the moments of inertia of the air carried along by the system at the two pressures, then

$$t_1 = 2\pi\sqrt{\frac{K+k_1}{c}} \quad \text{and} \quad t_2 = 2\pi\sqrt{\frac{K+k_2}{c}},$$

or since $t_2 - t_1$ is small compared to either t_1 or t_2,

$$k_2 - k_1 = \frac{2(t_2 - t_1)K}{t_1}.$$

From this expression and the numbers contained in the table above, we get the following results:—

	Difference of Pressure.	$k_2 - k_1$.	Moment of Inertia of Vibrating Body.	Increase of Moment of Inertia for 76 cm. Air-pressure.	Percentage increase of Moment of Inertia for 76 cm. Air-pressure.
Cradle	71·0 cm.	·073	70·88	·078	·11
Cradle + Bar No. 1	70·5	·288	615·68	·311	·05
Bar No. 1	...	...	544·80	·233	·04

As far as I am aware the only experiments on the effect of the inertia of the air carried by a swinging magnet on its period previously made are those of Lamont (*Pogg. Ann.* lxxi. p. 124, 1847). He found that the period of a cylindrical magnet 85 cm. long and weighing 8 grams* is given by

$$T_p = T\left(1 + \frac{p}{76} \times \cdot 00033\right),$$

where T_p is the period in air at a pressure of p cm. of mercury, and T is the period in a vacuum.

The above corresponds to a percentage increase of moment of inertia for 76 cm. of air-pressure of 0·066, which agrees in magnitude with the numbers I have obtained.

It is thus evident that the inertia of the air appreciably increases the period. This result will not influence the results obtained in the preceding part of the paper, since the bars were all of the same size and did not differ much in moment of inertia. The magnitude of the effect, however, was such as to make it worth while to investigate the effect of the air in the case of the magnets used in the Kew pattern unifilar magnetometer. Since the moment of inertia of the magnet is determined by means of a bar such as those which are being tested, when the effect of the air is neglected the moment of inertia deduced will be too small. The moment of inertia required is that of the magnet together with any air it may carry with it during its motion. Since, however, the inertia-bar itself will carry some additional air, if we assume that the inertia of the bar is that deduced from its dimensions, we shall underestimate the inertia of the magnet and the air which it carries.

To investigate this subject an exact copy of the magnet belonging to magnetometer No. 61 (Elliott Bros.) was made in brass, the moment of inertia of this dummy being almost exactly the same as that of the magnet, and this dummy magnet was swung in the instrument used for comparing the moments of inertia of the bars.

Three suspension-fibres of different diameters were used, that of intermediate thickness being chosen so that the period of the dummy was nearly the same as that of the magnet when swinging in the earth's field in England. The period of the dummy was determined both with and without the inertia-bar No. 1, and the results are collected together in the following table.

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* This corresponds to a diameter of about 0·4 cm.

	Period.	Temperature. °	Pressure. cms.	Logarithmic Decrement.
DUMMY ALONE.				
	2·5085	15·6	77·3	·0014
	2·5084	16·3	75·1	·0012
	2·5085	14·4	74·7	·0012
Means...	2·5085	15·4	75·7	·0013
	2·5074	15·4	5·8	·0006
	2·5071	16·4	5·7	·0006
	2·5072	16·5	5·5	
Means...	2·5072	16·1	5·7	·0006
DUMMY + BAR No. 1.				
	4·3527	16·5	75·8	·0012
	4·3528	16·5	75·8	
	4·3528	15·8	75·9	·0011
	4·3527	15·9	75·9	
Means...	4·3527$_5$	16·2	75·8$_5$	·0012
	4·3511	15·7	6·0	
	4·3510	15·7	5·8	·0007
	4·3512	15·7	5·9	
	4·3510	15·4	6·3	·0007
	4·3510	15·4	6·0	
Means...	4·3511	15·6	6·0	·0007
DUMMY ALONE.				
	9·4418	18·2	75·6	·0028
	9·4419	18·2	75·6	
	9·4419	17·2	75·4	·0029
	9·4419	17·3	75·4	
Means...	9·4419	17·7	75·5	·0028
	9·4361	16·9	6·4	·0021
	9·4356	16·9	6·3	
	9·4358	16·9	6·2	·0020
	9·4361	17·0	6·1	
Means...	9·4359	16·9	6·2	·0020

DUMMY + BAR No. 1.

	Period.	Temperature.	Pressure. cms.	Logarithmic Decrement
	16·3777	17·7°	74·9	·0023
	16·3779	17·7	74·9	
	16·3795	16·0	75·7	
	16·3795	16·0	75·7	·0023
	16·3792	16·0	75·7	
	16·3777	15·8	76·4	·0024
	16·3777	16·0	76·4	
	16·3778	16·1	76·4	·0023
Means...	16·3780	16·4	75·8	·0023
	16·3722	15·8	6·8	
	16·3715	15·8	6·8	·0020
	16·3723	15·8	6·9	
	16·3730	15·8	6·9	·0020
	16·3723	15·9	7·0	
	16·3722	15·8	6·9	·0020

DUMMY + BAR No. 1 (Box open at sides).

	Period.	Temperature.	Pressure. cms.	Logarithmic Decrement
	16·3814	16·8	76·7	·0026
	16·3808	15·1	76·8	
	16·3817	15·2	76·8	·0023
	16·3817	15·3	76·5	
	16·3820	15·5	76·5	·0023
Means...	16·3815 *	15·6	76·7	·0024
	16·3765	15·6	8·5	·0014
	16·3759	15·6	8·2	
	16·3763	15·6	7·9	·0014
	16·3763	15·6	7·7	
Means...	16·3762	15·6	8·1	·0014

DUMMY BAR No. 1 (in narrow box).

	Period.	Temperature.	Pressure. cms.	Logarithmic Decrement
	16·3784	15·9	76·6	·0093
	16·3786	15·9	76·6	·0090
	16·3778	16·1	76·6	·0090
Means...	16·3783 *	16·0	76·6	·0091
	16·3743	15·9	7·4	·0088
	16·3739	15·9	7·4	
	16·3740	15·9	7·4	·0087
	16·3734	15·5	8·2	·0087
	16·3732	15·5	8·1	·0089
	16·3734	15·5	8·1	
Means ...	16·3737	15·7	7·8	·0088

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* The inertia-bar was only adjusted in a symmetrical position by eye. This accounts for the variation in period and not the change in the box, as will be shown by some experiments mentioned later.

Dummy only.

	Period.	Temperature. °	Pressure. cms.	Logarithmic Decrement.
	14·6275	16·9	75·8	·0039
	14·6278	17·3	75·8	
	14·6273	17·7	75·8	·0038
	14·6275	17·9	75·8	
	14·6292	17·7	75·6	·0038
	14·6286	17·7	75·6	
	14·6292	17·8	75·6	·0038
	14·6290	17·8	75·6	
Means...	14·6281	17·6	75·7	·0038
	14·6215	17·3	6·6	·0028
	14·6192	17·3	6·4	
	14·6205	17·2	6·3	·0031
	14·6195	17·2	6·2	
	14·6196	17·2	6·2	·0028
	14·6194	17·2	6·2	
Means...	14·6198	17·2	6·3	·0029

Increase of Moment of Inertia due to the Air.

Dummy alone.

Period.	Diff. of Pressure.	$k_2 - k_1$.	Increase of Moment of Inertia for 76 cm. Air-pressure.	Condition of Box in which the System was swung.
2·51	70·0	·281	·305	Shut.
9·44	69·3	·362	·397	Shut.
14·62	69·4	·315	·345	Shut.
			Mean... ·349	
Dummy with Inertia Bar No. 1.				
4·35	69·8	·637	·693	Shut.
16·38	68·9	·618	·682	Shut.
16·38	68·6	·528	·594	Open.
16·38	68·8	·511	·511	Narrowed.
			Mean... ·620	

From these results it would appear that if the correction for the air effect varies with the period, the experiments are not of sufficient accuracy to detect it, at any rate over the range employed. Neither do they indicate clearly that there is any change with the size of the box containing the vibrating system, at any rate nothing of the same order as the change in the logarithmic decrement. Hence the mean of all the observations will be taken when calculating the correction to be applied to the moment of inertia of the dummy as deduced from observations of period with and without the inertia-bar made at ordinary atmospheric pressure.

Since it is the apparent moment of inertia of the magnet swinging in air which is required when determining H, no correction is required to eliminate the effect of the air carried by the magnet itself. When, however, the inertia-bar is used with the magnet in determining the moment of inertia of the magnet, the inertia of the bar is really greater than that calculated from its dimensions, on account of the air which it carries along with it as it swings. To obtain the moment of inertia of the air carried by the bar, we have to take the difference between the corrections for the dummy together with the bar and that for the dummy alone. Hence the moment of inertia of the bar must be increased by 0·271 grm.-cm.^{2} In the case of a brass bar 10 centimetres long and 1 centimetre in diameter, which will have a moment of inertia of about 550 grm.-cm.^{2}, the correction amounts to 0·049 per cent. That is, the calculated value of the moment of inertia of the magnet determined with such a bar, without taking any account of the air effect, will be 0·049 per cent. too small. If the horizontal component of the earth's field has the value 0·18, this corresponds to an error in H given by

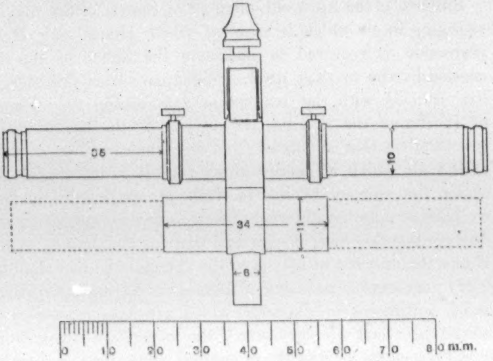
$$H = 0{\cdot}000044 \text{ C.G.S. or } 4{\cdot}4\gamma.$$

The above correction only applies to the form of magnet used in the particular pattern of unifilar, and of which the dimensions are given in fig. 2.

It will be noticed that when used in the instrument employed for comparing the moments of inertia of the bars, the correction to the moment of inertia of the bar 1 due to the air effect was 0·233 grm.-cm.^{2}; while when this bar is used with the dummy magnet, the correction comes out as 0·271 grm.-cm.^{2} A difference in the two cases is to be expected owing to the great difference in the shapes of the carriages which support the bar in the two cases, which will certainly modify the amount of the additional air carried when the bar is inserted.

Some experiments were made to see whether the period of the dummy magnet is the same when swinging in the box as when swinging in the open air. These were performed by taking the period with the dummy in the closed box, and also with the sides of the box removed. The mean of four determinations of period with the box closed was 9·4472, while the mean of four determinations of period with the box open gave 9·4474. It would thus appear that at any rate with a box of the size of that shown in fig. 1 the confinement has no appreciable effect on the period of the magnet.

Fig. 2.



§ 3. *Comparison of the South Kensington Bars with some belonging to the National Physical Laboratory.*

The National Physical Laboratory possesses a series of bars by different makers obtained for the purposes of comparison, and with a view of determining the degree of accuracy it was reasonable to expect in bars supplied with magnetic apparatus sent for test. I was enabled, through the kindness of the Director and Dr. Chree, to swing these bars in my instrument, and thus compare their moments of inertia with those of my bars. I had at first hoped by determining the dimensions of these bars, and hence calculating their moments of inertia, to obtain additional data for determining the moment of inertia of bar No. 10. On measuring the bars, however, I discovered that their form was so far from uniform as to make any argument based on the assumption that they were regular cylinders of relatively slight value. Thus in the case of bars 13, 14, and 15, the ends were inclined to the axis by at least 40′. In the case of bar No. 11 the diameter varied between ·9937 and ·9946. Bar No. 15 is the old bar belonging to the Kew Observatory standard instrument. Its diameter is very variable, being ·9962 cm. at the middle and near one end, and ·9934 cm. near the other. Although, on account of these irregularities, the numbers are of little value for determining the value of the moment of inertia of my standard bar, yet as the bars had been compared by the Observers at Kew with the standard magnet of their unifilar, all the bars were swung in the comparison apparatus, and the observed periods are given in the following table:—

No. 11.	No. 12.	No. 13	No. 14.	No. 15.
3·6841	3·6777	3·6998	3·6577	3·7925
3·6841	3·6777	3·6995	3·6577	3·7927
3·6841	3·6781	3·6998		3·7929
3·6838	3·6777			
3·6837				
3·6840	3·6778	3·6997	3·6577	3·7927

From these values and the moments of inertia calculated from the mean dimensions of the bars, the following values for $\frac{K_1}{t_1^2 - t_0^2}$ were obtained.

Number of Bar...	11.	12.	13.	14.	15,
Value of $\frac{K_1}{t_1^2 - t_0^2}$	45·936	45·934	45·937	45·955	45·971
Difference from mean (excluding bar 15)...	−18	+14	+11	−7	

Although, for the reasons given, these values are of no use as far as determining the value of the constant $c/4\pi^2$ to be used in determining the moment of inertia of my standard bar is concerned, yet they are of very considerable interest in that they show the kind of variation to be expected in the bars such as they have been turned out by makers hitherto, and incidentally the magnitude of the errors likely to arise if the moment of inertia of such bars is calculated from their dimensions, and such calculated moment used to determine the moment of inertia of the magnet.

Dr. Chree having been kind enough to supply me with the mean values of the moment of inertia of their standard magnet, as deduced by comparisons with these bars, also with the dimensions which they had used in calculating the moments of inertia, I have been able, taking 45·938 as the mean value of $c/4\pi^2$, to calculate from my comparisons of the Kew bars the moment of inertia of the Kew standard magnet. The results obtained for the moment of inertia of the magnet at a temperature of 0° C. by the intermediary of the different bars are shown below, no correction having been applied to allow for the effect of the air.

Number of Inertia Bar used.	11.	12	13.	14.	15.
Moment of inertia of the Kew magnet at 0° C.	268·61	268·47	268·61	268·61	268·54

In the case of the bar 15 the results are very doubtful, as the bar was damaged between the observations at Kew and those at South Kensington. Dr. Chree writes " . . . the bar suffered some abrasion . . . . subsequent to the swings whose results I send you, but prior to the time when it was sent to you." As a result of the abrasion the bar changed in weight from 68·770 grams to 68·756 grams. If, as is likely, the abrasion did not take place uniformly throughout the length of the bar, then it is impossible to allow for the effect it would produce on the comparison. If, for this reason, we omit the results with this bar, the mean value obtained for the moment of inertia of the Kew magnet at a temperature of 0° C. is

268·58.

Four of my own bars have been used by the Observatory for deducing the moment of inertia of the Kew magnet. The results obtained are as follows :—

Number of Inertia Bar used ...	1.	3.	8.	9.
Moment of inertia of the Kew magnet at 0° C.	268·48	268·65	268·42	268·41
Mean.........	268·49			

Taking the two sets of results together, the mean value of the moment of inertia at 0° uncorrected for the air effect is

268·54.

Or, applying a correction of 0·049 per cent., which has to be added on account of the air carried by the inertia-bar, the final value is

268·67.

DISCUSSION.

Dr. C. CHREE expressed his interest in the paper and remarked that the want of homogeneity in inertia bars had proved, on more than one occasion, a source of serious trouble in magnetometer work. The variation in density in an exceptional case might be as much as 3 or 4 per cent. He favoured the adoption of a standard bar which could be swung in various magnetometers. He expressed his interest in the size of the error which may affect the value of the horizontal force due to the increased moment of inertia caused by the air. A similar error entered into pendulum observations. With regard to the rigidity of quartz fibres, he was inclined to think that there would be a change in rigidity with a change in load.

Mr. W. A. PRICE asked if the difference in magnetic permeability between air and the material of the bars was sufficient to affect the period, and also if there were any other fibres the rigidity of which increased with rise of temperature.

Dr. R. T. GLAZEBROOK expressed his interest in the paper and said that he had followed the course of the work. In attempting to get an accurate value for the horizontal component of the Earth's field it was most important to check the homogeneity of the inertia bar. Various small corrections were applied in the determination of "H," and it was essential that the moment of inertia of the magnet should be known with great accuracy. Four or five years ago he suggested experiments similar to those described by Dr. Watson for measuring inertia bars, and although preliminary observations were made they were not carried far.

Dr. WATSON said he knew of no other substance but quartz which was suitable for the experiments he had performed. He had not found any variation of period due to magnetic permeability. With regard to Dr. Glazebrook's remarks, he said the measurement of the diameter of the bars was easy but that the determination of the length was much more difficult.

LIII. *The Electrical Resistance of a Conductor the Measure of the Current passing.* *By* W. A. PRICE, *Fellow Phys. Soc.*[*]

1. CHANGE in the length of a platinum wire is a convenient measure of the current, but the ordinary hot-wire instrument is not sufficiently precise for some purposes. The experiments in this paper were made by Mr. C. H. Wright, in Crompton & Co.'s works at Chelmsford, to ascertain whether the resistance of a conductor would be a better measure of the current than its length where considerable accuracy is required, the practical object in view being the calibration of alternating-current instruments. It was hoped to retain the advantage possessed by the dynamometer of calibration by direct currents, and thus of direct reference to the Clark cell and standard resistance ; to get the precision of other resistance measurements ; and to avoid the use of apparatus which is easily put out of adjustment.

2. To try whether the resistance of a conductor carrying alternating current could be easily measured, the connexions were arranged as in fig. 1. The mean fall of potential over R carrying a current from the alternating supply A was compared with that over S, a standard ; a small direct current being passed through the system from the accumulator H. The potential-differences of R and S were compared on the potentiometer. B, K were regulating resistances, and highly inductive coils at M, N the primaries of

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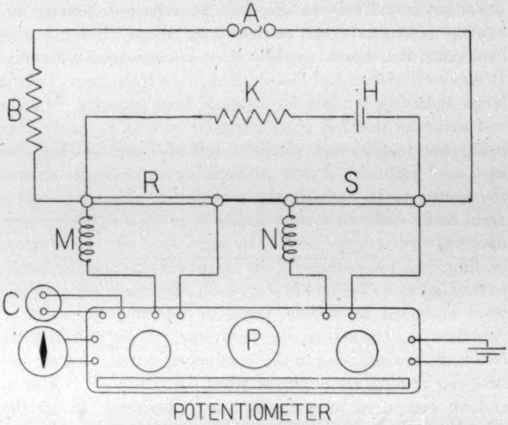
[*] Read May 12, 1905.

small high-voltage transformers reduced the alternating currents in circuits where only direct-current effects were required. If the alternating current in R be I, and the direct current *i*, the heat developed $\propto I^2 + i^2$; so that if I be large compared with *i*, say 100 times as great, the heating effect of *i* is negligible. The experiments of fig. 1, in which I was several hundred milliamperes, and *i* was 1 milliampere, were designed to show whether the resistance of R carrying alternating current could be measured by a superimposed direct current so small that it would not affect the heat developed. It was found that using the Chelmsford town-supply for the alternating source observations could be made without any difficulty, the widening of the galvanometer-line by the alternating current producing no appreciable inconvenience in reading. With the workshop supply readings were impossible, as the line was given a violent jerk every time a motor was started or stopped.

Fig. 1.



3. It was assumed that the temperature of R, a thin conductor, will be the same for a direct current as for an alternating current of the same root mean square; so that if R

can be constructed to measure direct currents, it can be calibrated with those, and used to measure alternating currents. All the subsequent experiments were accordingly made to ascertain the conditions for its use with direct currents.

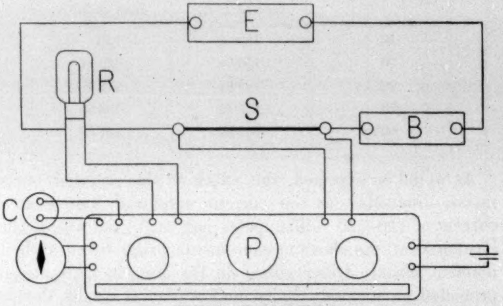
4. Silver was chosen for trial. Being a pure metal, its resistance changes rapidly with temperature, and its molecular condition is unlikely to change with repeated heating and cooling as might be the case with an alloy. Being a good conductor, it is heated rapidly by a low potential-difference. It was available as foil ·5 mil thick, in which form it has a large radiating surface for a small heat capacity. In the end we found this foil gave irregular results, probably from being too fragile, and platinum-foil of 1 mil in thickness was used instead. A few preliminary experiments showed the results to be exceedingly sensitive to draughts; and a great many different arrangements were tried to protect conductors from irregular cooling by currents of air. Conductors of thin foil were enclosed in large vessels and in small, packed in mica slips and in slag-wool. The containing vessels were protected in various ways, or immersed in water or paraffin-oil. The nett results were clear. If any considerable free body of air is near to the conductor, currents of air and irregular cooling are produced when the conductor is heated, and no consistent measurements are obtained. If, on the other hand, the containing vessel or any solid packing is near to the conductor, its heat capacity affects the temperature, which only slowly reaches a permanent value. These results indicate that the conductor must be contained in a relatively large vessel, exhausted of air, and suggest that an incandescent lamp may be suitable for small currents and large pressures, and a slip of metal foil for large currents and small potential-differences.

5. Experiments on the above lines were started on—

- A. Carbon-filament incandescent lamps.
- B. A platinum ligament in a metal box, exhausted by a Fleuss pump.
- C. A platinum ligament sealed into an exhausted glass bulb.

All these experiments were directed to the establishment of a relation between the resistance of the conductor and the direct current passing. The connexions are shown in fig. 2, where R is the conductor under test, shown, *e. g.*, as an incandescent lamp, S is a standard resistance of known value, B is a regulating resistance, E a storage-cell, and C a standard cell. The potential-differences of R, S, and the E.M.F. of C were compared on the potentiometer P for a series of different currents obtained by regulating B; and from the three readings the current through and the resistance of R are calculated. In each case there are four leads to the resistance, two carrying the current, and two which carry no current connected to the potential points, and the resistance measured is in every case that between the potential points. The potentiometer lends itself admirably to experiments of this class, since all the data are obtained from adjustments on one calibrated wire, while the whole of the conductor whose resistance is measured may be inside the exhausted vessel, so that the leading-in wires are not involved. With a bridge the current could not be measured with the resistance, while the resistance of leading-in wires would have to be taken into account.

Fig. 2.



6. The resistance of a 50 volt, 8 c.p. Stearn lamp was measured with a small testing-current immersed in a paraffin-bath warmed up to 56° C. to obtain an idea of the change of resistance with temperature. The mean change over a range of 40° C. was $-{\cdot}0224$ per cent. per degree. In the resistance was included in this case that of the leading-in wires; and it may be mentioned as a curiosity that a particular 4-volt

lamp tested at the same time showed a very nearly constant resistance over the same range, the increase of the platinum and the decrease of the carbon resistance just balancing one another. The combination of a carbon lamp and a metallic conductor might be useful for some purposes as having a small or negligible temperature-coefficient.

7. To determine the effect of external temperature a 50-volt 8 c.p. lamp was tested in a paraffin-bath, with the following results:—

Current in milliamperes.	Resistance in ohms.	
	Bath at 7° C.	Bath at 50° C.
10	160	159
30	150·35	149·97
50	142·84	142·6
70	137·14	136·83
89·98	131·96	131·82

As would be expected, the effect of the external temperature diminishes as the current employed increases. A current of 110–120 milliamperes just made the above lamp incandescent, the above measurements being taken with the filament black. Observations on the lamp in air, quite unprotected, gave closely similar and consistent results, showing that air-draughts outside an exhausted bulb are of no appreciable importance.

8. To ascertain how far the resistance of the lamp is affected by use, the following results were obtained, on a lamp when new, and after it had been run for 6 hours and 17 hours respectively at its full voltage of 50 volts.

Current in milliamperes	Resistance in ohms.		
	New in air at 11° C.	6 hrs. run, air 15° C.	17 hrs. run, air 15° C.
10	163	164·7	164·25
30	153·6	155·5	155
50	146·2	147·7	146·45
70	140·2	141·4	140·5
90	135·2	……	135·26

These experiments seem to indicate that these lamps might serve very usefully for measuring alternating currents from 50 up to a few hundred milliamperes, or even a thousand, but would not be instruments of much precision. The reason for the discrepancies was not then apparent.

9. Curves of three lamps of the same manufacture, taken to see how far they followed the same form, are given in

Fig. 3.

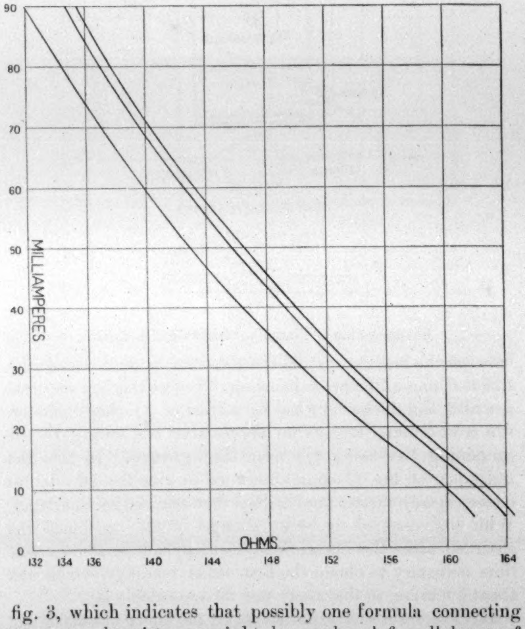
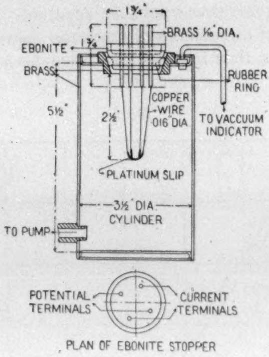


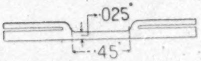
fig. 3, which indicates that possibly one formula connecting current and resistance might be employed for all lamps of one method of manufacture.

10. A platinum-foil ligament in a circular box, exhausted by a Fleuss pump, was then tried. The form of the ligament

and the arrangement of the box are shown in fig. 4. The joints between the rubber packing and the box, and also

Fig 4.


1¾"
BRASS 1/16" DIA.
EBONITE
1¾"
BRASS
RUBBER RING
COPPER WIRE ·016" DIA
5½"
2½"
TO VACCUUM INDICATOR
PLATINUM SLIP
3½" DIA. CYLINDER
TO PUMP
POTENTIAL TERMINALS
CURRENT TERMINALS
PLAN OF EBONITE STOPPER



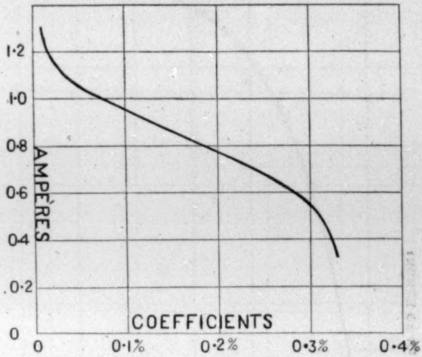
Enlarged Plan of Platinum Strip ·001 inch thick.

between the brass wires and the ebonite cap, were sealed with stiff machine-oil before exhausting. Before this box was used a smaller and thicker one had been tried, of 2 inches diameter, which was found to give an appreciable lag, owing to the containing box being too near the ligament. In this the resistance of the filament continued to rise for 20 minutes under a steady current, and had not then attained its maximum. With the vessel of fig. 4 no change in the resistance was observed after the current had been applied for the shortest time necessary to obtain the first set of readings, which was about $\frac{3}{4}$ minute, so that there was no appreciable lag.

11. The results of experiments on the effect of outside air temperature on the conductor of fig. 4 are given in fig. 5. The results were not very consistent, only a mercurial thermometer being used, and the exact position of the curve is a

little uncertain; but its character is well defined. In the diagram the abscissæ are the percentage corrections to be applied to the resistance for each degree by which the air temperature differs from 15°. The resistance thus corrected is the value for an air temperature of 15°.

Fig. 5.



12. The results of experiments on the degree of exhaustion of the vessel are given in the diagram of fig. 6. It will be noticed how rapidly the resistance for a given current, 1·2 amperes, air temperature 15° C., increases with the exhaustion when the latter is fairly perfect. This might form the basis of a useful gauge for high vacua. In the next set of experiments the vacuum was actually gauged in this way, the pumping being started before each set of readings and continued until the resistance of the strip for a given current, 1 ampere, corrected for the air temperature from the figures of fig. 5, took the same value as before. The curve at normal atmospheric pressures is too steep to be of any use as a barometer.

13. On the diagram of fig. 7 are plotted a number of observations made on four different occasions of the resistance of the platinum strip of fig. 4; the vessel being exhausted to about ·2 inch mercury, the exhaustion being fixed by the resistance at 1 ampere.

Fig. 6.

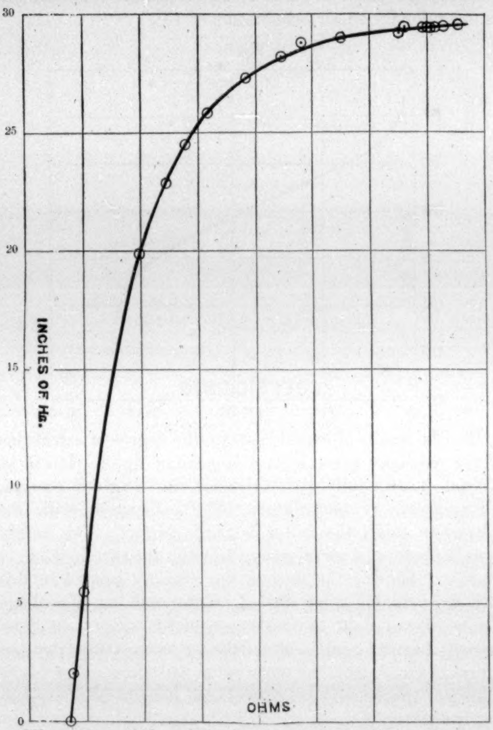
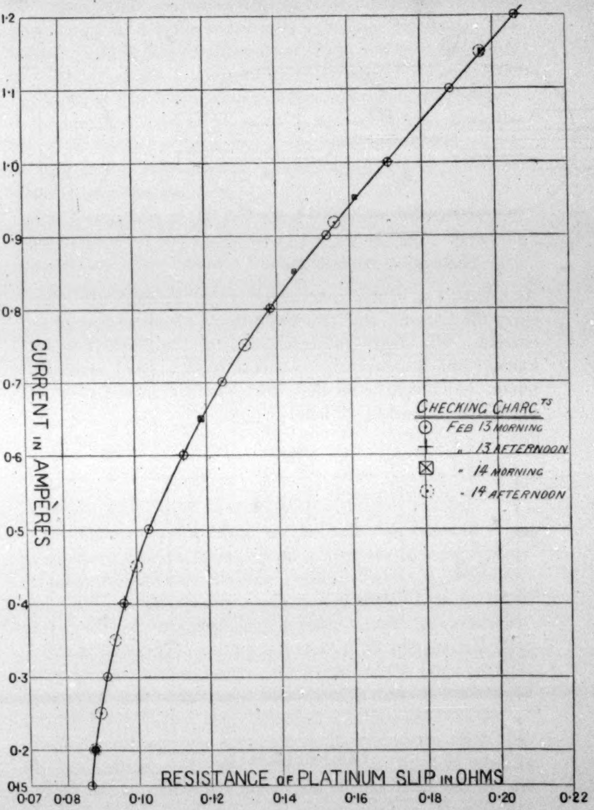
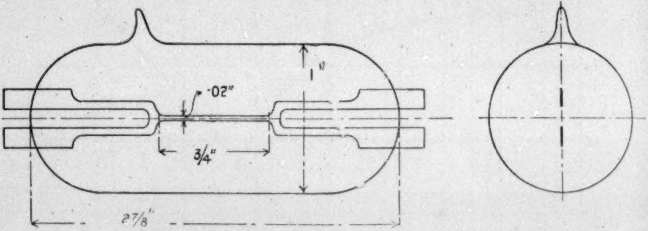

30
25
20
15
10
5
0
INCHES OF Hg.
OHMS
0·13
0·14
0·15
0·16
0·17
0·18
0·19
0·20

Fig. 7.


1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·15
CURRENT IN AMPÈRES
0·07
0·08
0·10
0·12
0·14
0·16
0·18
0·20
0·22
RESISTANCE OF PLATINUM SLIP IN OHMS
CHECKING CHARC.TS
⊙ FEB 13 MORNING
+ " 13 AFTERNOON
⊠ " 14 MORNING
⊙ " 14 AFTERNOON

14. Experiments were then carried out on a platinum-foil ligament sealed into an exhausted glass vessel shown in fig. 8.

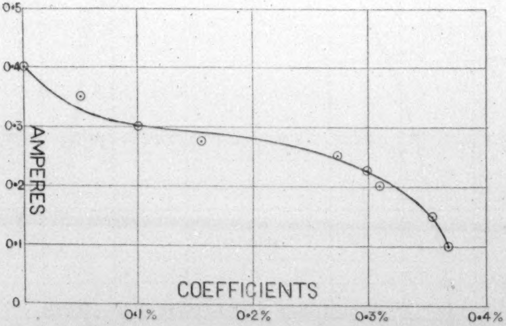
Fig. 8.



Platinum-foil ·001 inch thick.

As before, two of the connexions sealed through the glass carry the current, and two are potential-leads and carry no current. The degree of exhaustion of the vessel was not known; but it was a high vacuum from a good mercury-pump, and much better than that obtained in the vessel of fig. 4 from a mechanical hand-pump.

Fig. 9.



15. Fig. 9 gives the results of experiments on the effect of the air temperature on the resistance of the strip, the abscissæ being as in fig. 5 percentage corrections to reduce to an air

temperature of 15° C. Two points are noticeable in this diagram. The effect of the air temperature becomes negligible for much lower current-densities in the highly exhausted glass vessel than in the imperfectly exhausted brass one. The second point is that the extension of the curve to the x axis appears to indicate a temperature-coefficient of resistance for the metal of which the slip was composed, which was believed to be pure platinum, of about ·375, which is higher than the accepted figure; so that the exact position of this curve, though it was very carefully taken, could not be relied on without confirmation.

16. The figures of the following table are different sets of measurements of the resistance of the slip of fig. 8, at various currents and after different treatment.

Current in milliamperes.	Resistance in ohms.			
	A.	B.	C.	D.
·3	·32	·319	·318	·3155
·35	·368	·368	·363	·361
·4	·421	·420	·416	·413
·45	·479	·479	·474	·471
·5	·531	·534	·5295	·529

The figures A were taken as the bulb was received from the maker; B were taken ten days afterwards, a good many experiments with currents not exceeding ·5 ampere, alternating and direct, having been made in that time; C were taken shortly after B, the slip having been heated previously for one hour to a bright red by a current of ·9 ampere; D were taken after a further heating for $\frac{3}{4}$ hour of the same current, ·9 ampere. The resistance of the slip when cold was unchanged throughout. It had been raised to a bright-red heat before originally sealing into the glass bulb. It seems clear that the explanation of the changes between A, B, C, D must be looked for in the small quantities of gas occluded by the platinum before sealing in, and given off when subsequently heated *in vacuo*; and this effect may also explain the changes in the resistance of a carbon filament noticed in § 8 above.

If a conductor of this kind is to be used as a gauge for the measurement of current, either, as Prof. Callendar has suggested to me, the exhaustion of the vessel must be poor, so that the absorption or occlusion of gas by the platinum does not make any appreciable change in the amount in the vessel, giving the case of the results of § 13; or else the exhaustion and sealing must be performed when the platinum is at a bright-red heat, so that no occluded gases are left. Either course seems to promise a satisfactory result.

Discussion.

Prof. **Ayrton** pointed out that at any rate for small alternating currents a suitably wound instrument of the Kelvin-balance type was quite sufficiently accurate.

Mr. **Campbell** asked was any compensation introduced for the temperature of the walls of the enclosure, or was the temperature merely read and a correction made for it?

The **Author**, in his reply, said that in all cases corrections were applied for the external temperature.

LIV. *A Simple Method of Determining the Radiation Constant: suitable for a Laboratory Experiment. By* A. D. Denning, *M.Sc., Ph.D., Demonstrator in Physics in the University of Birmingham* *.

The following experiment was suggested to me by Prof. Poynting, F.R.S., as a laboratory experiment, and as it is by no means difficult to carry out and appears to give good results, it may be useful to give an account of it.

The principle of the method followed may be thus briefly described:—A hemispherical radiator, blackened inside, was quickly placed over a flat silver disk of known dimensions and which formed one of the junctions of a constantan-silver thermoelectric couple. From observations, at definite intervals of time, of the deflexions of the moving coil of a low-resistance d'Arsonval galvanometer included in the circuit, the initial rate of rise of temperature * of the disk was obtained and this result, when substituted in the equation (α), given below, gives σ the radiation constant.

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* Read May 12, 1905.

For, suppose m = mass of silver disk,

s = its specific heat,

dT/dt = initial rate of temperature change,

then $ms \cdot dT/dt$ = initial amount of heat received by disk.

But if A = exposed area of disk,

R = radiation from that area per unit of time,

and R_1 = radiation from blackened hemisphere,

then the gain of heat by the plate

$$=(R_1-R)A,$$
$$=ms \cdot dT/dt.$$

Using Stefan's Law, we may write:

$$R=\sigma T^4 \quad \text{and} \quad R_1=\sigma T_1^4$$

where σ = the radiation constant,

T & T_1 = the temperatures of disk and hemisphere, respectively, measured from −273° C.

Consequently

$$ms \cdot dT/dt = A\sigma(T_1^4-T^4),$$

or

$$\sigma=\frac{ms}{A(T_1^4-T^4)}\cdot\frac{dT}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

If the galvanometer readings were proportional to the current existing at the instant of reading, we should have dT/dt by simply noting successive deflexions at known times. But, owing to the moment of inertia of the moving part of the galvanometer and the viscous resistance, the reading will not always indicate the current at the instant, as will be seen from the investigation below (Appendix). In this case, however, the readings did indicate the temperature, when plotted to eliminate oscillations.

The actual arrangement of the apparatus is shown diagrammatically in fig. 1. A is a cylindrical vessel of

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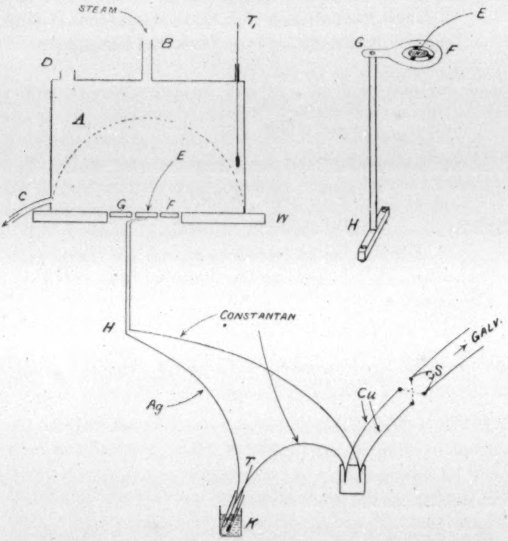
* The initial rate of temperature rise being taken to avoid gain of heat by disk from conduction and convection effects.

15·3 cms. internal diameter: to the underneath side of this vessel was soldered a polished copper hemisphere, indicated in the figure by the dotted line, and which, before an experiment, was evenly coated with a layer of lampblack produced by the combustion of camphor. The space between the cylindrical and hemispherical vessels was used as a reservoir for steam or the other substances which were

Fig. 1.



employed to maintain the hemispherical radiator at the temperature T_1. The tube B served as an inlet and the tube C as an outlet for the steam; the wider tube, D, was subsequently added for the purpose of covering the hemispherical radiator with ice. During the course of a series of observations this part of the apparatus rested on the wooden board, W. (The stands supporting this latter are not shown in the diagram.)

As mentioned above, the elements of the thermo-junctions were silver and constantan. In order to have an appreciably large surface to receive the radiation at the one junction, the silver and constantan wires were soldered on to a silver disk, E. This disk, after having been thoroughly well polished, was mounted in a vulcanite frame, FG, which fitted into the wooden board carrying A. The disk also was lamp-blacked before being used. If it be considered desirable to prevent the disk receiving radiation at the sides, felt or some other non-conducting material might be so wrapped round the disk that only the top surface of it was exposed to the hemispherical radiator. The wires coming from the disk were led down the arm GH to the binding-screws at the bottom. The other junction of the thermo-couple was placed, together with a thermometer, T, in a test-tube containing oil in the glass beaker, K. The two constantan wires were further soldered to copper wires leading through a mercury commutator to the galvanometer. These last two (the Cu-constantan) junctions were passed through rubber tubing fitting into two holes in the lid of a tin canister packed with cotton-wool.

The dimensions of the silver disk were:

Diameter $=2{\cdot}015$ cm. Weight$=8{\cdot}911$ gr.
Thickness$=0{\cdot}275$ cm. Sp. Ht. $=0{\cdot}0567$.

This value for the specific heat was the mean of several determinations made by means of a Joly steam calorimeter.

Inasmuch as the source of steam and also the outlet from the vessel A were on the side CD of the apparatus, a long piece of sheet nickel, polished on the one side, was tacked on to the wooden stand W, with the idea of screening the underside of the silver disk from the hot air currents. It was also found advisable to further protect the disk from draughts by loosely fixing under it a wide pad of cotton-wool, held in position by a wire passing between the retort stands supporting W, since otherwise the opening and shutting of doors, &c., was sufficient to occasion an irregular and jerky motion of the galvanometer-coil.

During the boiling of the water, the silver disk was covered by a screen with one edge resting on two thin corks to prevent actual contact between the disk and the screen. This screen was made by taking a piece of sheet nickel, approximately twice the size of the board W, doubling it into two with the dull side inwards and, after placing a layer of cotton-wool between the folds, wiring the two folds loosely together.

When steam had passed for some time through the vessel A, the latter was placed on the screen. At a particular instant the screen was removed, A was lowered over the disk and some six to ten readings of the position of the cross-wire image were taken at intervals of five seconds (in the majority of cases).

In order to find the temperature equivalent of the galvanometer deflexions, the silver disk was kept at a constant temperature, and the deflexions of the galvanometer were noted when the temperature of the other junction was altered by a known amount—subsequent reference is made to this point.

Before beginning an experiment it is necessary to ascertain that the two thermo-junctions are at the same temperature or to know the difference of temperature between them. In these experiments the following procedure was adopted:—By insertion of the shunt-key S the galvanometer was short-circuited and its zero-position found. Did the deflexions, consequent to the removal of the key, indicate that the temperature of the disk was too low, the hand was held over the disk until the cross-wire image returned to its initial position of rest; whilst if the temperature were too high, ether was poured on to a piece of cotton-wool resting on a crucible lid, and this was allowed to evaporate in the neighbourhood of the disk.

The results of a number of experiments extending over varying ranges of temperature are given in Table I. In the last column but one are given the values of the initial rate of change of temperature, whilst the last column contains the values found for $\sigma . 10^5$. It will be noticed that in some of these experiments alcohol and acetone were passed through the radiating vessel and afterwards condensed in a condensing worm (not shown in fig.). But subsequent experiments

TABLE I.

	Date.	Temperatures.		dT/dt.	$\sigma . 10^5$.
		T_1°.	T°.		
Steam	29/11/04	100°·0 C.	18°·7 C.	9·44	5·19
	12/12/04	99·5	15·6	9·71	5·30
	12/12/04	99·5	16·3	9·55	5·22
	17/12/04	98·5	17·4	6·47	3·36
Alcohol	19/12/04	78·6	14·8	4·95	3·92
	19/12/04	78·6	9·2	7·25	5·39
	20/12/04	78·6	14·5	6·71	5·32
Acetone	20/12/04	57	12·8	4·24	5·43
	20/12/04	57	13·1	4·24	5·46
Ice	21/12/04	0	15·2	1·06	5·24
	21/12/04	0	14·7	1·06	5·42

(results recorded in Table II.) showed that easier manipulation and equally satisfactory results were to be obtained by simply placing hot water in A; owing to the large bulk of water which A could hold (nearly two litres), its rate of cooling was comparatively slow; whereas the observations necessary for each experiment were made in less than a minute.

TABLE II.

Date.	Temperatures.		dT/dt.	$\sigma . 10^5$.
	T_1°.	T°.		
27/1/05	100°·5 C.	16°·6 C.	9·83	5·26
27/1/05	100·5	17·8	9·83	5·29
28/1/05	75·0	17·8	5·99	5·30
28/1/05	73·0	17·8	5·76	5·32
28/1/05	60·5	17·6	4·32	5·49
28/1/05	60·0	17·8	4·32	5·39
28/1/05	45·0	15·5	2·64	5·32
28/1/05	43·8	15·8	2·40	5·40
31/1/05	0	15·7	1·08	5·14
31/1/05	0	14·9	1·08	5·48

By quickly removing the radiator after sufficient readings had been taken and placing the cotton-wool saturated with ether near the disk, it was possible to take several sets of observations in fairly quick succession—especially if a current of air were maintained across the disk. Owing to the small range of temperature prevailing when the ice was used and the possibility of disturbances due to convection, such close agreement with the other results as was actually obtained was hardly to have been expected. It has been thought advisable to include all the experiments made to indicate what kind of accuracy can be expected.

A glance at the values for σ given in the last columns of both tables will at once show that two of the values are very much lower than the other nineteen. The most probable explanation for this is that somewhere in the galvanometer-circuit there was poor contact and a consequent increase in the circuit-resistance—most likely in the mercury commutator, since these at times are very unreliable. Indeed, determinations of the equivalent of the galvanometer readings made from time to time during the course of the first series of experiments showed variations in the sensibility of the galvanometer. Hence it were better to measure the sensitiveness of the galvanometer directly before taking a series of observations by arranging that the simple reversal of a commutating-switch might place the galvanometer in circuit with a low resistance, the fall of potential along which is a constant small fraction of the E.M.F. of a storage-cell.

Ignoring these two values, we find uniformity and agreement among the remainder—the average of the residual nineteen results being about 5·3. The mean of Kurlbaum's more elaborate experiments was 5·32. The simplicity of construction of the apparatus and the principles embodied in its use will, it is hoped, recommend the experiment as a laboratory method of illustrating an important and fundamental law.

APPENDIX.

Note on Motion of Galvanometer Coil.

In order to take into account the effect of the moment of inertia of the moving coil and the viscous resistance of the medium, let us suppose that a steady difference of 1° C. between the thermo-junctions gives a steady deflexion of the galvanometer-coil of λ radians, then a steady difference of temperature of T' would give a steady deflexion

$$\theta=\lambda T'.$$

If the torsional couple per radian exerted by the suspension is μ, then the couple on the fibre in a steady state would be

$$\mu\theta=\mu\lambda T'.$$

But the observations are taken during the motion of the galvanometer-coil, *i. e.* when the change of temperature is not steady.

In this case

$$\mu\theta=\mu\lambda T-I\frac{d^2\theta}{dt^2}-K\frac{d\theta}{dt};$$

where $I=$ the moment of inertia of coil,

$K=$ the viscosity coefficient of the air,

$T=$ the initial difference of temperature;

that is

$$I\frac{d^2\theta}{dt^2}+K\frac{d\theta}{dt}+\mu\theta=\mu\lambda T;$$

putting $K/I=k$ and $\mu/I=n^2$,

the equation of motion becomes

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+n^2\theta=n^2\lambda T.$$

Now, if T be approaching a steady value A, according to the exponential law we have

$$T=A(1-e^{-pt});$$

consequently

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+n^2\theta=n^2\lambda A(1-e^{-pt}),$$

the solution of which gives

$$\theta=\lambda A\left(1-\frac{n^2e^{-pt}}{p^2-kp+n^2}\right)-Be^{\frac{-kt}{2}}\sin\left(\sqrt{n^2-\frac{k^2}{4}}\,.\,t+\alpha\right).$$

The second term is evidently oscillatory so long as $n^2>\frac{k^2}{4}$; but if k be large it dies away rapidly, or to such an extent that it can be eliminated by plotting, and we have only to deal with

$$\theta=\lambda \mathrm{A}\left(1-\frac{n^2e^{-pt}}{p^2-kp+n^2}\right)$$

differentiating

$$\frac{d\theta}{dt}=\lambda p\mathrm{A}e^{-pt}\cdot\frac{n^2}{p^2-kp-n^2};$$

but from the differentiation of the exponential equation, given above,

$$\frac{d\mathrm{T}}{dt}=p\mathrm{A}e^{-pt},$$

and since we only want the initial rate of rise of temperature

$$\frac{d\mathrm{T}}{dt_{t=0}}=[p\mathrm{A}e^{-pt}]_{t=0}=p\mathrm{A},$$

but

$$\frac{d\theta}{dt_{t=0}}=\frac{\lambda p\mathrm{A}n^2}{p^2-kp+n^2}.$$

Hence

$$\frac{d\mathrm{T}}{dt_{t=0}}=p\mathrm{A}=\frac{p^2-kp+n^2}{n^2}\cdot\frac{d\theta}{dt}\Big/\lambda.$$

By plotting galvanometer deflexions against the times, we can eliminate the oscillations and obtain the initial rate of change of θ, *i. e.* $\frac{d\theta}{dt}\Big/\lambda$, from the curve. (And since we are only concerned with the ratio of $\frac{d\theta}{dt}\Big/\lambda$ we can use galvanometer scale-divisions instead of radians.)

Consequently, then, in order to obtain the initial rate of rise of temperature, the observed $\frac{d\theta}{dt}\Big/\lambda$ must be multiplied by

$$(p^2-kp+n^2)/n^2.$$

Should p^2-kp be small in comparison with n^2, this fraction may evidently be neglected. But to test this we must first find n, p, and k for the particular galvanometer in use.

(i) To find n :

If the circuit is open, k is negligible and $2\pi/n$ is the time of swing. Or, if the observed period of free swing be P then

$$n=2\pi/\mathrm{P}.$$

(ii) To find k:

For the deflexion θ on closed circuit we have

$$\theta = \mathrm{B}e^{\frac{-kt}{2}}\sin\left(\sqrt{n^2-\frac{k^2}{4}}\,.\,t+\beta\right),$$

and if successive deflexions or elongations θ_1 θ_2, θ_3, and θ_4 be observed it is easily seen that

$$\log\frac{\theta_1-\theta_2}{\theta_3-\theta_4}=\frac{k\pi}{\sqrt{n^2-\frac{k^2}{4}}}.$$

(iii) To find p:

From the exponential law we have

$$\frac{d\mathrm{T}_1}{dt}=\mathrm{A}pe^{-pt_1},$$

and therefore

$$\frac{d\theta_1}{dt}\propto \mathrm{A}pe^{-pt_1}=\mathrm{A}_1pe^{-pt_1}\ \text{say}.$$

Similarly for time t_2 (where $t_2=2t_1$)

$$\dot{\theta}_2=\mathrm{A}_1pe^{-pt_2}=\mathrm{A}_1pe^{-2pt_1},$$

that is

$$\dot{\theta}_1/\dot{\theta}_2=e^{pt_1},$$

or

$$pt_1=\log\dot{\theta}_1/\dot{\theta}_2.$$

But by plotting the deflexions against the times we get the deflexion curve $\theta=\mathrm{A}_1(1-e^{-pt})$,

and from this curve we may measure $\dot{\theta}_1$ and $\dot{\theta}_2$ for a pair of times such that $t_2=2t_1$, and consequently we find

$$p=\frac{1}{t_1}\log\frac{\dot{\theta}_1}{\dot{\theta}_2}.$$

With the galvanometer used in these experiments the following values were obtained for these quantities:

$$p=0{\cdot}0069;\quad n=0{\cdot}654;\quad \text{and}\quad k=0{\cdot}127,$$

which, when substituted, give

$$\frac{p^2-kp+n^2}{n^2}=\frac{427}{428},$$

which may be taken as 1, that is to say, the observed value of $\frac{d\theta}{dt}\Big|\lambda$ could be taken as a measure of the initial rate of rise of temperature of the silver disk.

Discussion.

Prof. Callendar asked what steps had been taken to protect the area of the edges of the silver disk, which appeared to be about half the area of the upper surface, from the oblique radiation of the hemisphere. The upper surface alone was taken as the effective area in the calculation. He had found this difficulty in applying the Crova disk method to the measurement of radiation in a previous case, although the radiation was nearly parallel and normal to the disk, instead of being distributed over a hemisphere. He thought that greater sensitiveness would be obtained by using a single short constantan wire for the couple. It was necessary that this wire should be exceedingly fine to prevent error in the temperature-measurement owing to cooling of the end near the junction by conduction. A copper disk appeared preferable to a silver disk as reducing the number of junctions of different metals. The constantan wires used by the Author did not appear to be fine enough to eliminate this source of error, which, however, would tend to compensate the error introduced by the neglect of the area of the exposed edges of the disk.

Mr. Campbell asked whether the convection-currents do not introduce considerable error. The effect of the rim of the disk could be made of relatively less importance by increasing the diameter of the disk.

The Author, in reply, said that from the form of the radiator the amount of radiation received by the edges of the disk was only a very small fraction of that received by the horizontal surface, whilst the method of placing the radiator over the disk would probably not cause any great changes in the condition of the air directly under the radiator.

LV. *On the Ratio between the Mean Spherical and the Mean Horizontal Candle-Power of Incandescent Electric Lamps.* *By* J. A. FLEMING, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London*[*].

A PAPER was read before the Physical Society on November 11th, 1904[†], by Mr. G. B. Dyke, B.Sc., in which, amongst other matters, he gave the results of numerous measurements made in the Pender Electrical Laboratory of University College, London, of the ratio of the mean spherical candle-power of incandescent electric lamps to the mean horizontal candle-power, taken when the lamp was rotating round a vertical axis. In all cases the ratio of the mean spherical to mean horizontal candle-power was experimentally found to be a number near to 0·78 for about nine different types of electric glow-lamps. This constant ratio must depend upon definite optical facts, and cannot be a matter of accident. That this is the case can be shown by considering a typical instance.

Let us suppose a short straight filament *a b* (fig. 1) rendered incandescent to be placed vertically in the centre of a sphere described by the revolution of a semicircle P H Q round a diameter P Q, coinciding with the direction of the filament. Let the length of the filament be small compared with the radius $FH=r$ of the sphere. Let I_H represent the intensity of the light emitted or the candle-power of the filament in a horizontal direction F H, and let I be the intensity in any direction FN making an angle $NFH=\theta$ with the horizontal direction. Let I_S be the mean spherical candle-power.

Let us then start with the assumption that $I=I_H\cos\theta$. The justification for this is found in the fact that if a long straight incandescent filament is placed behind a small rectangular-shaped hole in a metallic plate, then inclining the filament to the plane of the plate within certain angular limits makes no sensible difference in the quantity of light which comes through the hole*. Consider then a zone of the sphere swept out by an elementary arc NN′ of the semicircle. The surface of this zone is $2\pi r^2 \cos\theta\, d\theta$, where $d\theta$ denotes the angle N′FN.

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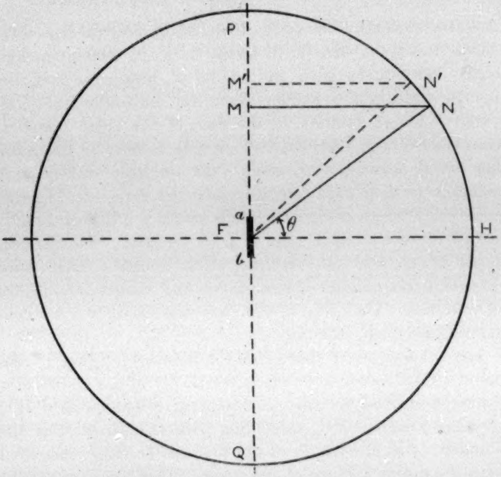
[*] Read June 16, 1905.

[†] See Mr. G. B. Dyke, "On the Practical Determination of the Mean Spherical Candle-Power of Incandescent and Arc Lamps," Phil. Mag. ser. 6, vol. ix. p. 136, Jan. 1905; also Proc. Phys. Soc. vol. xix. p. 399 (1905).

Fig. 1.



Hence the quantity of light falling on the whole zone is $I_H 2\pi r^2 \cos^2\theta\, d\theta$, and the total quantity of light incident on the entire surface of the circumscribing sphere is

$$4\pi r^2\, I_H \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta.$$

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* For the limits within which the above statement is confirmed by experiment the reader is referred to the Appendix to this Paper.

Hence by the definition of mean spherical candle-power I_S we have

$$4\pi r^2 I_S = 4\pi r^2 I_H \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta,$$

or

$$\frac{I_S}{I_H} = \int_0^{\pi} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{\pi}{4} = 0{\cdot}785.$$

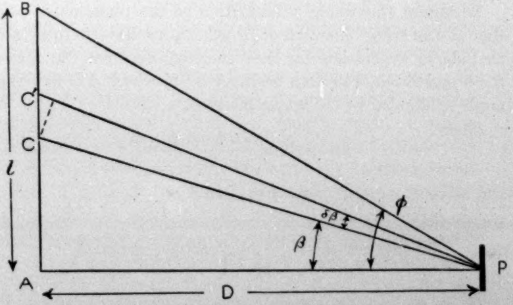
Accordingly for such a short straight filament the ratio experimentally found by Mr. Dyke, viz., $I_S/I_H = 0{\cdot}78$, holds good.

Conversely, the fact that experiment shows that for any incandescent lamp $I_S/I_H = \pi/4$ nearly, is a proof that the cosine law of luminous radiation from the filaments in different direction holds good, with considerable exactness.

The departure from the above rule when the filament is not short compared with the radius of the sphere can be found as follows:—

Consider a straight incandescent filament of length $2l$ the half of which is represented by the line AB (see fig. 2) placed

Fig. 2.



at a distance AP = D from a photometer-disk P, the filament being parallel to the photometer-plane. Let an element of length at C be denoted by dl. Let the angle APB subtended at

P by half the length of the filament be denoted by ϕ, and let the angle APC be denoted by β. Let the intensity of the light sent out or the candle-power *per unit of length* of the filament in a direction normal to itself be denoted by I_H, then the illumination on the photometer-disk, assuming it to follow the above mentioned cosine law, due to an element dl of the filament at C subtending an angle $d\beta$ at P will be given by the expression

$$\frac{I_H \cos^2 \beta dl}{(CP)^2} = \frac{I_H \cos^4 \beta dl}{D^2}.$$

Also from the geometry of the figure we have

$$dl = \frac{D \,.\, d\beta}{\cos^2 \beta}.$$

Hence the illumination on the photometer-disk due to the element dl of the filament at C is given by the expression

$$\frac{I_H D \,.\, \cos^2 \beta d\beta}{D^2}.$$

This is the same illumination as that which would be given by a vertical element of length $Dd\beta$ placed at A, but having a horizontal intensity or horizontal candle-power per unit of length equal to $I_H \cos^2 \beta$.

To obtain the whole illumination on the photometer-disk due to the whole filament of length $2l$, we have to integrate the above expression for one element, between the limits $\beta=0$ and $\beta=\phi$, and then multiply by 2, where ϕ is half the angle subtended by the whole filament.

Since
$$\int_0^\phi \cos^2 \beta d\beta = \frac{\sin \phi \cos \phi}{2} + \frac{\phi}{2};$$

and since
$$l \cos \phi = D \sin \phi,$$

we see that

$$2\int_0^\phi \frac{I_H D \cos^2 \beta d\beta}{D^2} = \frac{I_H}{D^2} 2l \left(\frac{\cos^2 \phi}{2} + \frac{\phi \cot \phi}{2} \right).$$

Hence the correcting factor when photometering a filament, the apparent length $2l$ of which subtends a finite angle 2ϕ at the photometer-disk, is $\frac{1}{2}(\cos^2 \phi + \phi \cot \phi)$. Accordingly the ratio of mean spherical to mean horizontal candle-power in this last case is given by the expression

$$\frac{I_S}{I_H} = \frac{\pi}{4}\left\{\frac{2}{\cos^2\phi + \phi\cot\phi}\right\}.$$

If the filament is not a straight filament but a loop as in an actual lamp, which may be so coiled that the different portions lie in different planes, we have an additional small correction to make. We may first notice the effect of the inclination of a straight filament in various directions.

Let three coordinate planes be drawn through the centre of the sphere of reference. Let one of these planes be horizontal. Let the photometer-disk be placed with its plane vertical. Let two other planes be described through the centre of the sphere which are respectively parallel and perpendicular to the photometer-disk plane.

Then consider a straight filament lying in the plane, perpendicular to the photometer-disk and having an inclination to it.

It is clear that if the cosine law of radiation holds good, the light sent out by each element of the inclined filament in a horizontal direction will be the same as that of its projection on a plane parallel to the photometric disk. Hence we may substitute for the inclined filament its projection on a vertical plane, and if we assume the filament and its projection have the same candle-power per unit of length, the horizontal illuminating power will be the same in the two cases. If, therefore, the filament lies in any plane we may consider it divided into equal elements of length, each of which makes its own contribution to the mean spherical and the mean horizontal candle-power. Disregarding for the moment the effect of distance from the horizontal plane on the illuminating effect of each element on the photometer-disk, we see that the ratio of the mean spherical to the mean horizontal candle-power must be greater for each element of length of the filament the more it is inclined to the vertical plane or the less its inclination to the line joining it to the photometer-disk.

Hence if the whole filament lies in one plane placed perpendicularly to the photometer axis the ratio I_S/I_H must be $\pi/4$. If it does not lie wholly in that plane the ratio will be somewhat greater.

In the case of filaments coiled in various ways a small correcting factor is also necessary, providing for the reduced illuminating effect of each equal element of the filament, into which the filament may be considered to be divided, the further the element is removed from the axis of the photometer, that is, from the horizontal plane. In the case of simple horseshoe-shaped filaments, so placed that the whole filament is visible, this correcting factor will be the factor

$$\frac{2}{\cos^2\phi + \phi\cot\phi},$$

where ϕ is half the angle subtended by the whole filament at the centre of the photometer-disk.

If the filament has a double or multiple loop, then the precise correcting factor for angular magnitude of the whole loop is not quite so sharply defined. It will, however, always operate to increase the fraction or percentage which represents the ratio of mean spherical to mean horizontal candle-power. It will always be a small correction if the whole filament subtends an angle, say, of not more than 10° at the centre of the photometer-disk, as in the ordinary photometry of an incandescent electric lamp.

These considerations show that incandescent lamp manufacturers can without much difficulty furnish the figures required to give the true photometric efficiency of any type of lamp. Instead of stating, as at present, the value of the maximum candle-power taken in one direction, viz., the direction perpendicular to the axis of the lamp in which the whole of the filament is seen exposed, they should furnish the mean horizontal candle-power taken in the same direction when rotating the lamp around its vertical axis, using for this purpose well-known mechanical devices.

If this reading is taken at such a distance from the lamp that the whole filament does not subtend a greater angle than 5° to 10° at the photometer-disk, then the mean spherical candle-power of the incandescent lamp is for practical purposes very nearly obtained by multiplying this mean horizontal candle-power by the factor $\pi/4$. From the measurement of the electric power in watts supplied to the lamp, we have at once the means of calculating the total flux of light per watt which fixes scientifically the efficiency of the lamp as an energy-transforming device.

APPENDIX.

To enable a judgment to be made as to the validity of the assumption that the intensity of the light proceeding from a filament in any direction varies as the cosine of the angle between the ray and the normal to the element, the following experiments were tried by Mr. G. B. Dyke at the suggestion of the writer.

A tubular incandescent carbon-filament lamp was prepared with a straight filament about 10 inches in length. This was placed on a photometric gallery at a distance of about 40 inches from a Lummer-Brodhun photometer, and two baffle-screens interposed having circular holes in them about 1 inch in diameter. These screens were placed so as to divide the space between the photometer and the lamp into three nearly equal parts.

The tubular lamp had a protractor attached to it, and could be tilted over at various angles to the vertical line, keeping it in one vertical plane normal to the photometer-disk. It is obvious that if the illumination sent out from unit of length of the straight filament obeys the cosine law above mentioned, then tilting the lamp as described to various angles should make no difference in the observed illumination on the photometer. As a matter of fact, the observed illumination decreased rapidly as the lamp was tilted over, in accordance with the figures in the following table.

Angle between the straight filament and the plumb-line through its centre.	Observed candle-power or illumination through the two baffle-screens.
0°	2·33
15	2·27
30	2·23
45	2·15
60	1·96
75	1·06

It is clear, however, that some part of the decrease of light at high inclinations is due to the loss of light by reflexion at the inner surface of the tubular glass bulb. To obtain the correction for this reflexion a sheet of glass was interposed between an ordinary lamp and the photometer, the plane of the glass being tilted at various angles to the plane normal to the ray. The photometric readings were as follows :—

Angle at which sheet of glass was placed to the plane normal to the ray.	Observed intensity of the light.
0°	26·1 c.p.
15	25·9 „
30	25·8 „
45	25·2 „
60	25·4 „
75	16·7 „

It follows therefore, that if the carbon filament in the tubular lamp could have been used without a glass envelope, the light sent by it through the baffle-screens would have been greater in the ratio of the first term in the right-hand column of the previous table to any succeeding term in the same column corresponding to the same angle of inclination of the filament.

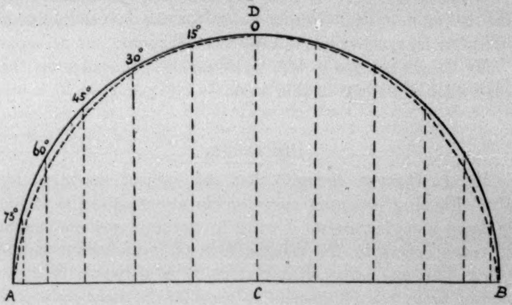
If therefore we call the intensity of the light sent through the screens unity when the filament is vertical, then its true intensity corrected for reflexion when the lamp is tilted to various angles is as follows :—

Angle of Inclination.	Corrected Value of Intensity of Light emitted per unit of length.
0°	2·33 × 26·1 ÷ 2·33 × 26·1 = 1·00
15	2·27 × 26·1 ÷ 2·33 × 25·9 = ·98
30	2·33 × 26·1 ÷ 2·33 × 25·8 = 1·00
45	2·15 × 26·1 ÷ 2·33 × 25·2 = ·96
60	1·96 × 26·1 ÷ 2·33 × 25·4 = ·86
75	1·06 × 26·1 ÷ 2·33 × 16·7 = ·71

The figures in the last column therefore represent the intensity of the light sent out by unit length of the incandescent filament at various angles of the normal. If the cosine law were strictly fulfilled these values should all be unity.

The easiest way to determine the effect of this departure from the true cosine law on the mean spherical candle-power, is to describe a semicircle (firm line) of unit radius and to set off on its diameter distances from the centre C (see fig. 3) proportional to the sines of the angles 0°, 15°, 30°, &c., and then through these points to draw ordinates (dotted lines) to the diameter of the semicircle. Fractions of these ordinates equal to the fractions in the last column of the above table are then set off, and the upper ends of these define a curve (dotted line) which is the Rousseau diagram of the photometric curve of luminous radiation of the filament.

Fig. 3.



If the value of the mean ordinate of this last (dotted) curve is taken, it gives us the ratio of the mean spherical to the mean horizontal candle-power of the straight filament. For the (dotted) curve as drawn in fig. 3 delineating the observations made with the above-mentioned carbon filament, this mean ordinate or corrected ratio comes out 0·77 instead of 0·785 as it should do if the cosine law had been strictly fulfilled. In this case the dotted line would have fallen on the semicircle and coincided with it.

Another similar set of measurements was made with a straight oxide filament taken from a Nernst lamp used without heating-coil, and with the supports of the filament all carefully painted dead-black to avoid errors due to reflected light. In this last case no correction due to loss of light by reflexion from a glass envelope was needed, as the Nernst filament was used in the open air. The ratio of the mean spherical to mean horizontal filament in this last case was experimentally found to be 0·785, or exactly the value it should have if the cosine law is obeyed.

We have therefore in these experiments a justification for the assumption made in this paper for the law of luminous radiation from a straight incandescent filament.

My thanks are due to Mr. Dyke for his assistance in the experimental portion of this work.

Discussion.

Mr. L. **Gaster** thought that the subject discussed by Prof. Fleming deserved considerable attention at the hands of lamp manufacturers. Owing to the improvements made in recent years in the manufacture of incandescent lamps using filaments other than carbon, it is advisable in comparing their relative merits to consider the mean spherical candle-power given out, and to determine the total flux of light per watt consumed thereby conveying an accurate meaning as to the efficiency of the lamp as an energy transforming device.

Dr. **Chree** pointed out that the quantity $\frac{1}{2}(\cos^2\phi+\phi\cot\phi)$ was based on a calculation which supposed the photometer-disk exactly opposite an *end* of the filament and required modification when this was not the case. When D/l was so large that ϕ was small, higher powers of ϕ being neglected, the quantity reduced to $1-\frac{2}{3}\phi^2$; but if the photometer-disk were opposite the *centre* of the filament ϕ had to be replaced by $\phi/2$ and the departure of the quantity from unity was reduced to a quarter. He remarked that the quantity $\frac{1}{2}(\cos^2\phi+\phi\cot\phi)$ had to be applied as a *divisor* to the observed illumination to deduce what the result would be under the ideal conditions when D/l was so large that ϕ was negligible; and he was doubtful whether the language employed, especially the description of the quantity as a "correcting factor," was sufficiently free from ambiguity.

Prof. J. A. **Fleming** said he had tried to impress upon lamp users the advantages of expressing candle-power in terms of the total flux of light per watt consumed. Now that a simple relation had been established between M.S.C.P. and M.H.C.P., there was no excuse for expressing the candle-power of a lamp as *its maximum* horizontal candle-power. With respect to Dr. Chree's remarks, it was quite true that in the proof of the paper sent out the case considered was that in which one end of a straight filament was opposite the centre of the photometer-disk. If, however, a filament of total length $2l$ was placed with its centre opposite to the centre of the photometer-disk and assumed to subtend an angle 2ϕ at the disk, then the formula given in the paper held good.

LVI. *The Pendulum Accelerometer, an Instrument for the Direct Measurement and Recording of Acceleration. By* F. W. LANCHESTER. (Communicated by C. V. BOYS.) *

ONE of the problems connected with the testing of locomotive vehicles is that of measuring and recording the magnitude of the starting and stopping efforts, and generally experimentally determining what may be appropriately termed the "ballistics of traction."

The instrument that forms the subject of the present paper has been designed to facilitate determinations of this description with a sufficient degree of accuracy for the ordinary purposes of the engineer, without involving any special and expensive preparation.

The fundamental principle on which the instrument works, is that the effect of the algebraic sum of tractive forces and resistances on the vehicle as a whole, is shared proportionately by every portion of its mass, and that consequently the tractive effort on the whole vehicle can be ascertained if that acting on any portion of its mass be measured.

It is evident that the actual measurement on a known mass carried on the vehicle could be made by means of a spring-balance, the mass being mounted so as to permit of its horizontal motion only; but it is also possible and it is easier to arrive at the same result by arranging the mass as the bob of a pendulum and reading the *coefficient* of traction directly in the tangent of the angle of its deflexion.

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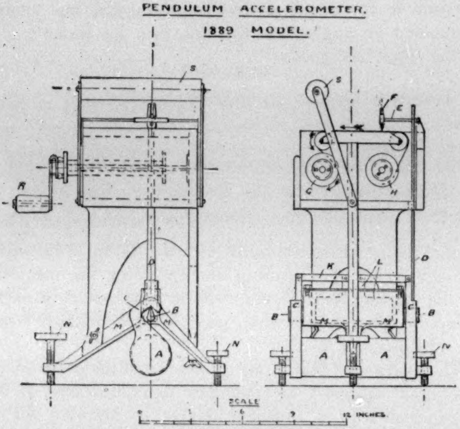
* Read June 16, 1905.

It will be recognized at once that the quantity being measured in this manner is the acceleration of the vehicle: hence the instrument is in reality an accelerometer, the measurement of gross tractive efforts and resistances requiring in many instances the instrument to be used indirectly, the direct reading giving the net difference between the propelling and resisting forces.

Fig. 1.



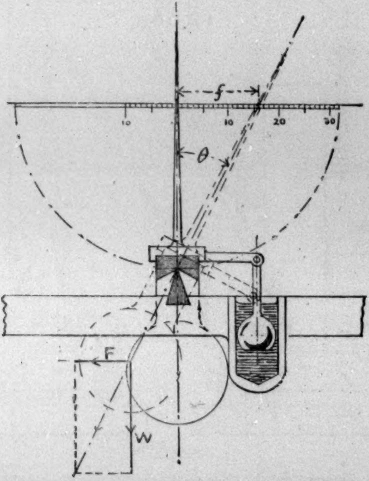
Note to figs. 1 *and* 3.

A. Pendulum-bob.
B-C. Knife-edge supports.
D. Motion arm.
E. Pencil arm.
F. Pencil or stilus.
G. Winding drum.
H. Spool of paper.
J. Datum marker.
K. Dash-pot bracket.
L. Dash-pot piston.
M. Frame casting.
N. Levelling-screws.
O. Anchorage screws.
P-Q. Clock-work.
R. Wind.
S. Handle.

The first pendulum accelerometer was made by the writer as far back as 1889; this somewhat rough model is still in existence (fig. 1), and many of the diagrams taken on the railway in 1889 have been preserved (fig. 8). This instrument was not fitted with clockwork, so that the time-scale is irregular.

Fig. 2 shows in diagram form the essential working parts of the pendulum accelerometer; and fig. 3 shows the actual proportions of my present (1904) model. Referring to fig. 2 in

Fig. 2.



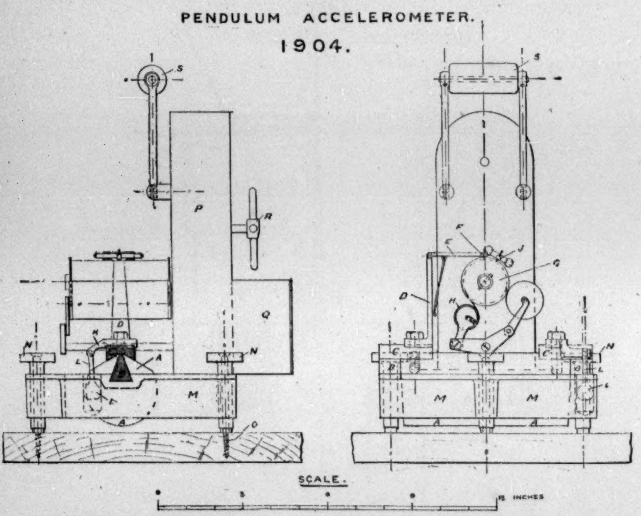
which a resolution of forces is given, let the angle of deflexion of the pendulum be denoted by θ, "W" = forces on pendulum-bob due to gravity, and "F" = force due to acceleration "f," then we have:—

$$f = g \,.\, F/W = g \,.\, \tan\theta.$$

It will be seen, therefore, that the motion of the recording pencil in order to give a uniform acceleration scale must be proportional to the tangent of the angle of deflexion. It is the form of "tan angle mechanism" that constitutes one of the improvements in the new model.

In the 1889 model the pencil was mounted on a sliding carriage (fig. 4) actuated by a forked arm, being an upward prolongation of the pendulum: in this arrangement, in addition to the pencil friction, two additional sources of sliding friction existed, those of the carriage mounting and forked operator.

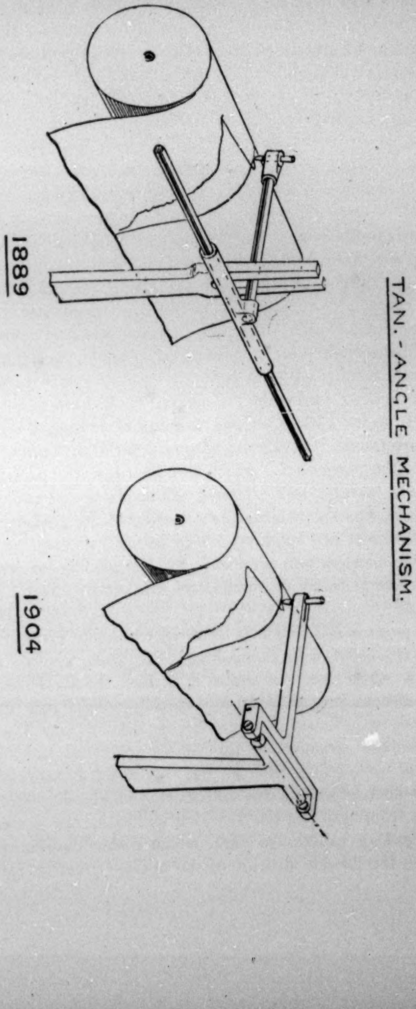
Fig. 3.



In the present form of instrument the pencil arm is pivoted directly to the pendulum continuation in such manner that the point of the pencil lies always in the plane of the pendulum axis. The only theoretical disadvantage of this arrangement is that a small time error is introduced, but provided the pencil arm be made reasonably long this error is infinitesimal.

In order that the accuracy of the instrument shall not be interfered with, the pendulum should be made as short as possible. It is assumed in the theory of the instrument, that the motion of the pendulum-bob is substantially that of the

Fig. 4.


1889
1904
TAN.-ANGLE MECHANISM.

rest of the vehicle, and consequently its motion of swing should be negligible in comparison with the motion of the vehicle; also its time of oscillation should be small in comparison with the time of change of acceleration that it is desired to accurately record. This point was not overlooked in the original model, in which the length of pendulum was 3 inches only; in the present model this has been reduced to 1¼ in., beyond which it is doubtful whether it is desirable to go.

The dash-pot, which is an absolute necessity for working on road vehicles, in the original instrument took the form of an oil-well of large dimensions in which dipped a vane or paddle of considerable area. On the present model the dash-pot is a simple cylinder with spherical piston making an approximate fit, the cylinder is filled with viscous oil and is adjusted to make the pendulum as nearly as possible "dead-beat."

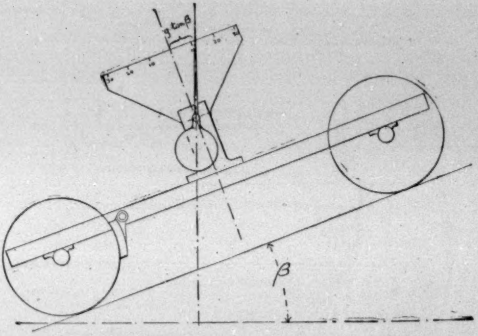
Considered purely as an acceleration meter, a correction would be necessary for any change of gradient; and if it were required, for instance, to integrate the diagrams to obtain velocity curves, either the precaution would have to be taken to work on level roads, or means found of making the necessary corrections. It is a point of great interest and importance that for the purposes for which this instrument is principally required, starting and stopping efforts, frictional and other resistance, &c., the readings are unaffected by gradient and no corrections are required. Put tersely, it may be said that the acceleration produced by the gravity component exactly compensates or neutralizes the direct effect of the gradient.

This point is one that it is quite easy to make clear. Let fig. 5 represent a car standing with brake applied on a gradient whose angle of slope is β, then the accelerometer will record an amount equal to $g \tan \beta$ (provided that it has previously been brought to zero on the level): now this will be a correct measure of the brake and frictional effect employed in holding the car up. Now let us suppose the brake removed, then the true acceleration (neglecting friction) will rise from zero to $g \tan \beta$, which will bring the accelerometer reading exactly to zero, which under these circumstances is the precise measure of the applied tractive forces.

This reasoning obviously extends to partial removal of the brake, and is in fact general.

In order to facilitate the accurate setting up of the instrument, it is fitted with a transverse level to ensure getting the knife-edges approximately level, the pendulum itself being relied on to obtain the correct zero position, a separate pencil or datum-line marker being fitted to the instrument, and the setting being correct when the datum-line and record-line correspond when the car is at rest on the level. When setting up on a motor-car, an approximately level place is found on the road, two trials of the setting are made with the car in opposite positions and the error divided. When on the railway a known level is chosen for setting up.

Fig. 5.



A few typical diagrams taken on the railway and on motor-cars are given in figs. 6, 7, and 8.

In fig. 8 we have a group of brake diagrams taken on the original model in 1889 on the Great Western and the Metropolitan Railways. These at once bring out a point of great interest, the sudden change of acceleration that takes place at the exact instant the train comes to rest; this is a characteristic of nearly all brake diagrams and will be referred to again. The max. negative acceleration recorded in these diagrams is about 4 ft./sec. sec. A starting diagram taken on

Fig. 6.

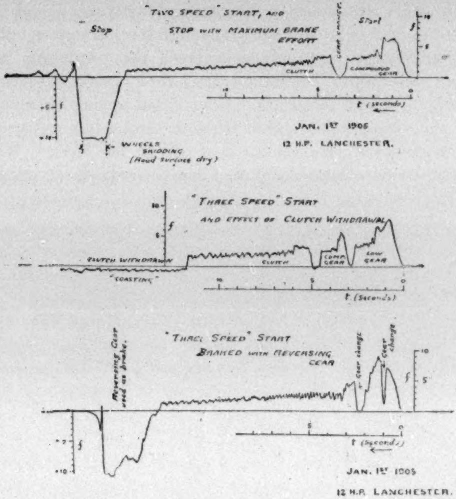
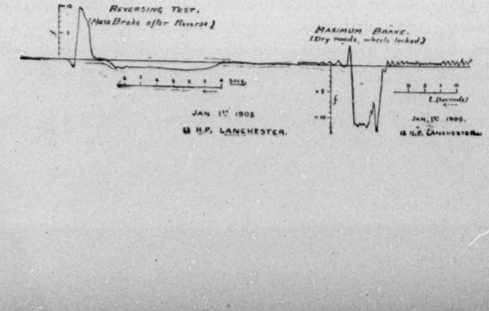

"TWO SPEED" START, AND
STOP WITH MAXIMUM BRAKE EFFORT
Stop
GEAR CHANGE
Start
CLUTCH
COMPOUND GEAR
WHEELS SKIDDING (Road surface dry)
f
10
5
t (seconds)
JAN. 1ST 1905
12 H.P. LANCHESTER.
"THREE SPEED" START
AND EFFECT OF CLUTCH WITHDRAWAL
CLUTCH WITHDRAWN
"COASTING"
CLUTCH
COMP. GEAR
LOW GEAR
t (Seconds)
"THREE SPEED" START
BRAKED WITH REVERSING GEAR
Reversing Gear used as brake.
Gear change
Gear change
t (Seconds)
JAN. 1ST 1905
12 H.P. LANCHESTER.

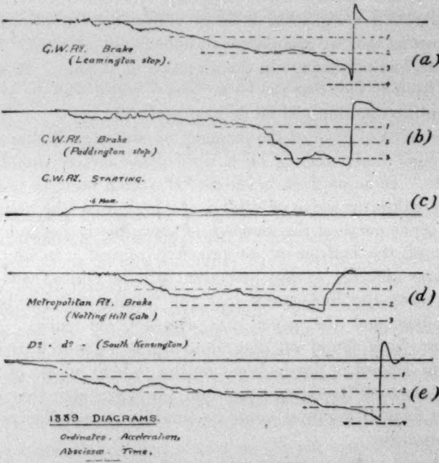
Fig. 7.


JAN. 1ST 1905
12 H.P. LANCHESTER.
THE USE OF THE BRAKE,
"OBVIATING 'JERK.'"
stop
Slowed for traffic
f
t (Seconds)
REVERSING TEST.
(Main Brake after Reverse)
secs.
JAN 1ST 1905
12 H.P. LANCHESTER.
MAXIMUM BRAKE.
(Dry roads, wheels locked)
t (Seconds)
JAN. 1ST 1905.
12 H.P. LANCHESTER.

the same journey is given. The maximum starting acceleration here recorded is about ·6. There is unfortunately no record of the type or weight of engine or the number or weight of the coaches.

Figs. 6 and 7 give a few typical motor-car starting and brake diagrams. The greatest negative acceleration here recorded amounts to 11·5 ft./sec. sec. against a maximum of 8·5 ft./sec. sec. positive. These give respectively about 800 lbs. per ton brake and 600 lbs. per ton starting effort.

Fig. 8.



The ordinate (acceleration) scale employed in the present instrument is 10 ft./sec. sec. per inch, and is not quite open enough to give pleasing diagrams in the railway. The original model with its scale of 3 ft./sec. sec. per inch was better suited to this class of work; but the much greater range and general compactness of the 1904 model renders it much handier and better for motor-car work, the purpose for which it is primarily intended.

In figs. 6 and 7 the time-scale (abscissæ) reads from right to left; in fig. 8 the time-scale reads in the opposite direction from left to right.

It has been remarked that a characteristic feature of brake diagrams is the suddenness of the drop at the instant of stopping. This is a very interesting and important point, inasmuch as it is the cause of the "jerk" nearly always experienced just as a train comes to rest; it was in fact in investigating this jerk in 1888 that the idea of the pendulum accelerometer occurred to the writer. At that time it was currently supposed that the jerk was the effect of the recoil of the buffer springs after stopping; whereas a very little consideration shows that it is in reality sudden *change of acceleration* that we recognize physiologically as "jerk," that is df/dt, and not change in the direction of motion. It suggests itself in fact that the term "jerk" might well be given a scientific meaning and be defined as d^3s/dt^3.

The value of df/dt at the moment of stoppage in the case of a rigid body sliding on a rigid plane surface would be infinite. In actual fact, in the case of a train brought to rest by its brakes the value of df/dt is so high that the correct form of the curve at the moment of stopping is beyond the power of the instrument to properly record. In all the diagrams the lag of the pendulum will be noticed, and in most cases the line shows that the "dash-pot" has been insufficient and the pencil has overshot the mark. The peculiar form found on diagram 3 fig. 6 is due to a partial vacuum formed in the dash-pot on the sudden release of the pendulum from its extreme position. It is probable that the correct form of the drop would be sensibly perpendicular to the datum-line.

It should be more widely recognized that a great part of the art of properly braking a train, or in fact any other vehicle, consists in taking the brake nearly off just before stopping. It is not easy to obtain a train diagram in which this has been done, but diagram 1 in fig. 7 is a good example taken on a car in which the driver was paying special attention to the point.

Many of the diagrams, figs. 6 and 7, are particularly interesting to the motor-car engineer: the loss of *starting* *effort area* (which is equivalent to velocity) during gear change, the exalted starting effort obtained whilst the energy of the flywheel is being drawn upon, the curve of falling effort (largely due to increasing wind resistance),—all these are points that repay the most careful study.

Other applications I anticipate for the pendulum accelerometer are its employment in the determination of the lateral (centrifugal) forces and coefficient of friction when rounding corners, the determination of wind resistance at different speeds and other resistances to traction, the determination of the thrusts and resistances of launches &c.; but these and the further discussion of the engineering aspect of the subject does not come within the purview of the present paper.

May 11th, 1905.

LVII. *A Method of illustrating the Laws of the Simple Pendulum. By* Mr. J. SCHOFIELD.*

[Abstract.]

A PENDULUM is fitted at its lower end with a narrow horizontal framework carrying vertical transverse wires. During the oscillations of the pendulum these wires are caused to cut a jet of mercury, and time signals are sent to the recording mechanism of a chronograph. The distances between the wires are known, and together with the time-measures they yield a displacement-time curve of the motion. From this the kinematical curves and equations of the moving system may be deduced by the usual methods. In the actual apparatus a tuning-fork arrangement with an accuracy of about 1/200 of a second is used as the chronograph, and the results obtained from the pendulum are accurate to about three per cent. The principle has also been applied to torsion pendulums.

Mr. SCHOFIELD also exhibited a set of String Models of Optical Systems, the lenses and prisms being made of celluloid, so that the paths of rays through them can be shown.

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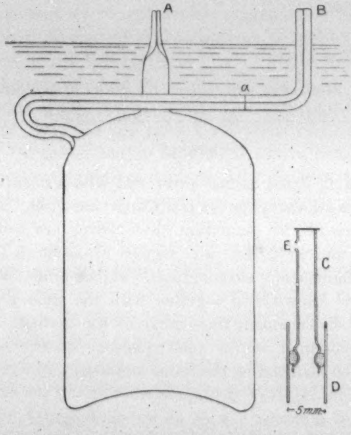
* Read February 24, 1905.

LVIII. *A new Form of Pyknometer.*

By Mr. R. V. Stanford*.

The construction of this modification of the common U-shaped pyknometer will be seen from the accompanying sketch. It may be readily made before the blowpipe, of any desired content from 1 c.c. upwards. It is filled with the liquid whose density is desired by attaching at B a tube which dips into the dish containing it, and applying suction at A. The apparatus, entirely filled with liquid at room temperature, is placed upon a stand in a thermostat, and the plunger CD affixed at B by means of rubber tubing. When the liquid has attained the temperature of the bath, a finger is placed over the aperture E in the piston-rod, and the piston depressed until the liquid meniscus has descended to the mark *a*. The finger is then removed from E, the plunger detached, and the instrument removed from the thermostat



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* Read June 16, 1905.

and weighed. The plunger consists merely of a piece of glass tubing some 0·5 cm. outside diameter, into which a smaller tube fits, as shown in the figure. The piston is made airtight by means of a small rubber ring.

This form of pyknometer has the advantages that it dispenses with a wire suspension, and that the adjustment to the mark *a* is much easier and more accurate with the plunger than is the usual one by means of blotting-paper applied at A, and in addition the horizontal arm permits of its employment in the common thermostats with opaque sides, while reducing to a minimum the amount of contained liquid outside the bath.

The University, Birmingham.

LIX. *The Specific Heat of Iron at High Temperatures.* *By* J. A. Harker, *D.Sc., Assistant at the National Physical Laboratory, Teddington**. (From the National Physical Laboratory.)

Although a knowledge of the specific heat of iron at high temperatures would seem to be of vital importance to the metallurgical industry, yet it would appear that no determination of this physical constant has been made since our high-temperature thermometric scale has been rendered definite and precise by the investigations of the last fifteen years.

Probably on account of its difficulty, the work done on the subject of specific heats at high temperatures is comparatively meagre. Apart from observations dating from more than forty years ago, a careful search in the literature of the subject has revealed the work of only one observer, who carried his observations up to really high temperatures—Pionchon, a pupil of Berthelot, who worked in Paris in 1886. His experiments were made by the usual mixing calorimeter method, but were carried out with exceptional skill and accuracy. His specimens were heated in a specially constructed gas-furnace, those such as iron, which were of a readily oxidizable character, being protected by enclosure in a thin platinum envelope. His temperatures were measured by the Violle method of calorimetric estimation, using a small ingot of platinum heated alongside the specimen. Employing the values previously determined by Violle for the specific heat of platinum, Pionchon found for the melting-point of silver by this method 907° C. We know from more recent determinations that the correct value for this melting-point lies between 955° C. and 962° C. From this it is obvious that before the results of Pionchon's experiments can be compared with those of modern work, his temperature-scale must be corrected, at all events at the higher ranges. For this purpose I have assumed that the scale is accurate up to 400° C., and from that point upwards have added a gradually increasing correction to reduce the value of the silver point to its proper figure. The results obtained are discussed at the end of the paper.

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* Read May 26, 1905.

The experiments now to be detailed were undertaken to test the accuracy of the results obtained with the iron ball pyrometer still employed in certain industries for the measurement of temperature. No protecting envelope was employed for the earlier experiments at lower ranges. The specimens, of which several of different sizes were used, were cut to the shape of a cold chisel, and were made of a sample of exceptionally pure metal containing only ·01 per cent. of carbon *, supplied to the laboratory by Mr. Hadfield of Sheffield. The largest weighed 197 and the smallest 7 grams. They were heated to the required temperature in a small vertical electric furnace conveniently mounted on a swinging bracket, which permitted the whole to be rapidly brought to an exact position over the calorimeter when required. The furnace was closed at the top and bottom by closely-fitting hinged lids. In the earlier experiments the temperature was

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* The complete analysis of the specimen is as follows:—

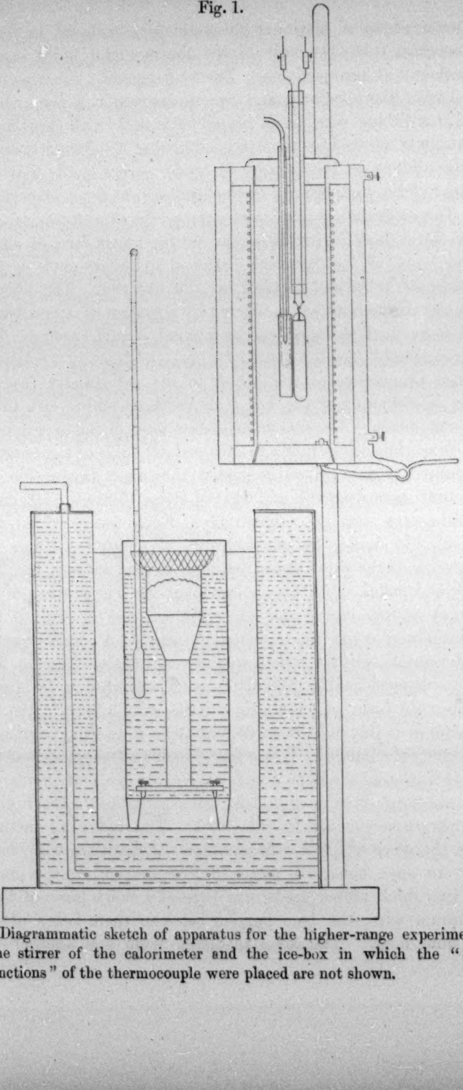
Carbon = 0·01
Silicon = 0·02
Sulphur = 0·03
Phosphorus = 0·04
Manganese = trace

measured by a platinum thermometer, enclosed in a thin porcelain tube, the bulb of the thermometer being in close contact with the specimen. To ensure equality of temperature between the specimen and the thermometer, a thick sheath of iron tubing was placed round both, and in all cases ample time was allowed for the establishment of a sufficiently steady temperature in the furnace to render errors in the measurement of the temperature of the specimen as small as possible.

The calorimeters employed were of the type formerly used by the author for investigations on the latent heat of steam, and were of the form of flattened cylinders made of thin brass, nickel-plated heavily on the exterior. The stirring was continuous and was effected by a system of screw-blades on a vertical shaft driven by a small electric motor. This arrangement has the great advantage of leaving a relatively large proportion of the space in the calorimeter free for the introduction of the hot body, without undue risk to the thermometer. The two calorimeters used in the experiments were of capacities chosen so as to make the rise of temperature obtained with a given specimen the most favourable for accurate measurement, and were of about 4500 and 1700 cubic centimetres capacity respectively. Each was provided with a suitable double-walled enclosure of about ten times the capacity of the calorimeter, having internal walls of brightly polished metal. By this arrangement the temperature of the jacket water, and hence the rate of rise or fall of the calorimeter, could be regulated at will. A diagrammatic representation of the whole arrangement is shown in fig. 1.

Some considerable difficulty was experienced in designing a suitable form of automatic release, which would free the specimen at the right moment and allow it to drop vertically into the calorimeter. It was found that various arrangements of spring tongs which acted reasonably well at moderate temperatures became unusable after exposure to the higher temperatures reached in this work. The following method was therefore adopted. A small loop of platinum wire, bent into an open hook, was attached to the top of the specimen. To two thick nickel leads was twisted a short piece of thin platinum wire, the free portion between them being about 10 mm. long. The leads insulated from one another by

Fig. 1.



Diagrammatic sketch of apparatus for the higher-range experiments. The stirrer of the calorimeter and the ice-box in which the "cold junctions" of the thermocouple were placed are not shown.

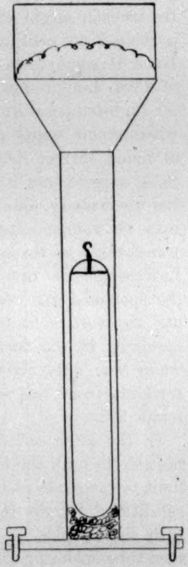
porcelain tubes are coupled to a key and battery, by which the thin wire can be instantly fused when required. The electric furnace with the specimen in position, suspended from the fine platinum by means of the hook, was arranged so that only a very few seconds were necessary to open the lower door, swing the whole into position over the calorimeter, drop the specimen, bring back the furnace and reclose. Special blank experiments showed that the heat gained by the calorimeter from the glowing walls of the furnace during the transfer might attain, under unfavourable circumstances, perhaps 2 per cent. of the amount given out by the specimen. Even if some form of the ingenious tramway calorimeter of Prof. Louguinine, with modifications to render it suitable for high-temperature work, had been adopted, it is doubtful whether this source of uncertainty would have been capable of much further reduction. It may be pointed out that in these experiments it is probable that a considerable part of the uncertainty, due to the gain of heat from this source, may be compensated for by the inevitable small loss in transference of the specimen from the interior of the heated furnace to the calorimeter. Since in these experiments the specimen was permitted to fall freely under gravity, and the vertical distance from the normal position of the specimen in the furnace to the surface of the calorimeter water was only about 20 cm., it is obvious that the uncertainty from this source is not great, even at the highest temperatures.

In the later series of experiments, where it was found possible to push the temperatures up above the safe working limit for accurate platinum thermometry, thermocouples were substituted for the thermometers. In the later experiments with the couples, to ensure accuracy the end of the thin steel tube enclosing the couple was inserted into a block of iron of similar shape and size to the sample.

Above 1000° C., however, prolonged exposure of the couple to contact with iron direct is not permissible in the oxidizing atmosphere of a furnace arranged vertically as in this case. The thin steel tube is therefore to be replaced by one of porcelain.

Above about 800° C. it was found that the oxidation of the specimen of iron in the furnace, and its disintegration when dropped suddenly into the calorimeter water, was so rapid as to introduce appreciable uncertainty into the experiments, and, indeed, at temperatures not much higher it was possible by prolonged exposure to transform the whole of a small piece of the same iron into oxide. A number of experiments were tried employing various means to prevent, or at least retard, this action, among others gilding the specimen by several different processes. Although by some of these it appeared possible to considerably moderate the oxidation, yet it was found that the sudden cooling always detached the surface layer with its protecting covering, when the whole was plunged directly into water. For work at the higher ranges it was obvious therefore that a sheath of some kind such as that used by Pionchon was necessary. A thin tube of fused quartz will stand plunging suddenly into water when at a fair red heat, but experiments showed that when enclosing a cylinder of iron, even when this is separated from it by a thin layer of inert material, the tube almost invariably breaks up, when dropped directly into cold water.

Fig. 2.



Section of magnesia tube.

The following modification of the process was then tried:—Instead of allowing the iron to fall directly into the water, it was caught in a vertical funnel-shaped tube of the form shown in fig. 2. A pad of asbestos-wool on the bottom of the tube was arranged to break the fall. The whole of the rest of the tube was filled with dried "light magnesia." This substance possesses the curious property of offering practically no resistance to the passage through it of any reasonably heavy body, even when the magnesia is in the form of a layer of considerable thickness. The hole formed in it by the falling of the specimen fills itself up almost instantaneously, leaving a layer of about 8 to 10 cm. in thickness to prevent radiation of heat from the hot specimen upwards.

A considerable number of experiments were tried using this arrangement and having the iron specimen protected by a quartz sheath. Experiments at low temperatures gave values agreeing within the limits of error with those got by the earlier method. It was found, however, that above about 900° C. there was a fairly rapid action of the quartz on the surface-film of the iron even when great precautions were taken in having no oxide present on the surface of the sample on its introduction. Besides, an additional objection to the use of quartz in the experiment in this form was that the time required for complete equilibrium to be established in the calorimeter after the specimen had fallen was so great, owing to the low heat conductivity of the quartz in the thickness compatible with strength, that appreciable uncertainty was introduced into the calculation of the true amount of heat given out in an experiment.

It was found, however, that by sealing up the iron in a thin tube of glazed porcelain, the oxidation difficulty could be avoided, and also that this substance did not possess the disadvantages named in the case of quartz. After prolonged use of one specimen at temperatures up to nearly 1200° C., the iron on examination was found to be only slightly tarnished, probably from some action of the vapour of the glaze which at these temperatures is quite soft.

It was found best for the experiment to use a somewhat long and narrow sample, and that the iron should not be in the form of a cylinder but of small pieces fitting not too tightly into the tube. This necessitated the weight of the porcelain being a rather large fraction of the whole—about two-fifths of the total weight. It was obvious therefore that an accurate determination of the specific heat of the kind of porcelain used was necessary. Accordingly a set of experiments was made with a specimen consisting of a porcelain tube filled with fragments of the same kind of porcelain weighing 11 grams in all. The results of the experiments are given in the following table :—

Mean Specific Heat of Porcelain between 15° C. and the temperature given.

Temp. cent.	Spec. heat.
912	·2582
958	·2563
1075	·2539

These experiments indicate a small fall in the specific heat over the range, through which the porcelain was used as protecting sheath in the iron experiments.

The results of the whole three sets of experiments on the iron, made with an interval of more than a year and employing several different specimens, are tabulated together. The most striking feature is the marked drop in the value of S_0^t above 900°. As the specific heat of both oxides of iron is much greater than that of the metal, it is probable that in those experiments where oxidation took place in the furnace, the value found would be too high. The specimen was cleaned from all but a thin firmly adhering film after each experiment, and was weighed both immediately before and after. The time of attainment of thermal equilibrium was of course much shorter in the earlier observations, but even in the series with the magnesia tube it is unlikely that the calculation of the quantity of heat given out should be in error more than about half a per cent. in most of the experiments.

The final result of the whole set of experiments is shown in the table appended, arranged in order of ascending temperature. Preliminary observations with the protected specimen at lower ranges showed that the method gave a result in agreement with that obtained by the earlier method, and when plotted together it is impossible to distinguish the observations made with the various sized specimens in the different series. Further, no systematic difference was shown between the earlier sets, using a platinum thermometer for temperature measurement, and those done later with thermocouples.

All the values of Q_0^T obtained from the individual observations were plotted on a large scale, and a mean curve drawn from which the results in the later tables were calculated.

	Temp.	Q_0^T.		Temp.	Q_0^T.
1	216	25·5	19	672	95·8
2	216	25·3	20	721	111·5
3	385	49·1	21	741	113·8
4	386	49·7	22	782	123·6
5	392	49·9	23	787	126·6
6	397	50·7	24	818	133·1
7	466	59·9	25	819	133·7
8	489	65·2	26	826	132·4
9	494	66·1	27	842	136·4
10	503	67·1	28	856	139·6
11	520	69·0	29	859	142·6
12	585	80·3	30	866	140·4
13	588	82·3	31	896	146·6
14	597	84·7	32	923	150·7
15	601	85·3	33	1006	156·0
16	613	87·1	34	1077	161·9
17	636	88·9	35	1111	174·4
18	637	91·0	36	1144	190·0

These show the value for Q_0^T obtained by the author, along with the original and recalculated values given by Pionchon's experiments*. It will be seen that the agreement between the two latter is almost perfect up to 900° C. In his paper Pionchon mentions the fact that above this temperature the experiments become very much more difficult, and that the platinum envelope used suffers deterioration at a fairly rapid rate.

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* Pionchon's experimental values are quoted in Le Chatelier and Boudouard's 'Mesure des températures élevées.' Both in the original French and in the new English translation by Burgess there are, however, two serious errors.

The value for 200° should be 23·5 instead of 22·5
and that for 400° „ 51·5 „ 41·5

to the nearest 0·5 unit.

The diminution at 900° and subsequent rise in the specific heat of iron, shown by the author's experiments at the higher part of the range, require confirmation, and some experiments now in progress indicate that it may be possible to construct envelopes of inert materials more refractory than porcelain, which would permit the extension of the range upward of the experiments to above the melting-point of the metal.

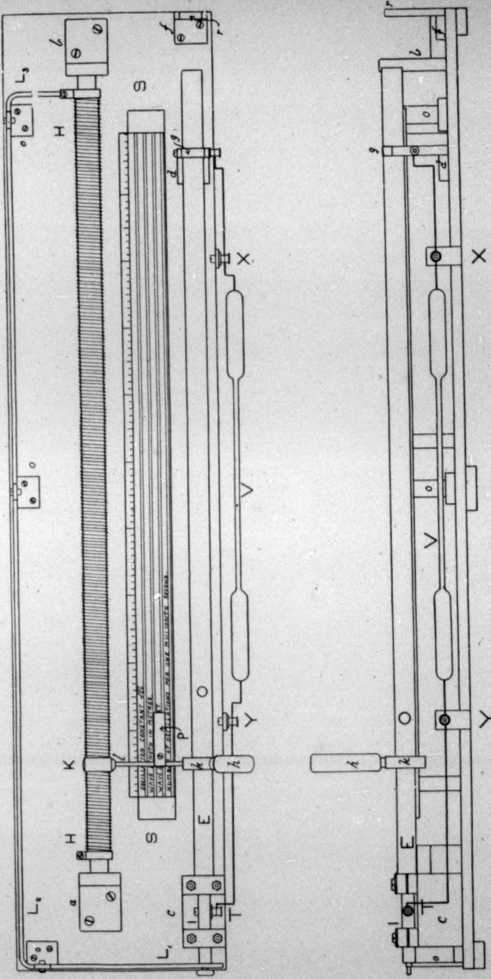
Total Heat between 0° and T°.—Q_0^T.

T.	Pionchon's value from formula.	Pionchon's value re-calculated.	The Author's value.
200	23·5	23·5	23·5
300	36·8	36·8	37·0
400	51·6	51·6	51·3
500	68·2	66·0	66·9
600	87·0	83·2	83·8
700	108·4	102·2	104·1
800	135·4	125·0	127·8
900	157·2	146·7	148·0
1000	179·0	166·0	155·7
1100	...	...	168·8

Mean Specific Heat between 0° and T°.—S_0^T.

T.	S_0^T.	T.	S_0^T.
200	·1175	700	·1487
250	·1204	750	·1537
300	·1233	800	·1597
350	·1257	850	·1647
400	·1282	900	·1644
450	·1311	950	·1612
500	·1338	1000	·1557
550	·1361	1050	·1512
600	·1396	1100	·1534
650	·1440		

FLEMING. *Proc. Phys. Soc. Vol. XIX., Pl. XIX.*



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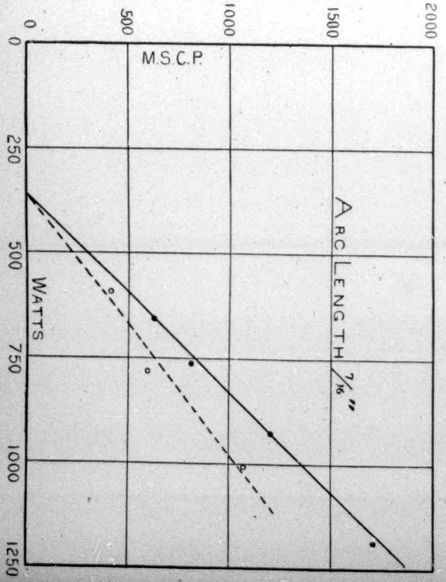
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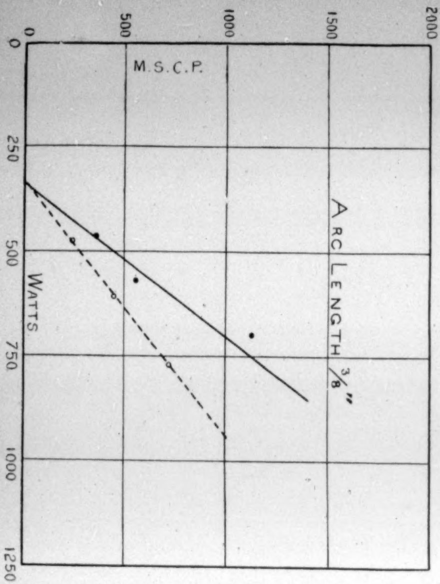
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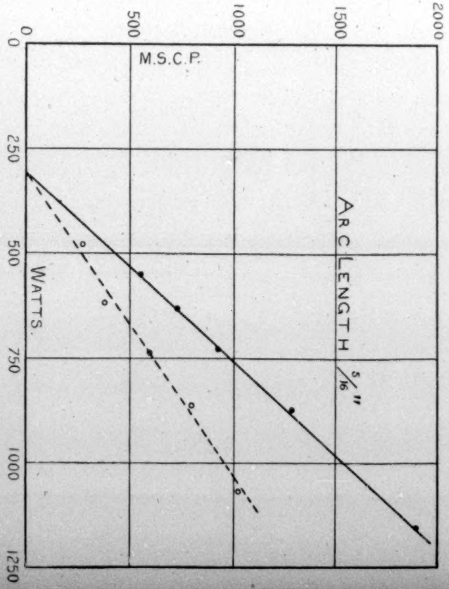
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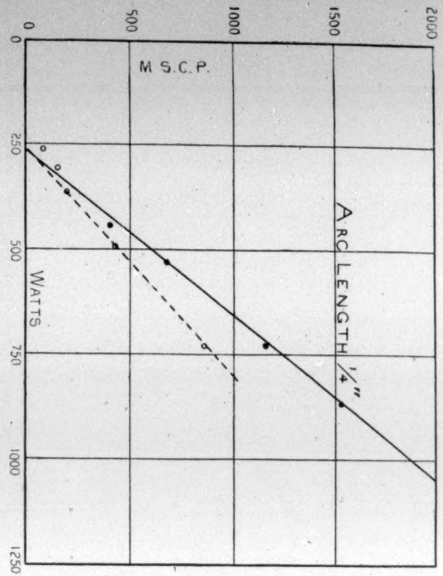
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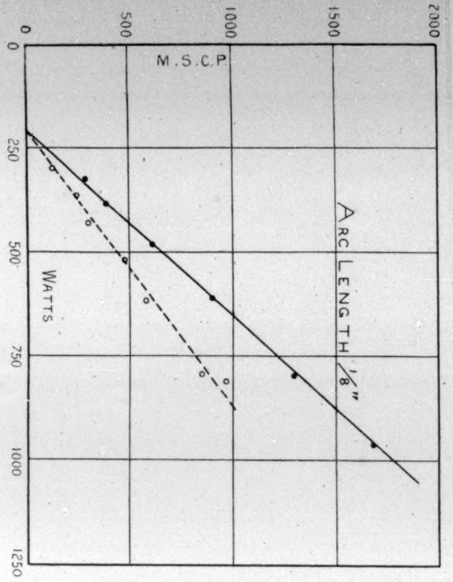
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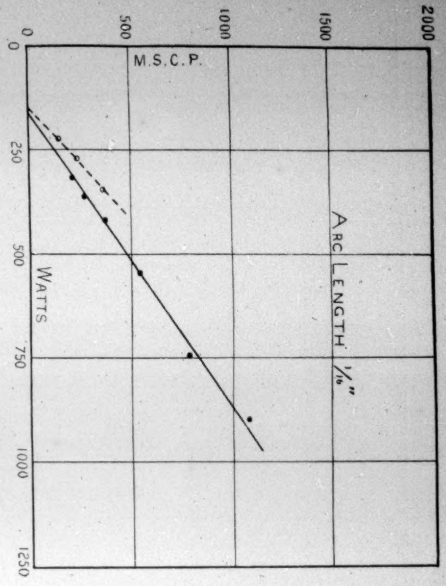
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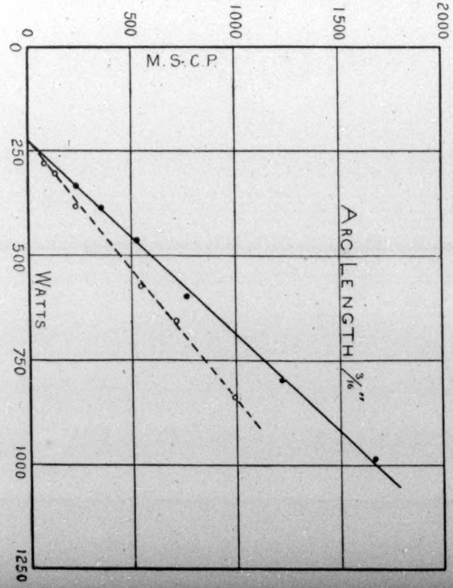
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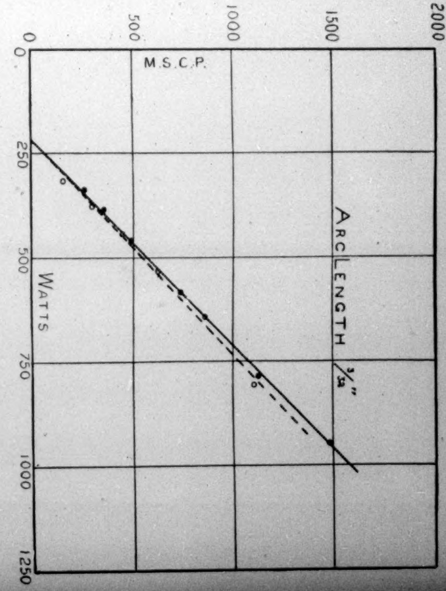
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LX. *The Electrical Conductivity of Flames.* By HAROLD A. WILSON, *M.A., D.Sc., Fellow of Trinity College, Cambridge, and Lecturer on Physics in King's College, London* [*].

(1) *Flames containing no Salt Vapour.*

THE following paper contains an account of a series of experiments on the conductivity of a coal-gas flame for electricity between platinum electrodes immersed in the flame [†]. In order to study the variation of the current with the distance between the electrodes and the fall of potential along the flame, a special burner was used to produce a long narrow flame. The burner consisted of a fused quartz tube 25 cms. long and 2 cms. in diameter, having a single row of holes each 1·5 mm. in diameter parallel to its axis. There were 40 holes uniformly distributed into a row 20 cms. long. One end of the tube was closed with an indiarubber stopper, and the other was connected by means of an indiarubber tube to a large Bunsen burner. The mixture of coal-gas and air supplied by the Bunsen burner was burnt at the row of holes, and formed a fairly uniform flame 20 cms. long and about 4 cms. high. As electrodes two parallel disks of platinum were used, each 1 cm. in diameter. One of these was supported near one end of the flame and the other was fixed to a stand sliding in grooves parallel to the flame, so that it could be moved along horizontally in the flame and so placed at any desired distance from the fixed electrode. Since the quartz tube was a good insulator, a current could be passed horizontally from one end of the flame to the other without fear of any of it going through the tube instead of the flame. This arrangement of apparatus is shown diagrammatically in fig. 1.

The current through the flame between the electrodes was measured by means of a moving-coil galvanometer, and the potential-difference between the electrodes by means of an electrostatic voltmeter. The electrodes were bright red hot. The variation of the current with the distance between the electrodes for different potential-differences will be first considered. The following table gives the observed galvanometer deflexions. Each number is the mean of several observations ($1 = 8{\cdot}8 \times 10^{-9}$ ampere).

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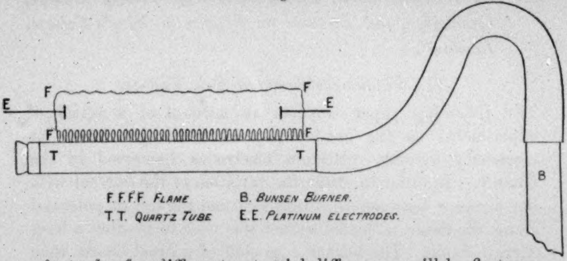
[*] Read June 16, 1905.

[†] An excellent account of the present state of our knowledge of this subject by F. L. Tufts is contained in J. Stark's *Jahrbuch der Radioaktivität und Elektronik*, 1 Band, 1 Heft (1904).

Fig. 1.



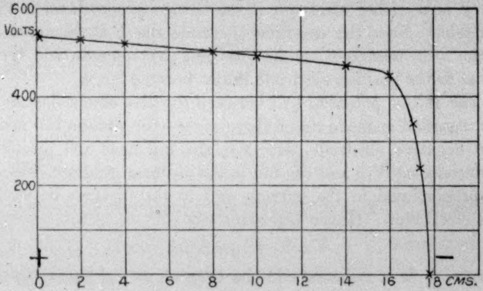
Potential Difference.	Distance between Electrodes.				
	1 cm.	4·5 cms.	9 cms.	13 5 cms.	18 cms.
600 volts.........	310	304	295	280	270
400 „	255	247	240	230	215
200 „	175	170	165	155	143
120 „	130	125	115	104	90
40 „	67	64	57	53	48
20 „	42	39	35	32	29
10 „	22	20	16	14	13
4 „	9	8	7	6	5·3
2 „	5	4·7	4	3·5	3

It will be seen that in every case the current falls off slowly with increasing distance between the electrodes.

The fall of potential between the electrodes was examined by means of a third electrode, consisting of a fine horizontal wire which was put in between the disk electrodes and could be moved along from one to the other. Its potential was measured by means of a Kelvin's multicellular electrostatic voltmeter.

Fig. 2 shows the variation of the potential along the flame obtained as just described. The distance between the disks was 17·7 cms. and the P.D. used 550 volts. It will be seen that there is a sudden fall of potential near each electrode and a uniform gradient in the space between the electrodes.

Fig. 2.



Let n_1 = number of positive ions present per c.c.
n_2 = ,, negative ,, ,, ,,
k_1 = velocity of positive ions due to one volt per cm.
k_2 = ,, negative ,, ,, ,, ,,
X = electric intensity in volts per cm.
e = charge on one ion.
i = current density.

Then we have the well-known equation

$$i = \mathrm{X}e(k_1 n_1 + k_2 n_2).$$

Where X is uniform $n_1 = n_2$, so that $i = \mathrm{X}en(k_1 + k_2)$.

According to this equation we should expect the current through the flame to be proportional to the potential gradient existing between the electrodes. To test this, two small electrodes were put in the flame 0·5 cm. apart, and the P.D. between them was measured by means of a quadrant electrometer. It was found that the P.D. between them was very nearly proportional to the current through the flame. The following are some of the results obtained :—

Current. (C).	P.D.	P.D. ÷ C.
270	4·0 volts.	0·0148
54	0·8 ,,	0·0148
18	0·25 ,,	0·0139

The P.D. between the two small electrodes varied to some extent as they were moved along the flame together, owing to the non-uniformity of the flame, but at any one point not very near either electrode it was always proportional to the current. Near the negative electrode the P.D. became too large to be measured on the quadrant electrometer, and very close to the positive electrode it also became large.

The fall of potential (V) between the disk electrodes may be regarded as made up of three parts—the sudden fall near the negative electrode, say V_2; the fall near the positive electrode, say V_1; and the fall in the uniform gradient, which is proportional to the current and to the distance between the electrodes. Hence approximately

$$V = V_1 + V_2 + ACd,$$

where A is a constant, C the current, and d the distance between the electrodes.

The following table gives the values of $(V_1 + V_2)$ got by subtracting ACd from V, using the results given above. The value of A was got from the measurements of the electric intensity between the electrodes by means of the equation $ACd = Xd$.

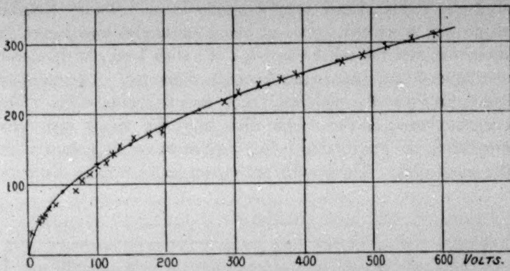
We have $\frac{X}{C} = A = 0{\cdot}03$, the current being expressed as before in terms of the corresponding galvanometer deflexion. Hence

$$V_1 + V_2 = V - 0{\cdot}03\ Cd.$$

P.D.	1 cm.	4·5 cms.	9 cms.	13·5 cms.	18 cms.
600 volts.........	591	559	520	487	454
400 „	392	367	335	307	284
200 „	195	177	155	137	123
120 „	116	103	89	78	71·4
40 „	38	31·4	24·6	18·6	14·0
20 „	19	14·7	10·6	7·0	4·3
10 „	9	7·3	5·7	4·3	2·8
4 „	4	2·9	2·1	1·6	1·1
2 „	2	1·4	0·9	0·6	0·4

In fig. 3 these values of $(V_1 + V_2)$ are plotted against the corresponding values of the current. The curve drawn is the parabola $C = 12{\cdot}8\sqrt{V_1 + V_2}$, and it will be seen that nearly all the points lie near this curve. It appears therefore that the current varies approximately as the square root of

Fig. 3.



(V_1+V_2). Hence we have $C=B\sqrt{V_2+V_1}$ nearly, where $B=12{\cdot}8$ for the flame used, or $V_1+V_2=\frac{C^2}{B^2}$.

But

$$V=V_2+V_1+ACd,$$

so that

$$V=\frac{C^2}{B^2}+ACd.$$

Prof. J. J. Thomson * has given the theory of the variation of V with C in a uniformly ionized gas. He finds that when the current is far from its maximum possible value so that there is a region between the electrodes where the electric intensity is uniform, then the relation between V and C is of the form $V=aC^2+bdC$, where a and b are constants. The significance of a and b in Prof. Thomson's formula is moreover precisely that which has been deduced above for $\frac{1}{B^2}$ and A.

It appears therefore that the conductivity of the flame is due to approximately uniform ionization taking place throughout the region between the electrodes, and, further, that the current is always very far from its maximum possible value even when a P.D. of 600 volts is applied to electrodes only 1 cm. apart in the flame.

To further test the applicability of the formula

$$V=\frac{C^2}{B^2}+ACd$$

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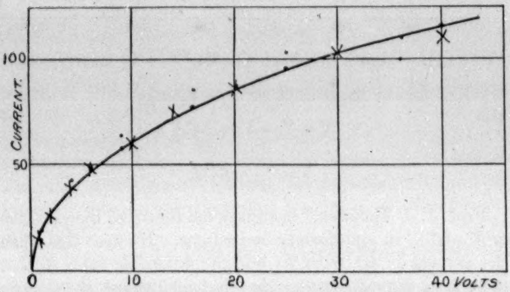
* 'Conduction of Electricity through Gases,' p. 73.

to flames, the current between two vertical parallel platinum disk-electrodes placed symmetrically in an ordinary Bunsen flame was measured. One of these electrodes was 3 cms. in diameter, and the other consisted of a disk 1 cm. in diameter surrounded by a guard-ring 3 cms. in diameter. The annular space between the disk and ring was 0·5 mm. wide. The current between the large disk and the small one was measured, the guard-ring being kept at the same potential as the small disk. The results are shown in fig. 4 for a distance of 1 cm. between the electrodes. The current was very nearly independent of the distance between the electrodes, which means that ACd in the formula $V=\frac{C^2}{B^2}+ACd$ was negligible. Consequently we should have $C^2=B^2V$. The curve drawn is the parabola $C=19{\cdot}25\sqrt{V}$, and it will be seen that the points fall very near to it. The term ACd in this experiment was small, partly on account of the smallness of d and partly because the flame was very hot, which of course diminishes A.

Fig. 4.



In previous experiments on the variation of C with V with the electrodes near together, no guard-ring was used, so that the relation obtained was probably complicated by the non-uniformity of the field at the edges of the electrodes.

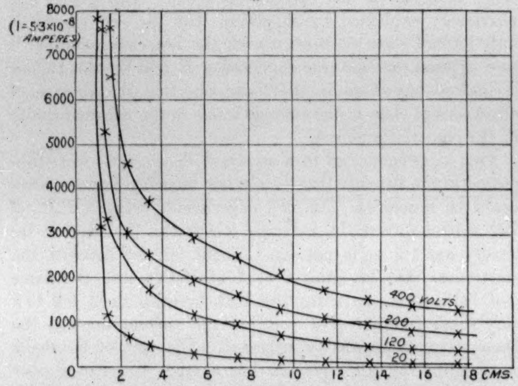
(2) *The Effects due to introducing Salt Vapours into the Flame.*

The quartz-tube burner enables the effect of putting salt into different parts of the flame to be easily studied. A bead of an alkali salt on a platinum wire was put into different parts of the flame, the electrodes being about 18 cms. apart. It was found, in agreement with earlier results, that the current through the flame was not affected unless the salt vapour came in contact with the negative electrode, when a large increase in the current occurred. This result was previously explained by supposing that the salt vapour is only ionized when in contact with the hot electrodes, but it now appears that the true explanation is that the salt vapour is ionized anywhere in the flame, and that the absence of effect except close to the cathode is due to the non-uniformity of the potential gradient.

Two wires connected to a quadrant electrometer were supported in the flame so that the electric intensity between them could be measured. In one experiment, using a P.D. of 700 volts between the principal electrodes, the electric intensity was 1·6 volts per cm. at about midway between the electrodes. On bringing a bead of K_2CO_3 into the flame just below the two wires, the P.D. between them fell to a very small fraction of a volt, but the current through the flame was not appreciably increased. The part of the flame occupied by the salt vapour is thus a much better conductor than the rest of the flame. In the experiment just described, the potential fall at the cathode was $700-18\times1\cdot6=671$ volts, and, as we have seen, $C=A\sqrt{V_2}$. Now the effect of putting in the salt is to diminish the electric intensity to practically zero in the part of the flame occupied by the salt, which was a length of about 2 cms. of the flame. Thus putting in the salt must have increased the cathode fall by 3·2 volts, which according to the formula $C=A\sqrt{V_2}$, since V_2 is 671 volts, should have increased the current by one part in 400. Now the deflexion due to the current in this experiment was about 200 mms., so that putting in the salt ought to have increased the deflexion to 200·5 mms.; but since the deflexion was never perfectly steady owing to small oscillations of the flame, so small an increase could not have been detected. It thus appears that the absence of effect on putting in salt, except close to the cathode, is not inconsistent with the view that the salt is ionized anywhere in the flame.

Fig. 5 shows the variation of the current with the distance between the electrodes for several P.D.'s when some K_2CO_3 was put on the cathode. The K_2CO_3 on the cathode fused and volatilized at a very constant rate, so that the current remained almost constant as long as any salt was left.

Fig. 5.

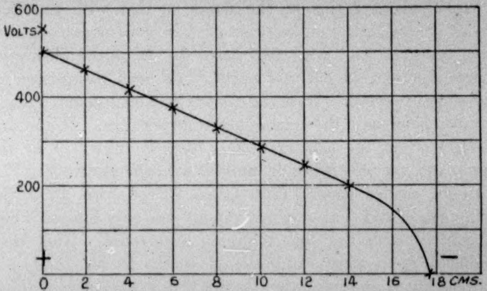


It will be seen that the current falls off rapidly as the distance between the electrodes is increased, this effect being much more marked than in the case of the flame alone.

Fig. 6 shows the effect of putting salt on the cathode on the potential gradient. It will be seen, on comparing with fig. 2, that the cathode fall is greatly diminished, while the uniform gradient between the electrodes is increased.

When the distance between the electrodes is large, the current with K_2CO_3 on the cathode is about 50 times greater than the current when no salt is present. The electric intensity in the region between the electrodes, however, is only three or four times greater, so that it seems as though putting salt on the negative electrode increased the conductivity of the rest of the flame. This effect is of a rather unexpected character, and the writer hopes shortly to carry out further experiments with the object of elucidating its real nature.

Fig. 6.



With salt on the cathode, most of the fall of potential occurs in the uniform gradient between the electrodes, so that the current which is proportional to the gradient varies roughly inversely as the distance between the electrodes and directly as the potential-difference.

It has been shown recently by F. L. Tufts* that coating the negative electrode with calcium oxide greatly diminishes the fall of potential at this electrode, and so increases the current. This effect is clearly exactly of the same nature as the effect of putting an alkali salt on the electrode. Tufts also finds that when the negative electrode is coated with CaO, then putting a bead of salt anywhere in the flame between the electrodes increases the current considerably. It was consequently to be expected that with salt on the negative electrode a similar effect would be obtained, and this was found to be the case. With the electrodes 18 cms. apart and K_2CO_3 on the cathode, putting a bead of K_2CO_3 anywhere in the flame increased the current about 10 per cent., and the same effect was obtained with beads of other alkali salts. Two beads put in at the same time in different positions increased the current about 20 per cent. The portion of the flame occupied by the salt becomes a much better conductor than the rest of the flame, so that the electric intensity in it becomes very small. The electric intensity in the rest of the flame is consequently increased, and the current is proportional to this intensity. The salt from one bead occupied about 2 cms. of the flame, so that with the electrodes 18 cms. apart an increase of $\frac{1}{8}$ in the current ought to have occurred, and with two beads $\frac{2}{7}$. With the electrodes nearer together, the effect of putting in a bead was greater, as was to be expected. With the electrodes 9 cms. apart, putting in a bead increased the current about 20 per cent. It appears, therefore, that the salt vapour is ionized anywhere in the flame, and not only when in contact with the electrodes.

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* F. L. Tufts, *Phys. Zeitschr.* v. p. 76 (1904). See also F. L. Tufts, & J. Stark, *Phys. Zeitschr.* v. p. 248 (1904).

In an earlier paper (Phil. Trans. A. vol. cxcii. 1899) the writer suggested that the salt vapour was only ionized when in contact with the hot platinum electrodes. This view enabled a great many experimental results to be simply explained, but it appears now that the true explanation of these results is the smallness of the potential gradient in the flame, except close to the electrodes. It was supposed previously that when the P.D. used was large, so that the current only increased slowly with increasing P.D., that then the current was nearly at its maximum possible value, and so was a measure of the number of ions formed between the electrodes. It is clear now that this supposition was incorrect, and that the current is really always very far from its saturation value. It is of course probable that some surface ionization on the hot electrodes does occur, but, at any rate, when the electrodes are not very near together, it does not appear to be sufficient to appreciably influence the conductivity of the flame.

The view that the salt vapour is ionized throughout the flame was adopted by Arrhenius (Wied. *Ann.* xlii. p. 18, 1891), E. Marx (*Ann. de Phys.* ii. pp. 768, 798, 1900), and by F. L. Tufts and Stark in the papers referred to above.

In conclusion, I wish to say that my best thanks are due to Prof. J. J. Thomson for valuable advice and kind interest shown during the carrying out of these experiments.

DISCUSSION.

Dr. J. A. **FLEMING** said that the paper was of practical interest, and described some experiments he had made to utilize the properties of flames rendered conducting by potassium salts in the construction of a sensitive telegraphic relay. He had found that the best results were obtained by using fused potassic sulphate introduced into a powerful oil-flame produced by a *roarer*. A platinum wire was wrapped round a porcelain boat, and this was filled with the fused potassic sulphate and placed in the hottest part of the flame. If, then, a movable-coil galvanometer had a long platinum wire attached to its coil, the movement of the coil by a small current could be made to dip the end of this wire into the flame and so close another circuit, say of a printing telegraph. In this manner he had made a relay working with about 0·01 of a milliampere.

Mr. W. **DUDDELL** drew attention to the similarity of the phenomena described to those exhibited by very small current electric arcs. He asked the Author if he had considered the possibility of using a conducting flame as a high-voltage rectifier. Also had the Author tried sodium salts and what were their effects compared with those of potassium.

Dr. W. **WATSON** asked if the drop of potential at the negative electrode depended upon the temperature of the electrodes.

Dr. H. A. **WILSON**, in reply to Mr. Duddell, said that the salts of cæsium, rubidium, potassium, sodium, and lithium imparted conductivity to flames in the order named. This order was also the order of the atomic weights. He had found that a flame could be used as a rectifier when using alternations about 100 ~ per second, but was not sure whether it would rectify rapidly alternating currents. In reply to Dr. Watson, he said that the temperature of the electrodes exercised a great influence. An increase in the temperature of the electrode diminished the drop of potential at the electrode.

LXI. *Contact with Dielectrics.* *By* Rollo Appleyard *.

Objects.

To examine:—

(1) Whether tinfoil electrodes, pressed against a sheet of dielectric by indiarubber disks, enable accurate determinations of dielectric-resistance to be made.

(2) The effect upon dielectric-resistance of change of load on such tinfoil electrodes, in the case of press-spahn.

(3) The effect upon dielectric-resistance of increase or decrease of voltage in the case of press-spahn between tinfoil electrodes.

(4) The rate and direction of the change of deflexion in direct-deflexion tests on press-spahn, and to determine in how far these changes result from surface conditions, and in how far from internal stresses.

(5) The effect of reversals of voltage upon dielectric-resistance.

(6) The effect of prolonged "electrification."

(7) To indicate the probable limits of accuracy with mercury electrodes.

(8) To point out that Price's guard-wire can be used in sheet tests to eliminate leakage over the sheet surface, as well as over the instruments.

(9) The retentive force between electrodes and dielectrics.

Conclusions.

(*a*) Except in the case of homogeneous dielectrics, it is misleading to deduce specific values referred to unit cube of the material, from the results of tests on sheets.

(*b*) With tinfoil electrodes, the apparent resistance of press-spahn diminishes as the load increases, and it attains a fairly constant value at a load of about 400 grammes per cm.^{2}

(*c*) If, with tinfoil electrodes, the load is gradually diminished after a load of 543 grammes per cm.^{2}, the resistance gradually rises, but the rise is less rapid than the diminution in the former case (*b*).

(*d*) When the full load with tinfoil electrodes is again restored, the resistance falls to its minimum value.

(*e*) For small loads, with tinfoil electrodes, the 2nd-minute deflexion is in general greater than the 1st-minute deflexion. As the load increases, a point is reached at which these deflexions become approximately equal. For loads greater than about 360 grammes per cm.^{2}, the 1st-minute deflexion is in general greater than the 2nd-minute deflexion.

(*f*) Increase of voltage, with tinfoil electrodes, especially with small loads, behaves like increase of load, apparently increasing the contact area, and diminishing the observed dielectric-resistance. Load, voltage, and the normal effect of "absorption" thus combine to determine the ratio of the 1st-min. deflexion to the 2nd-min. deflexion.

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* Read June 16, 1905.

(*g*) When mercury electrodes are used, the dielectric-resistance, as measured at different voltages, is sensibly the same, even for abrupt and great changes of voltage.

(*h*) When mercury electrodes are used, the 2nd-minute deflexion is in general never greater than the 1st-minute deflexion. The inference is that when, with tinfoil electrodes, the converse is the case, it arises from imperfect contact, and not from the material itself.

(*i*) When mercury electrodes are used, the dielectric-resistance, as measured with a voltage applied in a given direction, is sensibly the same as that measured with the voltage reversed, and this equality appears to become greater after a few reversals.

(*j*) There is a critical load at which tinfoil electrodes yield fairly accurate results. With greater loads there is danger of crushing the material. With a less load the contact is faulty.

(*k*) With mercury electrodes under the application of 750 volts, with earthing and reversing, respectively, at every fifth minute, the resistance of a sample of press-spahn fell from 43·2 to 41·7 megohms in 51 minutes, thus indicating the limits of precision of resistance tests on such material on a prolonged test where the conditions have time to change.

(*l*) The earth-readings, in such an experiment as that described under (*k*), fall steadily towards zero, and their pairs of maximum values are approximately equal, corresponding to the two directions of voltage. These maximum earth-readings are only about 0·2 per cent. of the maximum readings with the battery. Consequently, if only 1 minute is allowed for a sample to recover after the application of a given voltage, the degree of precision of the results will be of the order 0·2 per cent., on account of the previous charge left in the sample.

(*m*) The retentive force between a disk of tinfoil and a sheet of dielectric, forming part of a condenser, increases rapidly with the voltage; and it vanishes, or becomes extremely small, when the voltage falls to zero.

(*n*) Small variations between the observed capacity of a condenser tested at different voltages may be due to more intimate contact between the electrodes and the dielectrics at the higher voltages.

In the course of some experiments described by Mr. E. H. Rayner [*], the electrical resistance between the faces of sheets of dielectric was measured, using electrodes of thin tinfoil. The dielectric sheet to be tested was laid between two circular soft indiarubber disks, each disk being covered, on its contact-face, with tinfoil, and the whole was surmounted by a brass disk of the same diameter, upon which lead weights could be placed. This interesting arrangement may be regarded as a compromise between the use of unyielding contact-plates and mercury contact. It is here proposed to investigate the capabilities of this compromise; and as the material known as "press-spahn" was examined by Mr. Rayner in his tests, it is here selected as a convenient dielectric for the present purpose.

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[*] "Report on Temperature Experiments carried out at the National Physical Laboratory." By E. H. Rayner. Journ. Inst. Electrical Engineers, vol. xxxiv. p. 613, May 1905.

Specific Values.—Mr. Rayner reduces his results to kilo-megohms per cm.^{3}, but it is perhaps better to state the total megohms measured in each case, without reducing them to specific values. The reason for this is that many materials of this class have a varnished surface, and this varnish has a higher resistance than the middle layers; so that thin sheets yield higher specific values than thick sheets, and specific values cannot be said to be representative of the material generally. Specific values should therefore be avoided, except in the case of homogeneous dielectrics.

Size of Electrodes.—The disks used by Mr. Rayner were 50 cm.^{2} This corresponds to a diameter of 7·98 cms. (3·14 ins.). The electrodes used throughout the following tests are 153·3 cms.^{2}, corresponding to a diameter of about 14 cms. (5·5 ins.). The soft indiarubber disks are of the same thickness as those employed by Mr. Rayner, *i. e.* 1 cm.

Load on Electrodes.—The weight used by Mr. Rayner to press upon the electrodes was 20 kilogrammes, so that the average load was 400 grammes to the cm.^{2} In order to give an idea of this load expressed in British units, it may be noted that it is almost exactly the same mean pressure as would be exerted by 1 cwt. distributed over a 5-inch disk. To obtain the same mean pressure on a 5½ in. disk, it had to be loaded with 135·2 lbs. (61·3 kilogrammes). It is clear that with some dielectrics a load of this magnitude is likely to reduce the thickness of the sample or alter its texture, and so invalidate the results; as press-spahn is fairly hard, it is probably not seriously affected, but it will be necessary to examine the effect of changes of load.

Change of Resistance with Volts.—Mr. Rayner observed considerable change of resistance according to the volts applied. For example, using successively 200, 1000, and 200 volts he obtained respectively 22,000, 9600, and 18,500 megohms with a sample of press-spahn. These changes appear to be excessive, and it is necessary to consider whether they are in any way to be associated with the method of applying the test.

Absorption.—Another question of great importance is the change of resistance with time, when the volts are constant. Mr. Rayner, following ordinary usage, calculated the resistance from the deflexion observed after "electrification" had proceeded for 1 minute, and he states that the deflexion after 2 minutes did not differ by more than about 5 per cent. from the deflexion obtained after 1 minute. In view of the valuable information to be derived from observations of the rate and direction of the change of deflexion, the actual readings at the 1 minute and 2 minutes, respectively, should be recorded. In what follows, this will be done.

Surface Leakage.—In the apparatus employed for the following tests, every part of the system was provided, where necessary, with a Price's guard-wire. A preliminary test was made to examine whether there was surface leakage between the electrodes. For this purpose the edge of a sheet of press-spahn was covered with tinfoil, and this was connected to the guard-wire. It was found, however, that this did not affect the deflexion. The surface* leakage was therefore negligible.

Temperature.—The temperature of the room during all the following tests varied from 15° C. to 16° C.

Humidity.—Press-spahn, as is proved by Mr. Rayner's results, is very hygroscopic; consequently the resistance cannot be regarded as a fixed quantity. Each set of tests must therefore be considered as distinct from those which precede or follow it. It is assumed that the conditions in regard to moisture remain constant during a short test.

Method of Testing.—The direct-deflexion method was employed throughout, the readings being recorded at successive minutes. In order to indicate the rate of change of deflexion, the actual readings, or their equivalent, are in each case tabulated. The dielectric-resistance is worked out from the 1st minute reading, and is expressed in megohms.

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* It may be pointed out that when I first put a guard-wire into practical use, at Mr. Price's suggestion, he mentioned to me that one of its advantages would be that in sheet-tests the surface leakage could be eliminated in the manner here described.

Experiment 1. (*Tinfoil Electrodes.* 150 *Volts.*)

TABLE I.

Load in grammes per cm.^{2}	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Total, megohms.
22·3	1000	345	369	26·8
31·5	"	379	383	24·4
61·3	"	450	453	20·6
76·0	700	345	347	19·5
90·8	"	358	359	18·7
105·3	"	367	369	18·3
119·8	"	378	378	17·8
197·2	"	430	430	15·6
226·7	500	328	329	15·0
256·1	"	335	335	14·7
282·1	"	340	340	14·4
311·5	"	343	343	14·3
361·1	"	357	356	13·8
390·5	"	360	359	13·6
449·1	"	365	363	13·5
543·3	"	368	368	13·3
449·1	"	373	370	13·2
390 5	"	370	368	13·3
315·0	"	368	367	13·3
256·1	"	366	365	13·4
196·8	"	362	360	13·6
149·7	"	345	344	14·2
90·8	"	335	334	14·7
31·7	"	285	287	17·2
22·3	"	258	259	19 0
197·2	"	349	348	14·1
361·0	"	367	367	13·4
544·1	"	381	379	12·9
22·3	"	266	268	18·5

(*The above table shows the effect of increase of load. The electrical test at each load was begun within* 1 *minute of adjusting the load.*)

Comparing the first and fifth columns of Table I., the resistance is seen to diminish from 26·8 to 13·5 megohms as the load increases from 22·3 grammes per cm.^{2} to 449·1 grammes per cm.^{2}; and it is seen to attain a fairly constant value at that load, which is not far from that used by Mr. Rayner (400 grammes). Mr. Rayner took the precaution of allowing 5 minutes for the load to settle before making a test, so that his results would probably be rather more uniform than these. The increase of resistance as the load diminishes takes place more slowly, and the value 19·0 megohms at minimum load is considerably less than the initial value. The increase of load at this point to 361 grammes per cm.^{2} lowers the megohms again to 13·4 ; and the resistance is further reduced to 12·9 megohms when Mr. Rayner's load is exceeded. When the minimum load is suddenly restored, the megohms rise once more to 18·5. It is especially interesting to note that for small loads the 2nd min. deflexion is always greater than the 1st min. deflexion, whereas for loads from 119·8 to 90·8, in the order shown in Table I., the 1 min. and 2 min. readings are at first equal, then the first is greater than the second, and finally they are again equal. The same general result will be noticed in the tests with tinfoil electrodes, which follow. Increase of voltage, as well as increase of the load, appears to have the effect of improving the contact when contact is imperfect. With small loads, it may be assumed that the surfaces are partially out of contact, and that the effect of the current is to improve the contact, so that the resistance *diminishes*, and the deflexion increases. Meanwhile the normal effect of "absorption" tends to produce an apparent *increase* in the resistance, and a point is reached at which no change of deflexion is observed, the effects being balanced. Finally, with the greater load, the films are in more intimate contact and the "absorption" effect preponderates, the second-minute reading being then less than the first-minute reading.

Experiment 2.—In order to examine the change in resistance with change of voltage, using tinfoil electrodes, the experiment represented by Table II. was carried out. The dielectric was short-circuited for about one minute between each change of voltage.

In this table it should be observed that with the small load the first-minute readings are never greater than the second-minute readings ; and that the apparent resistance of the dielectric diminishes as the testing-voltage increases. With the greater load, the first-minute readings are never less than the second-minute readings, and although there is a diminution of resistance with increased voltage, the effect

TABLE II.—*Tinfoil Electrodes.*

Load in grammes per cm.^{2}	Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
22·3	150	2000	272	285	31·0
”	300	600	288	290	20·2
”	450	500	424	426	17·3
”	600	300	382	383	15·7
”	750	200	341	342	14·8
”	600	300	386	386	15·6
”	450	500	442	443	16·6
”	300	600	316	317	18·4
”	150	2000	344	347	24·5
”	750	200	336	336	15·0
”	150	2000	342	345	24·6
543	150	1000	334	334	14·1
”	300	400	306	304	13·0
”	450	300	354	352	12·8
”	600	200	325	324	12·5
”	750	140	292	291	12·2
”	600	200	326	324	12·4
”	450	300	360	356	12·6
”	300	400	308	308	12·9
”	150	1000	346	345	13·6
”	750	140	292	290	12·2
”	150	1000	348	347	13·5

is far less marked than in the case of the small load. Increase of load is again seen to have a similar effect to increase of voltage.

At the end of these tests the dielectric sheet was tested between mercury surfaces, proceeding exactly as when using tinfoil electrodes. The results are shown in Table III. For this purpose the dielectric sheet is held in a vertical position symmetrically between two flat rings of ebonite, faced on each side with soft indiarubber. Disks of iron, forming in effect the jaws of a large ebonite vice, are then clamped one against the outer face of each ring. Mercury is poured into each of the hollow spaces between the iron disks and the dielectric, through holes in the top of each disk. The temperature can be conveniently read by placing a thermometer in the mercury.

This table renders it doubtful whether the true resistance of press-spahn varies greatly with the voltage, and it shows

TABLE III.

Mercury Electrodes.

Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
150	700	325	325	10·5
300	400	387	386	10·3
450	200	299	298	10·2
600	200	397	396	10·2
750	140	350	348	10·2
600	200	396	394	10·2
450	200	298	297	10·2
300	400	387	385	10·3
150	700	328	327	10·4
750	140	348	345	10·3
150	700	330	328	10·3

that the results obtained with tinfoil electrodes may be misleading when imperfect contact is interpreted as high resistance of the dielectric. In confirmation of the foregoing results, the sheet was left for a day, and was then tested again between mercury electrodes, the current being reversed as indicated in Table IV.

TABLE IV.

Mercury Electrodes.

Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
+150	700	287	284	11·9
−150	700	294	294	11·6
+300	400	342	335	11·6
−300	400	351	350	11·3
+450	200	269	261	11·3
−450	200	273·5	271	11·2
+600	200	358·5	346·5	11·3
−600	200	364·5	360	11·1
+750	140	319	317·5	11·2
−750	140	324	319	11·0
+600	200	365·5	355	11·1
−600	200	367·5	362·5	11·0
+150	700	303·5	302	11·2
−150	700	304	302	11·2

After leaving the sheet for four hours, it was again placed between the tinfoil electrodes, with the following result:—

TABLE V.

Tinfoil Electrodes.

Load in grammes per cm.^{2}	Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
22·3	+300	1000	459·5	466·5	20·1
„	−300	1000	498	494	18·6
„	+450	400	366	367·5	16·3
„	−450	400	377·5	380	15·8
„	+600	200	291	286	13·9
„	−600	200	292·5	293	13·9
„	+750	200	393	384	12·9
„	−750	200	395	393	12·8
„	+150	1000	232·5	232	20·2
„	−150	1000	217	212	21·6
543	−150	1000	408	408	10·0
„	−750	100	249	246	10·3
„	−150	1000	423	422	9·7

The final three tests with the greater load were intended to indicate the true resistance, the current therefore was not reversed; they are seen to be fairly uniform, and they justify the use of tinfoil electrodes for rough tests where mercury cannot be obtained. The fact that these three values are all lower than those given with the mercury apparatus on the same afternoon (Table IV.) is probably accounted for by the reduction of thickness due to the heavy load. It was 1¾ cwts. on a 5½ inch disk.

The next experiment was made with a view to testing the resistance with tinfoil electrodes, as nearly as possible under the same conditions as those adopted by Mr. Rayner. For this purpose the sheet was allowed to rest for 17½ hours. It was then put into the apparatus, and the weights on the disk area of 153·3 cm.^{2} were adjusted to give a mean load of 400 grammes per cm.^{2} Five minutes was then allowed, as in Mr. Rayner's experiments, for the conditions to become steady, before applying the voltage. Only one minute was allowed between the successive voltages. The readings and results were as follows:—

TABLE VI.

Tinfoil Electrodes.

Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
+150	1000	266·5	266·5	17·6
+750	140	240·0	238·0	14·9
+150	1000	284·5	283·5	16·5
+750	140	237·0	235·5	15·0

The results are fairly uniform, but the diminution of resistance with increase of volts is still marked.

For purposes of comparison, the dielectric sheet was then immediately placed in the mercury apparatus, and tested in precisely the same way, with the following results:—

TABLE VII.

Mercury Electrodes.

Volts.	Shunt, ohms.	Deflexion, 1 min.	Deflexion, 2 min.	Megohms.
150	1000	342	339	13·7
750	140	259	255	13·8
150	1000	332	331·5	14·1
750	140	255	252·5	14·0

A comparison between Table VI. and Table VII. seems to indicate that although the tinfoil electrodes under a load of 400 grammes to the cm.^{2}, and with an allowance of 5 minutes for settling down, are brought into fair contact with the dielectric, the contact is not so perfect as when mercury is employed. It is curious to note here, as in all these experiments, that where tinfoil electrodes are used the resistance appears to decrease with the increase of voltage, whereas with mercury contact this effect practically vanishes. I have shown (Proc. Physical Soc. vol. xiii. p. 157, 1894) that with celluloid sheets between unyielding metal plates, the change of apparent resistance with voltage is remarkably great. It is a matter of considerable importance in the study of dielectrics, and I have purposely described the experiments in detail so as to endeavour to establish the fact that what is often called change of dielectric resistance with voltage is, as a rule, merely a surface effect, and that it can be greatly reduced, if not entirely eliminated, by ensuring perfect contact with the electrodes. Moreover, the effect is generally found associated with so-called "negative electrification," the second-minute reading being greater than the first-minute reading.

In order to see how little the true resistance is affected by abrupt changes of voltage, an examination may be made of Tables III., IV., and VII., where only one minute was allowed between each change for the dielectric to lose its charge. The residual effect can be estimated by switching off the current at the end of a given minute, and immediately connecting the insulated electrodes through the galvanometer to earth. With a view to avoiding too lengthy a table illustrating this, I have calculated the values of [(deflexions) × (multiplying power of shunt)] in the following case, omitting shunts, and omitting megohms, from the tabulated figures. It may be stated, however, that the megohms, worked out from the 1st, 11th, 21st, 31st, 41st, and 51st mins., are respectively 43·2, 42·6, 42·3, 42·2, 42·0, and 41·7. The procedure was to apply the battery for 5 mins., and at the end of the 5th minute to switch the battery out of circuit and connect the insulated electrode to earth through the galvanometer, taking earth-readings for five minutes. At the tenth minute the battery was reversed, and the same process was continued in a cycle of reversals and earthings up to the 53rd minute. In the following table the reversals and earthings are sufficiently indicated by the signs of the deflexions, and by the smallness of the earth-readings in comparison with the battery readings. The fall after each change of connexions was perfectly steady, and in no instance was there anything approaching a rise after the deflexion corresponding to a given change had been recorded. [On another day, after allowing the sheet to rest for 12 hours, a similar test was applied, using 150 volts. The resistance calculated from the 1st min. (+) was 46·8 megohms. The resistance calculated from the 11th min. (−) was 46·5 megohms, a change of only 0·3 megohm against the change of 0·6 in the corresponding test with 750 volts. The earth-readings at the 6th min. and 16th min. were respectively −3·5 and +2, as against −10·5 and +7·5 with 750 volts. The earth-readings therefore are not directly proportional to the applied voltages originating them.]

TABLE VIII.

Mercury Electrodes (750 volts).

Min.	Deflexion.	Min.	Deflexion.	Min.	Deflexion.
1	+4630	19	+3	36	+7·5
2	+4589	20	+2	37	+5
3	+4548	21	+4721	38	+3·5
4	+4524	22	+4679	39	+3
5	+4499	23	+4638	40	+2·5
6	−10·5	24	+4606	41	+4762
7	−7·5	25	+4565	42	+4729
8	−6·0	26	−9	43	+4696
9	−5·0	27	−6	44	+4655
10	−4·5	28	−5	45	+4622
11	−4696	29	−4	46	−9
12	−4679	30	−3·5	47	−6
13	−4671	31	−4737	48	−5
14	−4630	32	−4713	49	−4
15	−4606	33	−4696	50	−3·5
16	+7·5	34	−4663	51	−4787
17	+5	35	−4639	52	−4762
18	+3·5			53	−4729

The maximum earth-reading is 10·5 with 750 volts. The effect of this residual charge upon subsequent readings is only 0·23 per cent.; or in terms of resistance say 0·1 of a megohm in the 42 megohms here concerned. In comparison with the effects of changes of humidity, it is probably negligible. The uniformity of the earth-readings in the above table is instructive. The (+) current applied at the 1st min. strikes the dominant note, and gives an earth-reading which at the 6th min. is −10·5. At the 11th min. the (−) current is applied, and gives an earth-reading which at the 16th min. is +7·5, *i. e.*, 3 divisions less than the corresponding earth-current of the previous charge; and we may assume that if the earth-current at the tenth minute had been allowed to act alone it would have fallen from −4·5 at the tenth to −3 at the sixteenth minute. At the 21st min. the (+) current is again applied; the corresponding earth-current at the 26th min. is −9. And it is seen, taking the readings at the 6th, 16th, 26th, 36th, and 46th min., that we obtain the sequence −10·5, +7·5, −9, +7·5, −9; a further indication that with mercury electrodes the apparatus conditions are trustworthy. These earth-readings were taken without a shunt, so that they are actual records of direct deflexions.

Retentive Force.—It seemed desirable to obtain positive evidence as to the magnitude of the retentive force between the tinfoil and the press-spahn, and to determine whether it was proportional to the voltage. For this purpose, a flat circular disk of gutta-percha, of the same size as the electrodes used in the foregoing experiments, was perforated over about half its surface with holes 1 cm. in diameter. A smooth sheet of thin tinfoil, not perforated, was attached to one side of this disk by a little chloroform. This arrangement allowed a certain amount of freedom of movement to the tinfoil, so that it could conform to the surface of a sheet of press-spahn when placed in contact with it. The disk was then suspended above a sheet of press-spahn, in a horizontal plane, from the beam of a balance, and its weight was counterpoised. Beneath the sheet of press-spahn was a second sheet of tinfoil connected to earth, so that the combination formed a condenser, the upper electrode of which was thus in balanced contact with the dielectric. The weight necessary to counterpoise the disk was first determined with no voltage between the electrodes, and the weight necessary to relieve it from contact was then measured at successive voltages. Check readings of the weight, taken between the various applications of voltage, never exhibited a difference greater than 1 centigramme for the normal (no-voltage) counterpoise. If there was residual coherence, it was therefore very small. The readings in a characteristic case were:—

TABLE IX.

Volts.	Counterpoise. Total grammes.	Retentive force, grammes.	Ratio of Retentive forces.	Ratio of Volts.
0	22·04	0	0	0
750	72·00	52	104	5
0	22·04	0	0	0
600	50·00	28	56	4
450	25·00	3	6	3
300	23·00	1	2	2
150	22·50	0·5	1	1
300	23 00	1	2	2
450	25·00	3	6	3
600	49 00	27	54	4
750	66 00	44	88	5

The retentive force between the tinfoil and the press-spahn increases far more rapidly than the voltage, the contact must therefore be more perfect with the higher voltage; this accounts in some measure for the changes of apparent dielectric-resistance with voltage shown in Tables II. and V. It is possible that a similar effect may account for small differences of capacity observed with different voltage with some tinfoil condensers, and it may also serve to explain to some extent certain abnormal "electrifications" with cables, where for any reason there is imperfect contact between the conductor and the dielectric.

DISCUSSION.

Mr. DAVID OWEN, in reference to the variation of apparent resistance with the volts applied in the case of the experiments made by the Author where tinfoil electrodes were used, pointed out that an effect of the same kind takes place in the case of the leakage resistance of condensers of paraffined paper and tinfoil. If the insulation resistance be measured by observing the rate of leak of the condenser, the curve of volts and time drops more slowly than according to the exponential law. The same result is obtained if the current maintained by the application of a steady voltage across the condenser terminals is measured by a galvanometer in series. For such a condenser a recent test gave an apparent resistance of 100 megohms for a P.D. of 4 volts, but this value steadily dropped with rise of voltage and was 41 megohms for 20 volts. Though the voltage here used was much less than those applied by the Author, the effect in question is the same in kind and in degree.

Mr. A. Campbell said Mr. Appleyard's paper was an interesting contribution to our knowledge of a difficult but practically important subject. He had established the advantages of mercury electrodes. Mr. Campbell remarked that he had used tinfoil and indiarubber electrodes for the last fifteen years, and mentioned that before they were adopted at the National Physical Laboratory for Mr. Rayner's and other work, they were put through a series of tests with varying weights, and from the observations a working pressure was chosen which gave tolerably accurate results. He had also tried guard-wires with several kinds of sheets, and found that the surface leakage was negligible. In Mr. Rayner's experiments at high temperatures, the changes due to a fall of temperature of a few degrees were so enormous that any inaccuracy due to the clamps was of trifling importance. Mr. Campbell mentioned that when measuring the capacities of thin sheets, tinfoil electrodes gave results which in some cases varied with the applied pressure. For example, with dry paper as dielectric, increase of pressure gave considerable increase of capacity, and since in this case the contact does not matter, the effect must have been due to approach of the electrodes. With mica, on the other hand, a change of pressure from 50 to 400 kgms. per sq. cm. caused very little change in capacity when the thickness of the mica was 0·5 mm. If the paper be black-leaded the change of pressure has less effect. He therefore suggested that in cases where the mercury was inconvenient, the use of black-lead might be of advantage.

Dr. H. A. Wilson expressed the opinion that when using mercury electrodes the diminution in resistance might be due to compression of the dielectric caused by the pressure of the mercury, rather than to better contact between the mercury and the dielectric.

Mr. W. A. Price said that the question of insulation tests was an important and far-reaching one. Dr. Wilson had suggested that the pressure of the mercury might cause a change of form, but he did not think that pressures so small would produce any appreciable change. The Author had shown that provided you obtained good contact between the electrodes and the dielectric, you could obtain constant values for the insulation resistance.

Prof. W. CASSIE suggested that the pressure might be applied to the tinfoil electrodes by placing a sheet of india-rubber over each of them, or by putting the plate of dielectric with its tinfoil electrodes in a flat indiarubber bag and pumping out the air under the indiarubber. Thus the pressure of the atmosphere would produce more uniform and effective contact of the tinfoil with the dielectric. The pressure obtainable in this way might have any desired value up to about a kilogram per square centimetre.

LXII. *A Refractometer.* *By* ROLLO APPLEYARD*.

FROM the time that chemists realized the value of a knowledge of refractive index as a guide to detecting adulterants, and to establishing the composition of certain organic substances, a number of instruments have been designed for facilitating the tests, and for ensuring precision. The classical method of determining refractive index by the two-fold operation of measuring the refracting angle and the angle of minimum deviation of a prism of the substance, the so-called "goniometer method," is still in use for standard tests; but it is now supplemented, and to some extent supplanted, by such instruments as the Pulfrich refractometer, the Abbé refractometer, the butyro-refractometer, and the oleo-refractometer, nearly all of which may be regarded as developments of the direct-reading refractometer invented more than a century ago by Dr. William Hyde Wollaston †. As each of the methods corresponding to these various instruments have distinct advantages in testing the particular class of substances for which they are intended, and as all these instruments, including the goniometer, have certain parts in common, it suggested itself that a single instrument might be designed which could be adapted to the conditions required by any of the methods. With the assistance of Messrs. Hilger, who have most skilfully worked out the details for me, an instrument has been constructed which I believe will fulfil these requirements.

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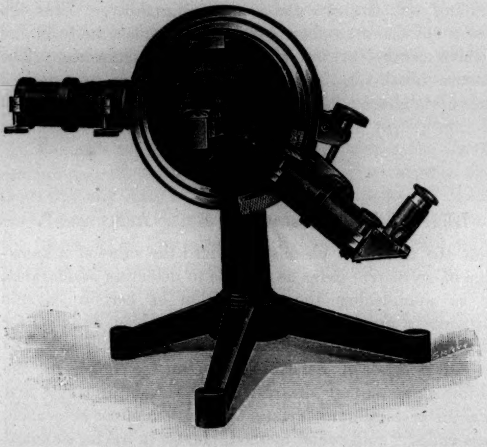
* Read June 16, 1905.

† Phil. Trans. Royal Society, pp. 365–380, 1802.



REFRACTOMETER.

As arranged for the "goniometer method," this instrument resembles an ordinary spectroscope, and it can be used for spectroscopic work. It has then a horizontal circle, provided, as usual, with a collimator and telescope. The circle is of platinoid, 8 inches in diameter; the divisions can be read by means of the telescope-vernier to 30 seconds of arc. A separate concentric levelling-table, 6 inches in diameter, the upper surface of which is worked optically flat, is provided for the prism; this table has three levelling-screws which fit into radial grooves in a prism-table beneath it. The prism-table can be rotated, and it carries a vernier, reading to 30 seconds on the circle.

To convert the instrument into one of the other forms of refractometer, the circle must be fixed in a vertical plane, as shown in the figure. It is provided with a cast-iron bracket, which enables the axis of rotation of the prism-table to be set horizontal. The special form of prism, corresponding to the required method, can then be fitted to the prism-table, and the operation of testing can be carried out, making use of the same circle and verniers as before. For example, in applying the Pulfrich method, a Pulfrich prism is mounted and fixed upon the prism-table, which can be rotated by a slow-motion mechanism, shown near the top of the illustration, for bringing the prism face into adjustment in the proper plane. For convenience of reading, the telescope has an eyepiece set at right-angles to its axis. This Gauss eyepiece is focussed upon a polished steel pointer, illuminated by a side mirror, the pointer being placed at the focus of the object-glass of the telescope. To use the instrument, the telescope is first set horizontal, its vernier then reading exactly 0 on the vertical circle. The prism-table is then adjusted at the slow-motion so that the vertical face of the Pulfrich prism reflects light from the steel pointer back along the axis of the telescope to form a real image, which is observed at the eyepiece. When the prism is in the right position, this image appears exactly in line with, and opposed to, the steel pointer itself. The instrument can then be used as an ordinary Pulfrich refractometer, the telescope being moved round the circle until the refracted image is seen in the eyepiece. The position and arrangement of the collimator for this method is shown in the figure.

LXIII. *The Magneto-Optics of Sodium Vapour and the Rotatory Dispersion Formula.* *By* R. W. Wood, *Professor of Experimental Physics, Johns Hopkins University*[*].

[Plate XXI.]

It has been shown in a previous paper[†] that the vapour of metallic sodium is an ideal substance for investigating the effect of a strong absorption-band on the magnetic rotation of the plane of polarization. The preliminary work was not very satisfactory, however, as the method employed did not admit of very accurate determinations of the wave-lengths, and the verification of the rotatory dispersion formula could only be considered as approximate. Improvements in the methods of observation and design of the apparatus have been accompanied by an increase in accuracy, which could scarcely have been hoped for, as will be seen by the following comparisons.

In the preliminary work no readings of the rotation for wave-lengths between the sodium lines could be obtained, while in the present case accurate readings have been made for as many as *nine* different values of λ between D_1 and D_2. Rotations as great as 1440° (four complete revolutions) have actually been observed, and this with a 10 cm. column of not very dense vapour, in a field of only 2000 C.G.S. units. This was for a wave-length midway between D_1 and D_2. In the earlier work it was only with the greatest difficulty that the bright lines which appeared in the region of the red and green channeled absorption spectrum could be seen. They have now been photographed to the number of about 160 with the 14-foot concave grating. Comparatively few of the absorption-lines show any trace of magnetic rotation, scarcely one per cent.; and the fact seems to have been established that these lines coincide with the bright lines of the fluorescent spectrum of the vapour, which has been found to be by no means the exact complement of the absorption spectrum as was at first supposed.

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[*] Read June 30, 1905.

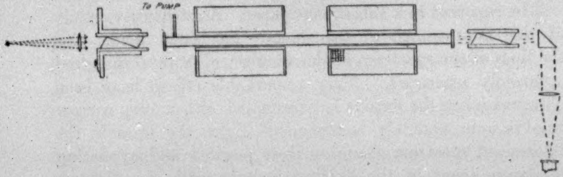
[†] Wood and Springsteen, "The Magnetic Rotation of Sodium Vapour," Phys. Review, July 1905.

In the present paper the magneto-optics of the vapour for light travelling along the lines of force will be discussed. The double-refraction of the vapour, which occurs when the rays are perpendicular to the field, has also been studied, and will be reported in a subsequent paper. An exhaustive study of the fluorescence of the vapour has been made, and the lines of the spectrum to the number of several hundred accurately measured. Very remarkable effects have been observed when the vapour is illuminated with a *very* narrow band of approximately homogeneous light, the lines in the fluorescent spectrum changing their position and appearing to dance about in the liveliest manner with the slightest change in the wave-length of the exciting light. The motion is of course only an illusion, lines disappearing and others reappearing, like the sparks of a spinthariscope. Stokes's law is violated in a most flagrant manner, bright lines coming out on both sides of the excited region. The behaviour of the spectrum indicates that we are dealing with a number of groups of electrons, each group containing a large number of vibrators. The excitation of one of the vibrators sets the whole group agoing, but does not start disturbances in the other groups. These and other remarkable phenomena will be fully discussed in the following paper ; they are mentioned here only on account of the apparent relation between magnetic rotation and the fluorescence.

In the earlier experiments referred to above, the sodium was heated in a tube of hard glass, the ends of which were closed with thin plate glass. The tubes lasted but a short time, owing to the chemical action of the metal, and in fact usually cracked on a second heating. In the present work a tube of thin steel has been used, the ends of which projected from the helices of the magnet. It was found that the field-strength within the steel tube did not differ greatly from that obtained when glass tubes were used.

The arrangement of the apparatus is shown in fig. 1. A piece of thin seamless steel tubing of such a diameter as to slip easily through the hollow cores of the electromagnet, from which the conical pole-pieces have been removed, is procured. A short piece of small brass tubing is brazed into one end, through which the tube is exhausted. It has been found that a good vacuum is essential, all trace of the rotation disappearing in hydrogen or nitrogen at atmospheric pressure.

Fig. 1.



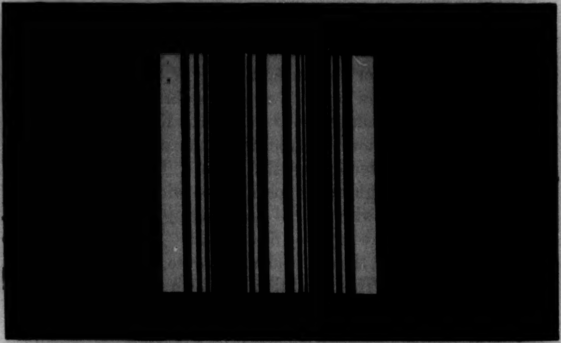
In the earlier work this fact was not known, and the tubes were exhausted and then sealed off from the pump. The hydrogen liberated from the sodium must have raised the pressure to at least 15 cms. in all of these experiments. In the light of what is now known, it is surprising that any results at all were obtained under these conditions.

A lump of sodium the size of a walnut is melted in an iron crucible, and poured out into a V-shaped trough made of thin sheet iron. As soon as the bar is solid it is placed in the iron tube, one end of which has been previously closed with a small piece of plate-glass cemented on with sealing-wax. The tube is introduced into the magnet, the sodium bar pushed to a position midway between the helices, and the other end closed with a piece of glass in a similar manner. The ends of the tube should be coated while hot with sealing-wax before the introduction of the sodium. One has then only to wave a Bunsen flame over them and press on the piece of glass, previously heated; the sealing-wax should come into optical contact with the glass to insure an air-tight joint. The tube is now connected with an air-pump which will produce a vacuum of a millimetre or two. If the air-pump leaks, it is a good plan to place a glass stopcock between the pump and tube to prevent the entrance of traces of air after exhaustion. For purposes of demonstration it is sufficient to heat the tube gradually with a Bunsen burner turned down low. In the present work, however, where constancy of temperature was essential, electrical heating was invariably used; the tube was wrapped with a thin sheet of asbestos board, around which was wound a spiral of iron wire, and the whole subsequently covered with an asbestos jacket.

The light from an arc-lamp, made parallel by a lens, is passed through a Nicol prism, the steel tube, and a second nicol, after which it is brought to a focus by means of a second lens upon the slit of a spectroscope. In the present case, the instrument in question was a concave grating of 14-foot radius, the observations being made both visually and by means of photography.

We will now consider briefly the phenomena which are presented when the sodium vapour is formed in the magnetic field. The nicols are crossed and the spectrum vanishes completely. The magnet is now excited and the Bunsen burner placed under the tube, the tip of the flame barely touching it. In a few minutes we see two bright yellow lines exactly in the position of the D lines. The light constituting these lines comes, however, from the crater of the arc, as we can easily see by intercepting the beam. The lines are in reality double, though they appear single with low resolving powers, while even with the 14-foot grating their duplicity cannot be made out when they first appear. As the density of the vapour increases, the components separate, four lines being distinctly visible. This condition is shown in Plate XXI. fig. 1 *a*. The lines are formed by the constituents of the white light bordering the two absorption-lines, which having suffered a rotation of 90° in the magnetic field, is enabled to pass the second nicol. The lines continue to separate, becoming broader, and presently we see between them a second pair of lines, for which the rotation is 270, the two dark regions between representing rotations of 180. This stage is shown in fig. 1 *b* and *c*, Plate XXI. In fig. 1 *b*, the two inner 90° lines are beginning to fuse together, the centre being partially dark however. In fig. 1 *c*, the fusion is complete and the centre of the system of lines is bright. With a further increase in the vapour-density, the outer lines (90°) separate still further, and widen out into broad flares of light, and other lines appear between them, corresponding to larger rotations, the system resembling a set of diffraction fringes, as shown in fig. 2. The centre of the system, as I shall designate a point midway between D_1 and D_2, becomes bright and dark in succession, as is shown in the succeeding figures of Plate XXI. fig. 1. Only a few of the inner lines show in the photographs, as they are very close together, and the vibration of the building prevented their resolution in the photograph. They could be distinctly seen with the eyepiece, and accurate settings of the cross-hair could be made. Their position with respect to the D lines was very accurately determined by means of a filar micrometer, settings being made on the dark lines between as well. A little practice was necessary before a complete series of readings could be taken without error, but consistent results were finally obtained. The method adopted was as follows. A current of given strength was turned into the iron spiral, and as the vapour-density increased the alternations of brightness and darkness at the centre were counted. No measurements were commenced until a steady state was reached, which was indicated by a fixed condition at the centre. The positions of the bright lines and dark spaces were then measured, the readings being taken down by an assistant. If at the end of the series any change was found to have occurred at the

Fig. 2.



centre, it was rejected. As soon as the series was completed, the heating current was shut off, and the tube allowed to cool. During this process the alternations of brightness at the centre were counted, and if the number of changes differed from that originally recorded the series was thrown out. The rapid changes which occur as the tube cools, especially an unjacketed tube, are most interesting; the centre of the system changes from bright to dark with great rapidity. The largest number of complete alternations observed was eight, corresponding to a rotation of 1440 degrees, one alternation (*i.e.* from bright to bright) corresponding to 180. The whole thing lasts but a few seconds, the bright band "winking" almost as fast as one can count. We can, as it were, see the plane of polarization actually turning around and around, for four complete revolutions. Beyond this point the absorption becomes too strong to admit of further observations between the D lines, but readings may be taken above and below them in the spectrum, with vapours of very great density.

When the vapour acquires a considerable density, a most magnificent bright-line spectrum appears in the red and green-blue region. Each bright line corresponds to a dark line in the absorption spectrum, but only a small percentage of the dark lines appear to exercise a rotatory power. Some of the strongest absorption-lines are absolutely unrepresented in the magnetic rotation spectrum, which indicates that there is some radical difference in the absorbing mechanism.

Much time has been spent in a further photographic study of the channeled absorption spectrum, a piece of work which was commenced three years ago by the writer in collaboration with Mr. J. H. Moore. It has since been found that the presence of hydrogen, nitrogen, or carbonic acid modifies the appearance of the absorption spectrum. The photographs previously obtained were made by passing the light through a steel tube containing sodium vapour in an atmosphere of hydrogen, under which condition the absorption spectrum has a most beautiful shaded appearance, being made up of fluted bands, each band containing a very large number of fine lines. If the hydrogen is removed, and the sodium vapour formed *in vacuo*, the fluted appearance disappears almost entirely, although the individual lines remain. Many new lines appear, however, which are broader and more distinct than the others, *and these lines appear* to coincide with the bright lines in the rotation spectrum. This point will be more fully investigated in the future, the work on the absorption spectrum not having been completed at the present time.

Returning now to the measurements obtained in the vicinity of the D lines, we will discuss them in their bearing on the theory of magnetic rotation.

The rotation has been measured over a considerable range of wave-lengths in the case of very dense vapours, by the method described in the previous paper. In brief this method consisted in rotating one of the nicols through various angles and measuring the position of the two dark bands which move, the one up, the other down the spectrum. The dispersion of the 14-foot grating was too great for this part of the work, and a large plane grating was accordingly combined with a pair of telescope objectives of six-foot focal length. A very brilliant spectrum was obtained in this way, and the dark bands were not so broad but that the cross hairs of the micrometer could be set on the centre of the band with a fair degree of accuracy. Readings of the rotation were obtained throughout the region comprised between $\lambda=5840$, and $\lambda=5932$. The dark bands cannot be pursued with accuracy to greater distances from the D lines, on account of their increased breadth. It will be understood from what has been said in the previous paper, that these dark bands retreat from the D lines when one of the nicols (originally parallel) is rotated towards the position of extinction. The bands enter the red and green regions when the nicols are very near the crossed position, under which circumstances the bright lines appear, and completely obliterate all trace of the dark bands, which by this time have become very broad. The rotation for the wave-lengths corresponding to the dark bands, is the angle through which one of the nicols has been rotated, measured from the crossed position.

The values found with vapours of various density are roecrded in the following table. The tables are designated by the magnitude of the rotation midway between the D lines.

Centre 30°.

5897·1	90
5896·8	180
5896·5	270
5895·4	270
5895·3	180
5894·9	90
5893·2	30
5891·4	90
5890·9	180
5890·6	270
5889·3	270
5889·0	180
5888·7	90

Centre 50°.

5898·4	30
5897·8	45
5897·6	60
5897·3	90
5897·0	180
5896·7	270
5895·3	270
5894·9	180
5894·6	90
5893·2	50
5891·9	90
5891·4	180
5891·0	270
5889·1	270
5888·7	180
5888·3	90
5887·8	60
5887·6	45
5886·9	30

Centre 90°.

5899·7	20
5899·3	30
5898·8	45
5898·3	60
5897·8	90
5897·45	180
5897·1	270
5896·8	360
5895·3	451
5895·0	360

Centre 90° *con.*

5894·5	270
5894·4	180
5893·34	90
5891·9	180
5891·5	270
5891·3	360
5891·1	450
5890·86	640
5889·0	450
5888·8	360
5888·5	270
5888·2	180
5887·5	90
5886·96	60
5886·5	40
5886·0	30
5884·8	20

Centre 180°.

5901·3	20
5900·3	30
5899·4	40
5899·0	60
5898·2	90
5897·7	180
5897·3	270
5897·1	360
5896·9	450
5894·9	540
5894·7	450
5894·5	360
5894·1	270
5893·2	180
5891·9	270
5891·6	360
5891·3	450
5891·2	540
5888·6	540
5888·4	450
5888·2	360
5887·9	270
5887·5	186
5886·8	90
5886·2	60
5885·4	40
5884·6	30
5883·2	20

Centre 450°.

5902·1	30
5902·2	40
5900·3	60
5899·4	90
5898·7	180
5898·0	270
5897·7	360
5897·5	450
5897·3	540
5894·8	900
5894·7	810
5894·5	720
5894·2	630
5894·0	540
5893·2	450
5892·3	540
5892·1	630
5891·8	720
5891·5	810
5888·3	720
5888·1	630
5887·9	540
5887·7	450
5887·4	360
5886·9	270
5886·1	180
5884·7	90
5883·2	60
5881·7	40
5880·3	30

Centre 540°.

5902·8	40
5901·5	60
5900·1	90
5898·8	180
5898·3	270
5898·0	360
5897·7	450
5894·4	720
5893·7	630
5893·5	540
5892·4	630
5892·1	720
5891·8	810
5891·7	900

Centre 540° *con.*

5888·0	630
5887·8	540
5887·4	450
5887·1	360
5886·7	270
5886·1	180
5884·6	90
5883·2	60
5881·5	40

Centre 720°.

5875·0	20
5876·3	25
5880·5	50
5881·4	60
5882·6	80
5883·2	95
5885·7	180
5887·0	270
5893·5	720
5899·0	180
5900·8	97
5901·8	80
5902·8	61
5903·6	50
5908·0	25
5909·7	20

Centre 960°.

5903·9	60
5902·0	90
5900·2	180
5899·6	270
5899·2	360
5898·8	450
5893·4	960
5886·6	540
5886·3	450
5885·9	360
5885·1	270
5884·5	180
5881·8	90
5880·0	60

A number of photographs of the spectrum with the dark bands in different positions were made with a large three-prism spectrograph, recently constructed for the study of the fluorescence spectrum (Pl. XXI. fig. 2). It was at first thought that by measuring the positions of the bands on the negatives, more accurate results could be obtained than by visual observations; but the reverse proved to be true, though fairly good curves were obtained. The density of the vapour was much greater than that employed for the visual observations, and the dark bands were found to be symmetrical with respect to the D lines, that is the rotation was the same at points in the spectrum at equal distances (measured in wave-lengths) to the right and left of the D lines. This was not the case with less dense vapours, the rotation being greater in the vicinity of D_2 than in the vicinity of D_1. The rotation constant of D_2 was found to be about double that of D_1; but since the direction of rotation is the same on opposite sides of the absorption-bands, the effects of the two bands are additive, and lack of symmetry will be less noticeable with very dense vapours, where the measurements are made in a region not very close to the lines.

Verification of the Rotatory Dispersion Formula.

Drude, in his *Lehrbuch der Optik,* has given two formulæ for the magnetic rotatory dispersion, the first of which, developed from the hypothesis of molecular currents, calls for an anomalous effect on crossing the band, and obviously does not apply to sodium vapour. The second, developed from the Hall-Effect Hypothesis, predicts rotations of similar sign and equal magnitude for wave-lengths symmetrically situated in the spectrum, with respect to the centre of the absorption-band. The formula deduced for the rotation is

$$\delta = \frac{1}{n}\left(\frac{a}{\lambda^2} + \frac{b\lambda^2}{(\lambda^2 - \lambda_m^2)^2}\right).$$

In the case of sodium vapour n differs so little from unity that it can be left out of account. Investigations of the refractivity of the vapour by the writer, have shown that with very dense vapour (comparatively speaking) the value of n may be as great as 1·3 in the immediate vicinity of D_2, but calculations showed that in all of the cases dealt with in the study of the magnetic rotation, n was practically equal to unity. When working very near the D lines the vapour was extremely rare, while observations made with denser vapours covered regions not very near the absorption-lines. Since, moreover, the rotation is zero for very short waves, the first term in the formula drops out, i. e., $a=0$. Precisely the same thing was done in the case of the ordinary dispersion formula, since the refractive index was found to be unity for very short waves*. The similarity in the sign of the rotation on opposite sides of the bands results from the fact that the quantity $(\lambda^2-\lambda_m^2)$ (which changes sign on crossing the absorption-band) is squared. An attempt was first made to verify the simplified formula by writing $\lambda_m=5893$, *i. e.*, assuming the D lines to be a single absorption-band. The value of the constant could then be determined from a single observation, and the rotations for different values of λ calculated. The calculated values agreed very closely with the observed on one side of the double absorption-band, but the agreement on the other side was very poor. This was at once seen to be due to the fact that the rotatory power of the two lines was very different, and the following formula was consequently adopted:—

$$\delta=\frac{a\lambda^2}{(\lambda^2-\lambda_{D_2}^2)^2}+\frac{b\lambda^2}{(\lambda^2-\lambda_{D_1}^2)^2}.$$

The constants a and b were calculated from two observations of λ, one greater than D_1, the other less than D_2. In almost every case the ratio $\frac{a}{b}$ was found to lie between the limits 1·8 and 2·03. Writing $D_1=5896$ and $D_2=5890$, the values of the constants a and b for vapours of different densities are recorded in the following table. The densities are defined by the magnitude of the rotation for a wave-length midway between D_1 and D_2.

Density	*a.*	*b.*	*a/b.*	Centre Rotation Calc.
Centre $\delta=$ 50°	1123	587	1·9	51°
" " 90°	2217	1350	1·65	98°
" " 180°	4309	1860	2·3	161°
" " 360°	5111	2571	1·98	205°
" " 450°	9279	4545	2·03	369°
" " 540°	10326	4607	2·24	392°
" " 720°	12750	7100	1·8	538°
" " 960°	23873	10333	2·6	893°

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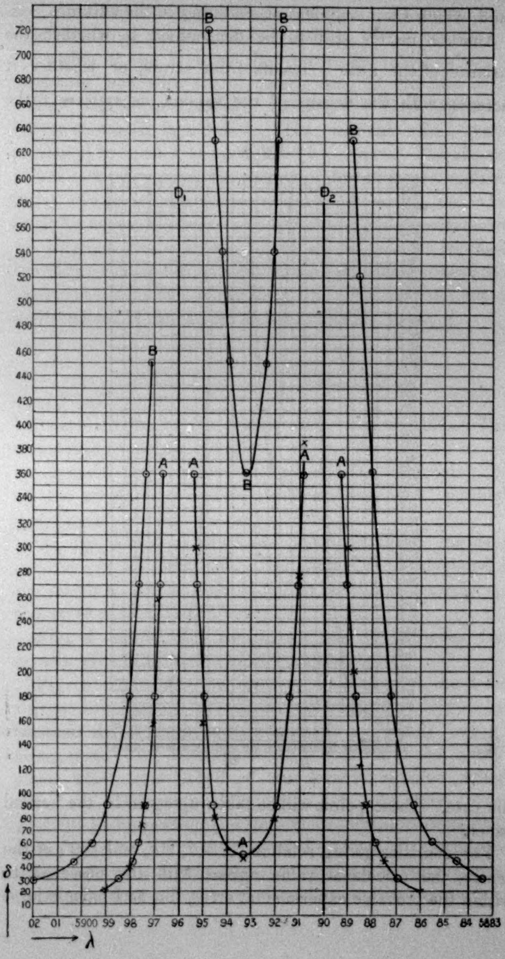
* See previous paper on the "Dispersion of Sodium Vapour," Phil. Mag. vol. viii. p. 293 (1904).

The "centre" as defined above does not in fact fall at $\lambda=5893$, but at 5893·5, *i. e.*, when the bright bands unite, the point of unison lies nearer D_1 than D_2. This is owing to the fact that the expansion of the bands from D_2 is more rapid than from D_1.

It was found that if the constants *a* and *b* were calculated from different pairs of readings, there was considerable difference between the values, a circumstance which indicates either that the formula is at fault, or that the observations were not sufficiently accurate. We are dealing with very steep curves, as will be seen by reference to fig. 3, and small errors in wave-length readings will make large errors in values deduced from them. The severest test of all is to calculate the rotation at the centre from values of the constants determined from two medium values of δ (say 90 or 180) outside of the D lines. This has been done in each case, and in almost every case the calculated value has turned out too small. I have given all of the tables of data thus far obtained, and some of them are doubtless less accurate than others. By means of an improved heating coil (which is in process of construction), of porcelain and strip platinum, I hope to be able to hold the conditions more nearly steady and obtain still more accurate results. It will also be necessary to secure greater rigidity of the grating. For large rotations at the centre, say above 180, we cannot be quite sure of the exact value, as the best that can be done is to regulate the heat so that the condition at the centre is either of maximum brightness or darkness. Errors of say 15 or 20 degrees can be made easily. It is quite possible too that the formula is incomplete. It seems more than likely that the molecular currents play some part, and that the formula built up on the hypothesis of the Hall effect is incomplete. As we shall see presently, however, it represents the rotation outside of the D lines with a surprising degree of accuracy, while between the lines it gives in most cases a curve which is elevated somewhat above the experimental curve. It is possible that the refractive index plays some part, though I regard this as very doubtful. Midway between the D lines the refractivity is equal to unity, rising and sinking rapidly to the right and left of this point. It is not worth while to experiment

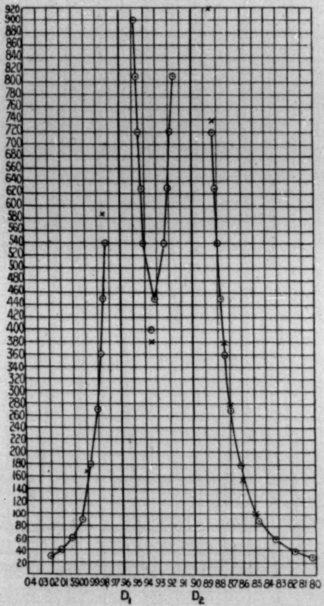
Fig. 3.


B
B
B
B
B
B
A
A
A
A
A
D₁
D₂
720
700
680
660
640
620
600
580
560
540
520
500
480
460
440
420
400
380
360
340
320
300
280
260
240
220
200
180
160
140
120
100
90
80
70
60
50
40
30
20
10
δ
02
01
5900
99
98
97
96
95
94
93
92
91
90
89
88
87
86
85
84
5883
λ

further with the formula until still more accurate data are at hand, which I am confident of obtaining in the autumn.

Another matter still to be determined is whether the variation of the ratio $\frac{a}{b}$ results from errors, or whether it increases with the density of the vapour, as appears to be the case. These points will be taken up again in the second paper.

Fig. 4.



For densities giving rotations of 50°, 90°, and 180°, at the centre, the calculated value agrees with the observed to within the probable experimental error, for, as I have said, it is impossible to fix the rotation at the central band to within say 15. In the case of denser vapours the discrepancies are much too large to be attributed to errors of this sort.

Outside of the D lines the calculated values agree remarkably well with the observed, even in the case of the densest vapour used, as will be seen by referring to the curves shown in figs. 3 & 4. The observed values are represented by circles, the computed by crosses. The observed values are, of course, for values of δ which are multiples of 90. In the computations, values of λ were selected, which could be accurately represented by five figures, consequently we judge of the accuracy with which the formula represents the dispersion by seeing whether the calculated values fall on the curve determined by experiment. The computations are easily made. One has but to determine the value of

$$\frac{\lambda^2}{(\lambda^2-\lambda^2_{D_2})^2} \quad \text{and} \quad \frac{\lambda^2}{(\lambda^2-\lambda^2_{D_1})^2}$$

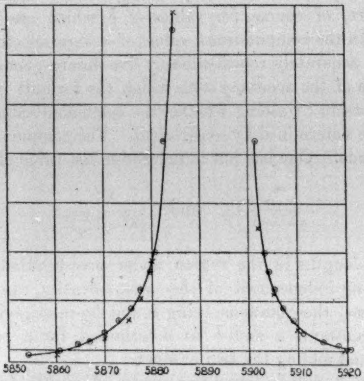
for wave-lengths in the region under investigation. These terms being independent of the vapour-density, can be used in all cases, the rotations being found by multiplying them by the constants *a* and *b* as determined for a particular density, and adding the two products.

On fig. 3 will be found two curves, A and B, for centre rotations of 50° and 360°. The values of δ calculated for the curve A, will be seen to fall in most cases exactly upon the curve of observed values.

Fig. 4 shows that the agreement is also good for a density giving a rotation of 450° at the centre.

Fig. 5 represents the rotatory dispersion of the densest vapour investigated by visual methods. The bright band between the D lines had disappeared completely owing to absorption. This corresponds to a rotation at the centre of about 1500, since 1440 is the largest rotation actually observed. The observed values in this case were obtained with the plane grating and long focus telescope-lenses of five inches aperture. The positions of the broad, dark bands were measured with a micrometer eyepiece for different settings of the nicol, and readings were also taken on the narrow bright and dark bands lying between the broad ones and the D lines, whenever they were distinct enough. In this case, the calculated values lie on a curve slightly lower than the observed curve. This may be due to the fact that the value of λ for which the rotation is equal to that of the nicol may not lie exactly at the centre of the band. On the whole, the formula may be said to represent the rotatory dispersion, exclusive of the region between the D lines, with a fair degree of accuracy. If, however, the constants a and b are calculated from different pairs of readings on the same curve, different values are found. On this account, it seems desirable to redetermine the curves with still greater precision, which will be done in the autumn.

Fig. 5.



Vapour of a still greater density was investigated by a photographic method.

The spectrum of the light transmitted through the two nicols and the sodium tube was photographed with the three-prism spectrograph. Seven exposures were made on a trichromatic plate, with the second nicol in seven different positions. A sodium flame, burning continuously in front of the slit of the spectrograph, recorded the D lines on the plate.

The tube was heated electrically, and the density of the vapour remained constant throughout the total time of exposure. Each exposure amounted to less than half a minute. One of these multiple photographs is reproduced on Plate XXI. fig. 2. The picture may be said to show the dispersion curve. The positions of the dark bands, which appear on each side of the D lines, were measured on the dividing engine, and the wave-lengths determined from the calibration curve of the spectrograph. For these dense vapours it was found that the rotatory dispersion curves were symmetrical with respect to a point midway between D_1 and D_2. The dispersion should therefore be well represented by a single-term formula

$$\delta = \frac{a\lambda^2}{(\lambda^2-\lambda_D^2)^2},$$

in which $\lambda_D = 5893$. This was found to be the case, as the following table shows.

Value of constant $a = 135600$.

λ.	δ Observed.	δ Calculated.
	°	°
5980	5	4·47
5950	10	10·4
5933	20	20·2
5923	40	38·0
5917	60	58·9
5912·5	90	89·2
5874	90	93·1
5869	*43*	*43·0*
5864	40	40·6
5852	20	20·2
5833	10	9·2
5814	5	5·2

The value of the constant was determined from the value of δ at wave-length 5869, as given by the curve.

The Bright Line-Spectrum produced by Magnetic Rotation.

Attention has already been drawn to the remarkable bright line-spectrum which presents itself when the nicol prisms are crossed. This spectrum, which at first could only be seen with the greatest difficulty, was finally obtained of such brilliancy that it could be photographed with the 14-foot concave grating. A good vacuum was found to be the most essential condition. In the earlier work with tubes sealed off from the pump, the pressure due to the liberated hydrogen was probably responsible for the faintness of the lines. The density of the sodium vapour must be just right. If it is too rare, the rotation is insufficient to bring out the lines, if too great the absorption weakens the spectrum. When the conditions are exactly right, the lines are almost as bright and numerous as the metallic lines in the arc. Photographs of the complete spectrum were made on trichromatic plates, made by the Cramer Dry Plate Co. of St. Louis. These plates were found to be sensitive up to the red lithium line ($\lambda=6705$). An exposure of fifteen minutes was sufficient with a small concave grating of about a metre radius. Two of these photographs are reproduced natural size on Plate XXI. fig. 4.

I have never observed a doubling of any of these lines such as occurs in the case of D_1 and D_2. So great is the quantity of the light transmitted by the second nicol, that a brilliant orange-coloured image of the crater of the arc appears on the slit of the spectroscope the moment the current is turned into the magnet, notwithstanding the fact that the nicols are crossed.

Inasmuch as it seemed desirable to investigate these lines under higher dispersion and determine their wave-lengths as accurately as possible, the apparatus was transferred to the concave-grating room.

The Ruhmkorff magnet was abandoned at this point, as it was found that an exposure of four or five hours would be necessary, and the magnet became dangerously hot in half an hour, when fed with the necessary current. In its place a very large magnet, built by Professor Rowland for the study of the Zeeman effect, was used. This magnet could be fed with a 110-volt current without resistance, and operated continuously for any necessary length of time.

A steel tube nearly 3 cms. in diameter could be used with this magnet, which increased in no small degree the amount of light available. After a number of failures, a very satisfactory negative was secured with the 14-foot grating showing the rotation spectrum and the absorption spectrum side by side on the same plate. The plate showed over 60 lines in the blue-green region, some of them so faint that they could scarcely be seen under the microscope of the dividing engine, while others were strong enough to yield prints capable of reproduction.

Another plate was exposed and a comparison iron spectrum recorded by the side of the rotation spectrum. The wave-lengths of the rotatory lines were measured on both plates, using the iron lines and the sodium absorption lines as standards. Some of the latter were redetermined, as it was found that they were much sharper when the sodium was heated in a high vacuum, than when heated in hydrogen, as in the previous work by Dr. Moore and the writer. A print obtained from the plate on which the rotatory spectrum and absorption spectrum were recorded, is shown on Plate XXI. fig. 5. The absorption spectrum is above. Many lines are, of course, visible on the negative which do not show on the print. The rotatory lines in many cases coincide with the heads of the groups of absorption-lines, though the centre of the line appears to be slightly displaced beyond the head of the group of absorption-lines. The displacement is, however, very slight, not more than half the width of the line. A list of the wave-lengths of all the lines visible on the negative follows. The approximate intensities are represented by numerals, 10 indicating the maximum intensity and 1 the minimum. Lines marked with an interrogation point were so faint as to be doubtful.

Green Rotation Spectrum.

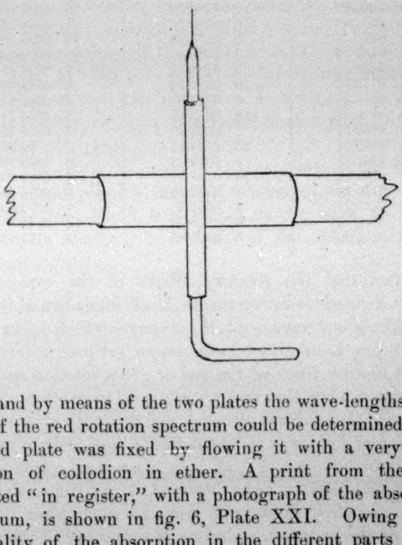
1	5225·34	1	5033·54		2	4839·56
1	5218·49	1	5025·66		9	4837·49
1	5212·02	1	5003·12	broad	2	4819·43
1	5186·70	10	5001·57		1	4814·60
1	5179·71	5	4979·34		1	4812·68
1	5172·98	1	4970·85		3	4810·16
	5171·98	1	4967·10		3	4802·62
1	5169·04	1	4964·39		5	4792·67
	5165·85	9	4962·85		3	4782·89
1	5140·71	1	4958·62		1	4777·00
2	5133·73	1	4933·93		1	4766·94
2	5126·54	3	4932·64		6	4756·69
1	5119·34	2	4924·32		1	4752·04
2	5094·78	4	4912·10		2	4738·51
7	5087·31	1	4904·67		4	4727·52
8	5079·78	1	4903·38		1	4716·90
1	5071·58	1	4896·65		1	4715·63
1	5052·83	1	4894·58		1	4703·78
1	5049·56	1	4892·77		2	4692·54
3	5048·49	2	4883·81		2	4670·30
7	5040·65	2	4865·59			

It is especially noteworthy that many, in fact most, of the strongest absorption-lines are not represented at all in the rotation spectrum. This fact is of fundamental importance, for it indicates that the absorbing mechanism is different in the two cases. Just wherein the difference lies cannot as yet be definitely stated. On either hypothesis as to the cause of the magnetic rotation, the effect will diminish as the mass of the electron increases. The lines of the rotatory spectrum may correspond to the negative electrons of small mass; the other lines in the absorption spectrum which show no rotatory power may be due to positive electrons, or at all events to some form of vibrator having a much larger mass than the negative electrons. The fact that the bright lines in the fluorescence spectrum coincide with the lines of the rotatory spectrum appears in accord with this assumption; for the agitation of the electrons by the light vibrations will be greatest for those having the smallest mass. It will doubtless be possible to speak more definitely in regard to this point, after the fluorescence spectrum has been more carefully studied. The matter will be discussed further in the paper on the fluorescence of the vapour.

The rotation spectrum in the red and orange region is more brilliant than the green-blue one, but it was found impracticable to photograph it with the 14-foot concave grating. Excellent photographs of it were secured with a small concave grating of about a metre radius. One of these, together with the absorption spectrum, is reproduced in fig. 3, Plate XXI. As the red lines could be *seen* without difficulty with the 14-foot grating, a method was devised for mapping them which may prove useful in other lines of work. A brass rod, on which a short brass tube had been fitted, was fastened into the grating camera just below the groove which held the plate. To the movable tube was soldered a second tube of 2 mm. bore, into which was fitted a short piece of brass rod. This rod carried a needle point. The whole arrangement is shown in fig. 6. A plate of thin glass was carefully smoked over a gas-flame, and then wiped clean with the exception of a narrow strip 4 mms. wide along the centre.

This plate was inserted in the camera in place of the photographic plate, and the spectrum brought into such a position that the lines crossed the smoked strip. The positions of the lines could be accurately recorded by bringing the needle-point in contact with the glass exactly on the centre of a line and then pushing it up across the soot-film. Intensities could be recorded approximately by varying the length of the lines. The sodium lines were also recorded on the plate in the same manner to serve as reference marks. An iron arc spectrum was then photographed together with the sodium lines, and by means of the two plates the wave-lengths of the lines of the red rotation spectrum could be determined. The smoked plate was fixed by flowing it with a very dilute solution of collodion in ether. A print from the plate mounted "in register," with a photograph of the absorption spectrum, is shown in fig. 6, Plate XXI. Owing to the inequality of the absorption in the different parts of the spectrum above the D lines, it was necessary to take photographs, at three different vapour densities, to represent the entire set of lines. The prints have been carefully registered

Fig. 6.



in mounting. The small percentage of lines which exercise rotatory power is very marked in this case. The position of the D lines is indicated on the left-hand end of the plate, the red being to the right. The wave-lengths of the lines in the red rotation spectrum are given in the following table:—

Red Rotation Spectrum.

8	6005·26	3	6218·94	2	6427·45
8	6019·22	5	6230·59	7	6434·23
4	6023·45	7	6235·96	2	6435·05
5	6027·47	3	6240·19	3	6438·30
7	6031·79	5	6252·18	3	6441·62
4	6060·25	7	6255·73	3	6445·24
6	6063·91	2	6260·32	10	6449·86
7	6067·47	10	6262·55	1	6470·58
2	6075·42	6	6272·78	10	6481·89
10	6107·78	3	6283·40	10	6490·36
9	6108·25	3	6283·84	7	6501·45
4	6113·59	10	6314·14	3	6515·69
2	6113·81	6	6317·09	4	6545·91
10	6125·67	10	6323·81	3	6555·71
10	6139·17	2	6358·65	5	6556·66
8	6162·74	10	6374·11	7	6579·55
2	6166·10	4	6379·24	8	6609·80
6	6172·32	10	6386·50	7	6623·96
4	6179·92	6	6399·79	3	6676·88
8	6183·93	3	6405·84	4	6692·60
9	6196·93	8	6419·69	4	6745·61
8	6216·56	3	6426·85	5	6761·19

The fact that the rotatory effects of the vapour only manifest themselves in a vacuum is of fundamental importance. The exact nature of the changes which occur when an inert gas is admitted have not as yet been determined. The fact that no trace of the red or green rotation spectrum could be seen when the vapour was formed in an atmosphere of hydrogen at atmospheric pressure, and that only a very slight restoration of light was visible at the D lines, led me to infer that the rotatory power had been destroyed. On experimenting with helium light, however, it was found that the rotation measured in degrees was the same after the admission of the hydrogen as before, the intensity of the transmitted light being however very much less. On this account it seems probable that the effect of the chemically inert gas is to modify the absorbing power of the gas. It has been found that the presence of hydrogen or nitrogen interferes in a very marked degree with the fluorescence of the vapour.

The same thing is true in the case of iodine vapour, which has been found to possess a very great rotatory power for green light, when the vapour is formed in an exhausted tube. A small bulb, containing a few crystals of iodine, is exhausted and sealed off from the pump. On placing it between the poles of the magnet between crossed nicols, a most beautiful emerald-green light is restored the moment the current is turned into the coils. The spectrum of this light has been photographed and observed visually. It resembles the absorption spectrum so closely as to lead to the belief that the narrow black lines in it are produced by absorption. Little or no rotatory power is exhibited for red or yellow light, though this portion of the spectrum is filled with fine absorption-lines, and the blue region is also wanting in the rotation spectrum, though the absorption is strongest at this point. These effects are at present under investigation, and will be reported in a subsequent paper.

The investigation reported in the present paper is one of a series made possible through substantial aid received from the Rumford Fund, two grants having been made by the American Academy for a study of the optical properties of sodium vapour.

Much assistance has also been rendered by Mr. A. H. Pfund, for whose services I am indebted to the Carnegie Institute.

DISCUSSION.

Mr. A. CAMPBELL asked if the apparatus could be used to measure magnetic fields; the sensitivity would be high.

Prof. R. W. WOOD said there would be difficulty in determining the density of the vapour. It could be used, however, to determine the density of the vapour from a knowledge of the magnetic field.

LXIV. *The Fluorescence of Sodium Vapour and the Resonance Radiation of Electrons. By* R. W. WOOD, *Professor of Experimental Physics, Johns Hopkins University**.

[Plate XXII.]

IN the modern theories of absorption we find frequent use made of the conception of a re-emission of radiant energy by an electron which is vibrating in unison with the incident light-waves. The emitted energy should, however, be of the same wave-length as that of the exciting waves, and while we have plenty of examples of photoluminescence, it seems probable that in these cases the phenomenon is extremely complicated, for the emitted radiation consists of a heterogeneous mass of waves, usually of slower period than that of the incident light. Repeated efforts have been made by various investigators to detect a lateral emission of yellow light by sodium vapour when in the act of absorbing sodium light. These efforts have been unsuccessful for reasons which will be presently given. The phenomenon has at last been observed: a dense mass of non-luminous sodium vapour radiating a brilliant yellow light when illuminated by the light from a very intense sodium flame. The same phenomenon has been observed in the region of the channelled absorption; in this case, however, radiations of other wave-lengths are emitted, as well as those of the same period as that of the exciting light. This seems to be the first case found of the phenomenon, which it may perhaps be well to style *resonance radiation*, to distinguish it from fluorescence. The intimate connexion between fluorescence and phosphorescence, and the almost indisputable evidence that the latter phenomenon is associated with chemical changes produced in the substance, makes it appear probable that fluorescence and resonance radiation are two entirely different phenomena, though the former is doubtless caused in some roundabout way by resonance.

In the present paper, however, I shall, for convenience, use the term fluorescence, though we may eventually have to distinguish between fluorescence spectra and resonance spectra.

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* Read June 30, 1905.

The very remarkable fact has been ascertained that the stimulation of the vapour with approximately monochromatic light, furnished by the monochromatic illuminator (spectroscope), results in the emission of light, the spectrum of which consists of a number of more or less regularly spaced bright lines of different intensities. The slightest change in the wave-length of the exciting light results in the disappearance of these lines, and the appearance of another set; the phenomenon indicating that the molecule contains a number of groups of electrons, the excitation of any one of which sets the entire group in vibration, without however disturbing the other groups. As the wave-length of the exciting light is slowly changed, the fluorescent spectrum presents a most remarkable appearance. The lines appear in rapid motion, the luminous bands moving in a rippling manner, like moonlight on water. The motion is of course an illusion, due to the continual disappearance and reappearance of the bright lines, the phenomenon reminding one forcibly of the scintillations produced on a screen of zinc sulphide by the electron bombardment produced by radium.

In the earlier work on the fluorescence of the vapour, carried on by the author in collaboration with Dr. Moore, the metal was vaporized in a tube of seamless steel. Owing to the distillation of the metal to the cooler parts of the tube, it was necessary to open the tube and introduce a fresh charge after two hours' heating. This was a great gain on the exhausted glass bulbs used by Wiedemann and Schmidt, which blackened, and became useless in half a minute or less, but something better was obviously necessary for the large number of long exposures necessary for a careful study of the remarkable changes in the fluorescence spectrum, which accompanied changes in the wave-length of the exciting light. For purposes of demonstration a plain tube of steel, or even brass, is all that is required. Steel is preferable, for it is probable that the sodium would eventually eat its way through brass. The latter metal can, however, be used if a large steel tube cannot be procured. The tube should be 2 or 3 inches in diameter and about 2 feet long. A small piece of brass tubing is brazed into one end, through which the tube can be exhausted to a pressure of a few millimetres, for it is only at low pressures that a fluorescence of any brilliancy can be obtained. A piece of clean sodium, obtained by melting the metal in a crucible and pouring it out on a plate, is introduced into the tube, and the ends closed with plate glass, cemented on with sealing-wax. One window should be attached before the introduction of the sodium. The following method gives the best results:—Heat the end of the tube and apply the melted sealing-wax smoothly to the rim, building it up to a depth of about 2 mms. Stand the tube on end, heat the glass plate, wave the flame over the wax until the surface melts, and then immediately apply the plate. The wax must be in optical contact with the metal and the glass, otherwise the joint is sure to leak. It is a good plan to go around the joint with a very minute pointed gas flame, heating the wax until it runs freely over the surfaces. A mercurial or mechanical pump can be used for the exhaustion. If the pump leaks ever so slightly, a glass stopcock should be put between it and the tube, and all the joints made of glass and sealing-wax. With a little practice the whole apparatus can be set up in fifteen minutes; and the tube, once prepared, can be put away for future demonstrations, care being taken to close it hermetically with wax.

Sun or arc-light is focussed in a slightly oblique direction, by means of a lens, directly over the lump of sodium, which should be brought to the centre of the tube. On heating the tube, the brilliant spot of green fluorescent light can be observed through the opposite end, care being taken to keep the eye out of the emergent beam of white light. Tubes of this description, while perfectly satisfactory for demonstration, are unsuitable for long continued spectroscopic study, and various devices were accordingly tried.

The apparatus in its final form consisted of a seamless tube of thin steel 3 inches in diameter and 30 inches long, with a steel retort at its centre in which a large amount of sodium could be stored. The retort was made by fitting two circular disks of steel to a short piece of tubing, just large enough to slip snugly into the larger tube. The circular ends of the retort were provided with oval apertures as shown in fig. 1. The retort was half filled with sodium, the molten metal being poured in through one of the apertures. It was then

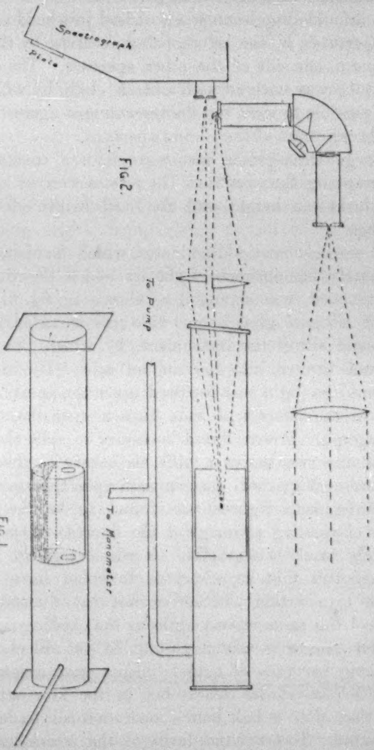


Fig. 2.

Fig. 1.

introduced into the tube and pushed down to the centre, after which the plate-glass ends were cemented on as shown in the figure. This arrangement prevented the rapid diffusion of the vapour, and enabled a large supply of metal to be kept at the centre of the tube. The tubes used in the earlier work required re-charging after two hours' continuous operation, while the retort-tube could be operated for several hundred hours on a single charge.

The illuminating beam was focussed just inside one of the oval apertures in the retort, being received by the opposite wall just to one side of the other aperture. The end of the large tube was covered with a black cloth, by which device it was possible to view the fluorescent spot against the dead-black background of the second aperture.

A large three-prism spectrograph was constructed for photographing the spectra. The prisms were of clear dense flint 4 inches in height, and the focal length of the lenses 36 inches.

The monochromatic illuminator, which furnished the approximately homogeneous light by which the fluorescence was provoked, was arranged as shown in fig. 2. A small strip of silvered glass 2 mm. wide was mounted in front of the second slit of the instrument, by which the emergent beam was thrown into the sodium tube. By means of a split lens the light was focussed upon the aperture, and an image of the fluorescent spot thrown upon the slit of the spectrograph. It was found necessary to split the lens and separate the two halves a trifle, as otherwise the image of the fluorescent spot was thrown back upon the small mirror. The arrangement figured was found to be the only one capable of yielding an image of the fluorescent spot upon an absolutely black background. It was extremely important to accomplish this, as otherwise it would have been impossible to ascertain whether or not the fluorescent light contained the same wave-lengths as the exciting light. For the same reason it was important to get rid of all traces of floating particles of oxide. These gave a good deal of trouble in the earlier work, but in the new tube it was found that after a half hour's operation all trace of them disappeared. To prove the purity of the atmosphere in the tube and the blackness of the background, it was only necessary to adjust the prisms of the monochromatic illuminator for some wave-length incapable of exciting fluorescence. The brightest part of the spectrum in the yellow region, not far below the sodium lines, answered the purpose admirably.

By means of two right-angled prisms it was possible to substitute a powerful beam of sunlight for the monochromatic beam, by which the spectrum of the fluorescence provoked by white light could be easily photographed or observed.

With white light illumination the fluorescence spectrum is made up of a large number of hazy lines, which in the yellow and yellow-green region are arranged in groups or bands, which lie close together in the vicinity of the D lines, widening, however, as the blue region is approached. Coincident with the D lines there appears a hazy band, the surrounding region being nearly devoid of light, which, if the vapour is not too dense, can be resolved into a double line, the components of which coincide with D_1 and D_2. These lines only appear when the vapour is stimulated with light of the wave-length of the sodium lines, *i. e.* the fluorescent light can be regarded as an emission of light by the electrons, in virtue of the vibrations excited by the incident light. This was proven in two ways: first, by illuminating the vapour with the light of a very intense sodium flame, which provoked a bright yellow fluorescence; secondly, by illuminating it with light from a spectroscope, and varying the wave-length continuously. The yellow band only appeared when the spectroscope furnished light of the wavelengths of the D lines. Further work, it is hoped, will show whether the D_1 vibration is independent of that giving rise to D_2. To solve this interesting question, it will be necessary to illuminate the vapour with the light of D_2 only, and ascertain whether both lines are present or not in the fluorescent spectrum, a difficult observation, but one which can doubtless be made. There is in addition a red fluorescence, the spectrum of which Wiedemann and Schmidt believed to be devoid of bands or lines. This, however, is not the case, for the spectrum in the red was found to be crossed by a large number of dark bands. They are much more difficult to observe than the green bands, and are only very conspicuous when the density of the vapour is considerable, which accounts for their having been overlooked previously. The general appearance of the fluorescence spectrum stimulated by white light is shown in Pl. XXII. fig. 1. In this photograph the red region and the double band at the D lines are not shown.

In the blue and green-blue region the lines and bands of the fluorescent spectrum are irregularly spaced. From $\lambda=5050$ to $\lambda=5071$ we have a regularly spaced system of lines, which get closer and closer together as we approach the less refrangible end of the spectrum. These lines collectively form a system of bands, the spectrum having a fluted appearance. The spacing between the bands also becomes less as we approach the more refrangible end. With white-light illumination the bands cannot be distinguished in the yellow region, but by exciting the fluorescence with blue-violet light, the bands can be traced up to the very end of the spectrum ($\lambda=571$). The bands change their position slightly as the wave-length of the exciting light is altered, but the lines of which they are composed remain fixed in position. What actually happens appears to be a redistribution of intensity among the individual lines.

The yellow-green-blue fluorescence spectrum can be divided roughly into three parts, which we will designate A, B, and C. Region A, comprised between wave-lengths 571 and 500, is made up of regularly spaced lines which collectively form a system of bands, owing to the periodic fluctuations of intensity. Region B, between 500 and 477, is made up of irregularly spaced lines, often hazy on one side, and of varying width. This region contains the first three bands of the group already alluded to, though special precautions are necessary to bring them into view. With dense vapour, and illuminating light of the proper wave-length (about 460), the region is seen to break up into three very broad patches; in other words, there is a periodic fluctuation of intensity as in region A. Region C between wave-lengths 477 and 468 is made up of very broad lines, spaced with a fair amount of regularity and arranged for the most part in pairs.

These three regions are indicated by brackets in fig. 1, Pl. XXII.

The positions of the bands, which apparently have their origin in the periodic fluctuations of intensity of the lines, are recorded in the following table :—

4791
4882
4971
5050
5120
5185
5246
5316
5361
5415
5465
5510
5551
5585
5620
5645
5665

The centre of the bands cannot, of course, be very exactly located, especially in the case of the first three which are found in region B.

These bands apparently obey the same law as the oxygen bands in the solar spectrum, *i. e.* if we plot their wave-lengths as abscissæ, and take ordinates which increase by equal amounts, we obtain a curve which is approximately a parabola. In this way we can calculate the position of the head of the group, which was found to fall at wave-length 5715. Two bands were therefore missing, one of which at 5676 was subsequently found in the spectrum obtained with blue-violet excitation, dense vapour, and long exposure : still longer exposures would probably record a trace of the first band at 571.

As has been said, these bands shift their position as the wave-length of the exciting light changes. The shifts are greatest in the region between $\lambda=515$ and $\lambda=545$, where it may amount to the full width of the band or 25 A.E. The bands in the vicinity of $\lambda=555$ remain fixed in position, as is probably true of all the less refrangible ones. It was at first thought that the head of the group remained fixed, the group opening out, or expanding so to speak, but this appears not to be the case ; and the law governing the motion of these bands, if one exists, cannot be determined until a larger number of photographs have been obtained. The probable cause of the motion will appear presently. The wave-lengths of the lines which collectively make up the fluted bands were determined from the calibration curve of the spectrograph, and are recorded in the following table. It was at first thought that they would obey the law of constant second differences, but such was not the case. It has not yet been possible to measure the lines above $\lambda = 555$, though the bands extend to 571.

Wave-lengths of bright lines in Fluorescence Spectrum of Na *Vapour.*

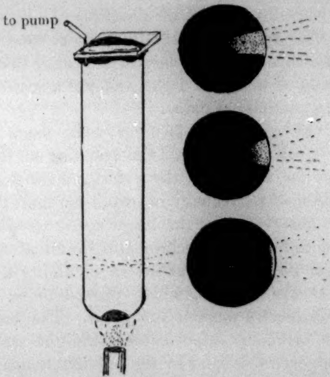
5553	5315	5040
48	09	34
43·5	02	25
39	5296	5000
35	89	4995
31	83	88
14	76	80
10	70	64
05	64	59
5499	58	34
95	52	25
91	46	12
86	39	04
81	32	4895
77	26	85
62	19	75
58	13	67
53	06	60
47	Faint 52	50
42·5	5194	39
37	88	21
31	80	12
26	73	03
21	66	4795
5399	Faint 59	85
93	49	78
87	41	69
81	34	59
76	27	53
70	20	43
64	13	39
59	03	33
Faint 51	5096	28
44	88	16
38	80	04
32	72	4695
27	49	84.
21		

Stimulation with Monochromatic Light.

Having described briefly the general appearance of the spectrum when the fluorescence is provoked with white light, we will now consider the case of monochromatic stimulation. In all of the earlier work with Mr. Moore, and during the first stages of the present investigation, no trace of any emission-band at the D lines was observed, neither was any fluorescence whatever observed when the prisms of the monochromatic illuminator were adjusted so that the instrument delivered light of wave-length approximately that of the D lines. A bright band at the D lines had, however, been observed by Wiedemann and Schmidt, and independently by the author in the case of sodium vapour formed in exhausted glass bulbs, which observations have been confirmed by Pucianti, who has recently published an interesting paper on the subject*.

It seemed probable that in the case of a large mass of vapour in a steel tube, the radiations capable of exciting the D line fluorescence were absorbed in the outlying mass of comparatively rare vapour before coming to a focus in the dense mass. As it was found difficult to observe the phenomenon in glass bulbs on account of the formation of a sodium dew on the walls, and the speedy discoloration of the glass, the arrangement shown in fig. 3 was adopted. A small piece of clean sodium was placed in a large test-tube, hermetically sealed with a glass plate, and furnished with a tube for exhaustion. Since a much more powerful D line radiation can be obtained from a bright sodium flame than from a selected portion of the spectrum, the monochromatic illuminator

Fig. 3.



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* *Accad. Lincei,* Atti. xiii pp. 430–440 (1904)

was abandoned, and a small oxy-hydrogen flame, heavily charged with sodium, substituted for it. An image of this flame was formed by means of a lens, at the centre of the tube immediately above the lump of metal. On gently heating the tube and viewing the interior through the cover-glass, a faint cone of yellow light was observed which, however, only extended to the centre of the tube, owing to the inability of the effective radiations to penetrate to a greater depth. This stage is shown in the upper diagram of fig. 3. As the temperature rose and the density of the vapour increased the fluorescence increased in brilliancy, the region retreating towards the wall, until only a bright skin of yellow light remained, which lined the wall of the tube at the point where the exciting radiations entered, as shown in the lower diagram. This experiment proved conclusively that sodium vapour is capable of re-emitting a yellow light, when in the act of absorbing the light of the soda flame. The appearance of the yellow cone and its behaviour as the vapour-density increased proved that it was a true re-emission of absorbed light, and not a scattering of light by floating particles of oxide.

An attempt was next made to observe the same phenomenon in the steel retort tube. On focussing an image of the flame upon the window of the retort a distinct spot of yellow light was observed, though it proved too faint to admit of satisfactory observation in the large spectrograph. Returning once more to the monochromatic illuminator, I was surprised to find no difficulty at all in obtaining a bright emission-band at the D lines, which proved to be double when the vapour-density was not too great. The band was of considerable brilliancy when direct sunlight was used. The failure to observe this band in the previous experiments may have been due to an insufficiently high vacuum, or to insufficient density of the sodium vapour. It has been found that long-continued work results in a steady improvement of the spectrum as to brilliancy, though it is quite impossible to ascertain the precise reason. It is probable that the proper conditions are unconsciously found, and the detrimental ones discarded, by what Prof. James would call the "Sub-conscious-self." The band at the D lines only appeared when the light from a corresponding part of the spectrum fell upon the vapour, showing that the electron system which produces these lines is independent of the system which gives rise to the channelled spectra. Whether D_1 and D_2 are connected has not yet been determined. To do this it will be necessary to free the illuminating beam from the D_1 radiations, and observe whether or not the fluorescent band is double. An attempt will be made to accomplish this.

We will now consider more in detail the appearance of the fluorescence spectrum, when the illuminating light consists of a more or less monochromatic beam from the illuminator. As has been said, the yellow end of the spectrum appears only when the stimulating light is blue-violet. It is probable that the yellow light will also be radiated when the stimulating light is of the same wave-length, though such a radiation has not yet been detected. The absorption for radiation in this region ($\lambda=571$ to 560) is very small, and the vapour appeared practically black in all cases thus far observed. The region of maximum fluorescence moves down the spectrum as the stimulated region moves up, as is well shown in the series of photographs in fig. 3 (Pl. XXII.). In this series of photographs the stimulated region is indicated by the luminous region to the left. The lowest one, *e*, shows very clearly that when the stimulation is at the violet end only the first five bands at the yellow end appear. In spectrum "*a*" the stimulation is at the centre. The change in the position of the groups or fluted bands is best shown in figs. 4 and 5. The same thing can be seen in fig. 3, by moving a vertical straight edge along the series. That the spectra are in perfect "register" can be seen by noting the position of the D lines at the right. These were recorded by holding a soda flame in front of the slit for a few seconds during each exposure. Radiations both above and below the excited region are present, showing that Stokes's law is not obeyed. This is especially noticeable in fig. 2 (Pl. XXII.), where the excited region is indicated by a white rectangle. The plate-holder of the spectrograph was arranged so that it could be moved up and down, and as many as seven spectra could be photographed on a single plate if necessary. After recording a fluorescent spectrum, the plate was elevated and the beam from the monochromatic illuminator thrown back into the instrument by means of a mirror. In this way it was possible to obtain a record of the extent of the region of fluorescence *below* the excited region. It was thought desirable to obtain a more homogeneous type of exciting light than that furnished by the monochromatic illuminator, and a mercury arc in a quartz tube was accordingly tried. Though the light was extremely intense, and though its radiation contained a number of lines in the region desired, it was found impossible to obtain the slightest trace of fluorescence. No one of the periods happened to be quite right; and the electrons remained quiet. The light of the cadmium spark, though much less intense, excited the fluorescence; but it has not yet been ascertained to what degree the " continuous background " of its spectrum was responsible for the phenomenon. Work along these lines will be continued during the coming year.

The shifting of the bands, which results in all probability from a change in the distribution of the intensity among the individual lines, when the wave-length of the exciting light is altered, has been already alluded to. In the earlier stages of the investigation, it was necessary to work with the slits of the monochromatic illuminator wide open. Later on, as the methods improved, it was found possible to narrow them by degrees, until finally a fluorescent spectrum of sufficient intensity for observation was obtained with the slits not much wider than a hair. A remarkable phenomenon was at once observed : the slightest change in the wave-length of the exciting light caused the spectrum-lines to flutter about in a most extraordinary manner. It was quite impossible to follow the changes by the eye, they were so complex. As a line disappeared or appeared, the sharp and shaded edges often changed place, the appearance reminding one of a flag flying first on one side of the mast and then on the other. Obviously a vast amount of information regarding the nature of spectra is to be obtained by a careful study of these changes, which can only be done by photography. I have estimated that fifty or one hundred photographs will be necessary for a complete record of the changes which occur when the wave-length of the exciting light is shifted over the range 450 to 550, and as each picture requires an exposure of four or five hours, the task is seen to be no small one. If the spectra were then printed in succession on the film of a kinematograph and projected on a screen, it would be possible to follow at leisure the wonderful changes which occur.

At the time of writing this paper only half a dozen photographs have been obtained, three of which are reproduced in fig. 6, Pl. XXII. Much of the detail is lost in the process of printing and reproduction, and on this account a careful drawing has been made on coordinate paper of all the details which are to be made out on the original negatives, fig. 7, Pl. XXII. The white-light fluorescent spectrum A is recorded on the lower line, and the spectra obtained with nearly homogeneous radiation above, as is the case in the photograph as well. A number of the lines in the complete spectrum shown in fig. 6 have been lettered A, B, C, D, etc. These letters correspond with the letters in the diagram, fig. 7. The letters at the left-hand side enable the spectra to be identified with the diagrams.

The line at which the excitation occurs is indicated by a double-headed arrow. This line always appears in the fluorescent spectrum. For example, in spectrum B the excitation is at wave-length 4962. In this case three lines of shorter wave-length appear, as well as the line at 50, and the lines at 504 and 5071, which belong to the groups of lines or bands already referred to, indicated by brackets. In spectrum C we are stimulating the vapour at 4935, and we obtain the second line in each of the first two groups. In spectrum D we stimulate another member of the same series, and also get the second line in the first two groups; while immediately above (E) by stimulating 4866 we obtain the third line in the second and third groups, this line being wanting in the first group, even when the vapour is excited by white light. In spectrum F, with a stimulation at 4839, we get another set of lines. The cause of the shift of the bands previously observed is at once apparent, the individual lines belonging to different groups of electrons. The number of groups has not yet been ascertained, but the photographs indicate that there are at least four, and probably more. It has not yet been possible to study the minute details which accompany the appearance or disappearance of a line, which are even more interesting, but another year's work on the subject will doubtless yield fruitful results. Longer exposures will doubtless enable these relations to be traced throughout a wider range of wave-lengths, but from what is already known it is clear that the shift in the apparent position of a group of lines results from a change in the distribution of the intensity among the individual lines constituting the group.

These experiments show in a striking manner the complexity of the piece of machinery which we will call the sodium molecule. Professor Rowland once said that a molecule is much more complicated than a piano. In most cases, all that we have been able to do is to strike the entire keyboard at once, but in the case of sodium it seems possible to strike one key at a time. A study of the fluorescent spectra of other vapours will doubtless do much to clear up the mystery of the mechanism of molecular radiation.

These investigations have been made possible through substantial aid received from the Rumford Fund of the American Academy, before which the paper has been presented. Much assistance has been rendered by Mr. A. H. Pfund, whose services were secured though a grant made by the Carnegie Institute.

Baltimore, June 1905.

LXV. *On the Method of Transmission of the Excited Activity of Radium to the Cathode.* *By* WALTER MAKOWER, *B.A., B.Sc.; John Harling Research Fellow in the University of Manchester.*

1. *Introduction.*

IT has been shown by Rutherford* that when a negatively charged rod is exposed to the emanation from thorium in a closed vessel, the quantity of excited activity deposited upon the rod in a given time is independent of the pressure of the gas with which the emanation is mixed as long as this pressure exceeds a certain value, but that below this limit the quantity of excited activity deposited on the negative electrode diminishes, at first slowly, and then more rapidly, as the pressure of the gas is reduced.

This phenomenon gives reason to suppose that at the moment of its formation the excited activity is uncharged, and that it is only by some subsequent secondary action upon the gas in which it is produced that it acquires the positive charge in virtue of which it is carried to the negative electrode. It was with a view to investigating the nature of this secondary action that the present experiments were undertaken.

No experiments regarding this point seem as yet to have been carried out with radium emanation; for this reason, and on account of the rapid rate of decay of thorium emanation and the consequent inconvenience of working with this substance, radium emanation was employed in the present research.

The variation with pressure of the amount of excited activity deposited in a given time on a negatively charged rod when exposed to a constant quantity of emanation was carefully investigated, and it was found that at low pressure the excited activity deposited on the rod depended, not only on the pressure of the gas, but also on the distance between the positive and negative electrodes.

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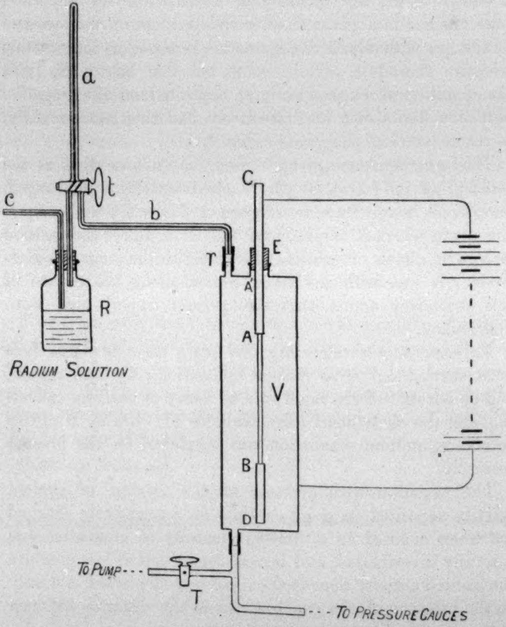
* Phil. Mag. [5] vol. xlix. (1900).

As the result of some preliminary experiments it was shown that the activity acquired by the rod was independent of the difference of potential between the electrodes over a considerable range. In these experiments a potential-difference of about 60 volts was used throughout.

2. *Description of Apparatus.*

The cylindrical metal vessel V (fig. 1), which could be filled with emanation when desired, was connected to the

Fig. 1.



positive pole of a battery of thirty small storage-cells. The wire AB (14·5 cm. long) was screwed into a stouter rod AC connected to the negative pole of the battery, and passing through a rubber stopper E fitting air-tight into the neck of the vessel V. The portion AA′ of the rod AC projected about 6 cm. into the vessel V, and another rod BD 6 cm. long was screwed to the lower extremity of AB, which was therefore exposed to a uniform electric field.

To obtain a constant supply of emanation, a strong solution of radium salt was placed in the glass bottle R provided with a stopper, through which passed a long capillary tube C reaching to the surface of the solution but not dipping into it.

The air inside R was thus always kept at atmospheric pressure, and by making C sufficiently long the rate of escape of emanation from the bottle by diffusion was rendered small. After standing for some weeks, the concentration of the emanation in the air above the radium solution was found to have become sensibly constant.

To transfer a definite volume of gas from R into V, the closed limb *a* of the three-way capillary stopcock *t* was connected by the capillary *b* with the vessel V, to which it was attached by rubber *r*. By means of a pump the whole was exhausted to a low pressure, when the pump was disconnected by closing the tap T. The stopcock *t* was then turned so as to connect *a* with R, allowed to stand for a short time and then again turned into its original position. In this way the volume *a* (about ·2 c.c.) of gas was transferred from R into V, and by repeating the operation any required volume of emanation could be introduced into V.

As the volume of *a* was always small compared with that of V, practically all the emanation in *a* was discharged into V at low pressures; at higher pressures it was easy to apply a small correction for the quantity of emanation left in *a*.

The pressure in V was registered on a mercury manometer at the higher pressures, and on a McLeod gauge at the lower pressures.

The rod AB was exposed for 30 minutes to the emanation, and then transferred to another metal cylinder in which its activity was tested in the usual manner by means of a quadrant electrometer. As under these conditions the activity of the rod remains constant for about 10 minutes*, it was possible to take several observations of the activity, the means of which are recorded in Tables I., II., and III.

3. *Observations and Results.*

Experiments were performed with three vessels V of diameter 1·1 cm. (vessel I.), 2·6 cm. (vessel II.), and 8·3 cm. (vessel III.) respectively. The pressure of the air with which the emanation was mixed was varied and the activity acquired by the rod in 30 minutes recorded.

With vessel I. it was found difficult to prevent the rod CD from touching the sides of the cylinder. The arrangement described above was therefore slightly modified by making the cylinder V somewhat shorter and fixing the rod BD into a rubber stopper fitting into the end of the cylinder.

In the case of vessel I. it was necessary to work with very small quantities of gas in order to get to sufficiently low pressures, and the capillary *a* was discharged once into V. With vessels II. and III., *a* was discharged twice and three times respectively.

The maximum value to which the activity deposited on AB tends at high pressures is not the same for the three vessels, on account of the difference in their dimensions and of the variation in the quantity of emanation used.

The results obtained are given in Tables I., II., and III., in which the pressures are given in cm. of mercury and the quantity of activity on the rod in arbitrary units.

TABLE I. (Vessel I.) diameter 1·1 cm.

Pressure in cm. mercury.	Activity on rod.
15·3	180
8·9	174
6·35	167
4·3	161
2·65	134
1·7	120
0·95	93

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* Mis Brooks, Phil. Mag. [6] vol. viii. p. 380 (1904).

Table II. (Vessel II.) diameter 2·6 cm.

Pressure in cm. mercury.	Activity on rod.
9·3	209
8·7	219
6·2	224
5·0	214
3·3	198·5
2·7	183
2·6	198
2·2	191
1·7	177
1·3	182
0·95	158
0·8	150
0·44	100
0·32	87
0·12	50

Table III. (Vessel III.) diameter 8·3 cm.

Pressure in cm. mercury.	Activity on rod.
5·3	394
5·15	410
4·15	413
2·0	409
1·7	398
1·4	385
1·05	386
0·8	367
0·41	332
0·38	346
0·17	268
0·10	178
0·086	170

4. *Discussion of Results.*

The experiments described above show that, at low pressures, the excited activity produced from radium emanation contained in a closed vessel is not confined to the cathode, as in the case at high pressures, but is distributed over the walls of the containing vessels and appears both on the anode and cathode, even in a strong electric field. This is precisely what has been previously found with thorium emanation *, and shows that the method of transmission is probably the same.

It will be further noticed from the results given above that in a large vessel the influence of pressure on the concentration of the excited activity on the cathode becomes appreciable only at low pressures, whereas with smaller vessels this influence is noticeable at much higher pressures. If, therefore, we assume that at the moment of its production the excited activity is uncharged, it is evident that the proportion of particles of excited activity which can traverse the gas in which it is formed, and reach the anode without becoming charged, is a function of the length of its path through the gas. As has been pointed out by Rutherford †, there are two ways in which the excited activity may be supposed to acquire a positive charge.

(1) The excited activity condenses on the positive ions existing in the gas and is thus transferred to the cathode; and

(2) The excited activity possesses the property of expelling negatively charged particles, and so becomes positively charged.

To decide between these two hypotheses, Rutherford performed experiments in which the number of ions in the vessel containing the emanation was increased by external means, and found that the amount of activity deposited on a cathode in a given time was not increased by this means. For this reason he rejected hypothesis (1) in favour of (2).

The experiments were, however, carried out at *atmospheric* pressure; and it is possible that, under these conditions, the number of ions produced by the emanation is sufficiently great to remove the excited activity as fast as it is formed, in which case any further increase of the ionization would be without further effect. To test this point still further, some similar experiments have been made at a *low* pressure, using vessel III. The walls of the vessel were made the cathode and the central rod the anode; 20 volumes of the capillary *a* (fig. 1) of emanation were introduced into the vessel and allowed to stand for 3¼ hours, the pressure being atmospheric. The emanation was then completely pumped out as quickly as possible, and a new inactive rod AB introduced and the field reversed. One volume of the capillary *a* was then introduced into the vessel, and the rod exposed to the emanation at a pressure of 1 mm. of mercury. Although the ionization in the vessel must have been increased about five times by the excited activity deposited on the walls of the vessel during the previous exposure to the emanation, the activity collected on the rod was found to be the same as when the walls were inactive. In a subsequent experiment 7 milligrams of pure radium bromide were brought close up to the vessel during the exposure of the negatively charged rod, without affecting the quantity of excited activity deposited on the cathode in 30 minutes. Supposition (2) is therefore the only one capable of explaining the observed phenomena.

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* Rutherford, *loc. cit.*

† Rutherford, Phil. Mag. [6] vol. v. p. 111 (1903).

It is of interest to speculate as to the mechanism by which a negatively charged particle is expelled from the excited activity during its passage through the gas. At the moment of formation of excited activity, the emanation from which it is produced expels an α particle* whereby the residue must itself be projected with considerable velocity. In their flight these particles of excited activity would collide with the molecules of the gas in which they are produced, and it is suggested that in a certain percentage of these collisions a negative particle is expelled, leaving the excited activity positively charged. To explain the results obtained in this investigation on this view, it is necessary to assume that the particles of excited activity, when mixed with air at 1 cm. pressure, travel a distance comparable with 1 cm. before becoming charged. This distance is great compared with the mean free path of a molecule of air at the same pressure, and therefore only a very small fraction of the collisions can be effective in causing a negative particle to be expelled by the excited activity.

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* The negative charge left on the excited activity by the expulsion of an α particle may be neutralized by a simultaneous expulsion of slowly-moving electrons similar to those which Prof. J. J. Thomson (Proc. Camb. Phil. Soc. vol. xiii. part 1, p. 49) has shown to be given off by radio-tellurium.

I am indebted to Professor Schuster both for placing the resources of his laboratory at my disposal and also for much valuable criticism during the course of these experiments.

LXVI. *Note on a Paper by* W. Makower *entitled* "*On the Method of Transmission of the Excited Activity of Radium to the Cathode.*" *By* W. H. JACKSON, *M.A., Assistant Lecturer in Mathematics in the University of Manchester*[*].

As a result of some experiments on the method of transmission of the excited activity of radium, Mr. Makower has arrived at the following hypothesis:—The molecules of excited activity, initially uncharged, are projected from radium emanation with velocities large compared with those of gas molecules, and at a certain percentage of encounters with the gas molecules a negative electron is shot off, leaving the activity positively charged.

It gives additional force to his argument to further show that the deviation of the experimental results obtained by him from those required by the above theory are actually less than the errors of observation.

Since the number of ions present in a gas ionized by a small quantity of radium emanation is exceedingly small compared with the number of molecules of gas, it is safe on this view, even though the positive ion influences a much larger field of force than an ordinary molecule, to assume that the importance of encounters with positive ions is negligible.

The amount of activity which, in the experiments described by Mr. Makower, is deposited on the cathode, is to be taken proportional to the number of excited activity molecules which have encounters with air molecules on their way to the sides of the cylindrical vessel.

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[*] Communicated by Mr. W. Makower.

Consider a number of particles travelling a distance x through air in which the average distance between molecules is λ, and the effective radii of molecules is σ. The distance may be divided into N layers each of thickness λ, in which each layer contains, on the average, $\frac{1}{\lambda^2}$ molecules per unit area, and the proportion, on the average, of encounters in each layer is $\frac{\pi\sigma^2}{\lambda^2}$. Hence the proportion of the particles which have no encounters is $\left(1-\frac{\pi\sigma^2}{\lambda^2}\right)^N=\left(1-\frac{\pi\sigma^2}{\lambda^2}\right)^{\frac{x}{\lambda}}$, and this for small values of λ is equal to $e^{-\frac{x}{L}}$, where $L=\frac{\lambda^3}{\pi\sigma^2}$ and is the "mean free path" of the particles in air.

Supposing now that the proportion of encounters effective in displacing a negative electron be R, the above result will be modified and the proportion of particles experiencing no "effective encounters" will be $e^{-\frac{R}{L}x}$.

In the experiments with which we are dealing, the central rod on which the activity was deposited was shielded at both ends, so that the calculation may be simplified by considering the problem as a two-dimensional one, that is by taking the cylinder to be of unlimited length. It is convenient to calculate the proportion of particles which reach the side of the cylinder without having any effective encounters, and then to subtract the ratio from unity.

Consider the number of particles which reach a small area of magnitude A, containing a point O. Use polar coordinates r, θ, ϕ, with O as origin; if P be any point of space let $OP=r$, and if Q be the projection of P on the right section of the cylinder through O, of which C is the centre, let $C\hat{O}Q=\theta$, and $Q\hat{O}P=\phi$.

Let M be the number of particles projected by the emanation per unit volume. The number of particles reaching the area A from an element of volume δV at P is proportional to the solid angle subtended by A at P, which is $\frac{A}{r^2}\cos C\hat{O}P$. In particular, the number reaching A from P is

$$e^{-\frac{\mathrm{R}}{\mathrm{L}}r}\frac{\mathrm{MA}}{4\pi r^2}\cos\theta\cos\phi\,\delta\mathrm{V}.$$

This expression is to be integrated over the infinite cylinder, and then it is clear that the deposit on any portion of the sides is proportional to its area. The area of unit length of the cylinder is πd, where d is the diameter, and the value of $\delta\mathrm{V}$ expressed in terms of r, θ, ϕ is $r^2\cos\phi\,\delta\theta\,\delta\phi\,\delta r$. Denote the number of particles reaching unit length of the sides by N, we have

$$\mathrm{N}=\tfrac{1}{4}\mathrm{M}d\iiint e^{-\frac{\mathrm{R}}{\mathrm{L}}r}\cos^2\phi\cos\theta\,d\theta\,dr\,d\phi \quad . \quad . \quad . \quad (1)$$

The limits of integration being

$$\theta,\ -\cos^{-1}\left(\frac{r}{d}\cos\phi\right)\text{ to }\cos^{-1}\left(\frac{r}{d}\cos\phi\right)$$

$$r,\quad 0\quad\text{to}\quad d\sec\phi,$$

$$\phi,\quad -\frac{\pi}{2}\quad\text{to}\quad +\frac{\pi}{2}.$$

Performing the first integration with respect to θ, we have

$$\mathrm{N}=\mathrm{M}d\int_0^{\frac{\pi}{2}}\int_0^{d\sec\phi}e^{-\frac{\mathrm{R}}{\mathrm{L}}r}\cos^2\phi\left(1-\frac{r^2}{d^2}\cos^2\phi\right)^{\frac{1}{2}}dr\,d\phi. \quad . \ (2)$$

In this integral substitute $r=xd\sec\phi$, whence

$$\mathrm{N}=\mathrm{M}d^2\int_0^{\frac{\pi}{2}}\int_0^1 e^{-\frac{\mathrm{R}}{\mathrm{L}}xd\sec\phi}\cos^2\phi\,(1-x^2)^{\frac{1}{2}}\,dx\,d\phi. \quad . \quad (3)$$

The maximum value of this integral is obtained by omitting the exponential factor, and corresponds to a vacuum when all the activity would reach the sides. Denoting the maximum value of N by N_0, we find, as is otherwise obvious, that

$$\mathrm{N}_0=\tfrac{1}{4}\pi\mathrm{M}d^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let the ratio $\mathrm{N}:\mathrm{N}_0$ be denoted by s.

An effective approximation to the integral expression for the value of s may be found by expanding $(1-x^2)^{\frac{1}{2}}$ in powers of x. Introduce the notation

$$\gamma_n(z)=\int_0^1 e^{-zx}x^n dx, \quad . \quad . \quad . \quad . \quad (5)$$

and write

$$\mu=\frac{R}{L}d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (3) may now be written

$$s=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\cos\phi\left[\gamma_0(\mu\sec\phi)-\tfrac{1}{2}\gamma_2(\mu\sec\phi)-\tfrac{1}{8}\gamma_4(\mu\sec\phi)-\ldots\right]d\phi \quad (7)$$

We also have

$$\left.\begin{aligned}\gamma_0(z)&=\frac{1}{z}\left(1-e^{-z}\right),\\ \gamma_2(z)&=\frac{1}{3}-\frac{1}{4}z+\frac{1}{2\,!\,5}z^2-\frac{1}{3\,!\,6}z^3+\ldots,\\ \gamma_4(z)&=\frac{1}{5}-\frac{1}{6}z+\frac{1}{2\,!\,7}z^2-\frac{1}{3\,!\,8}z^3+\ldots\end{aligned}\right\} \quad . \quad . \quad (8)$$

Also $\gamma_n(z)$ decreases continuously as either z or n increases.

If $\mu=\frac{10}{3}$, we have, if $z=\mu\sec\phi$,

$$\frac{1}{z}e^{-z}<\frac{1}{\mu}e^{-\mu} \quad <\cdot 012,$$

$$\frac{1}{2}\gamma_2(z)<\frac{1}{2}\gamma_2(\mu)<\cdot 015,$$

and with an error of order 3 per cent. we may write

$$s=\frac{4}{\mu\pi}\int_0^{\frac{\pi}{2}}\cos^2\phi\, d\phi=\frac{1}{\mu} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This simplified result should therefore hold if the defect from the maximum deposit of activity on the cathode is less than 30 per cent. Hence, with this restriction, the deposit of activity y should be given by the formula

$$y=N_0(1-s)=N_0\left(1-\frac{L}{Rd}\right). \quad . \quad . \quad . \quad (10)$$

But the mean free path varies inversely as the pressure, and hence, if L_0 be the mean free path for a pressure of 1 cm. of mercury, and x cm. denote any other pressure, we obtain

$$y=N_0\left(1-\frac{L_0}{Rd}x\right). \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This equation affords a means of determining the values of N_0 and $\frac{L_0}{R}$ which best suit each of the Tables I., II., III. obtained by Mr. Makower. Owing to the restriction that the value of y must be greater than $\cdot 7N_0$, it is found that this is an inexact method of determining the value of $\frac{L_0}{R}$. In the accompanying figure, the maximum activities have all been reduced to 100, and the three curves are calculated by assuming the *mean* value of $\frac{L_0}{R}$ to be ·7 cm.

The values of N_0 and $\frac{L_0}{R}$ were determined as follows :— Assuming that the set of n points in question, *i. e.*$(x_1, y_1), \ldots (x_n, y_n)$ belong to a curve of form

$$y=a-\frac{k}{x},$$

we make

$$\Sigma\left(y_r-a+\frac{k}{x_r}\right)=0,$$

and

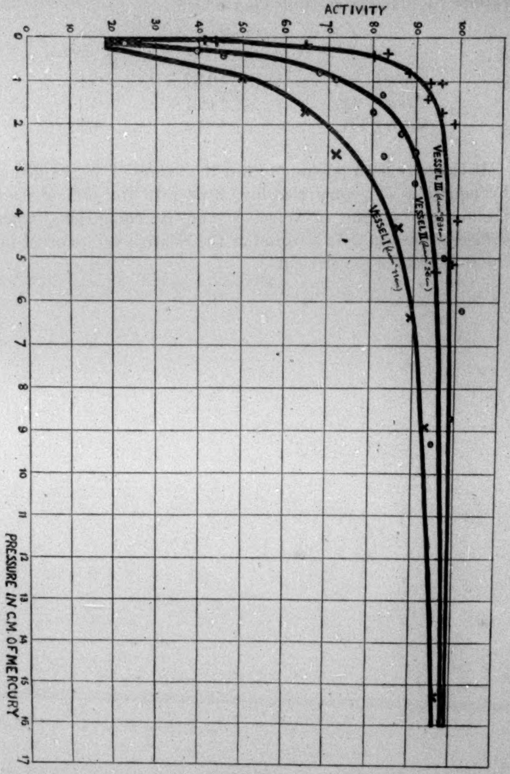
$$\Sigma\left(y_r-a+\frac{k}{x_r}\right)^2 \text{ a minimum.}$$

By differentiation this leads to the equations

$$\left.\begin{aligned} k&=\frac{\Sigma y\Sigma\frac{1}{x}-n\Sigma\frac{y}{x}}{n\Sigma\frac{1}{x^2}-\left(\Sigma\frac{1}{x}\right)^2} \\ na&=k\Sigma\frac{1}{x}+\Sigma y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the curves for vessels II. and III. all the points for which the deposit was greater than 70 per cent. of the maximum were used in the above formulæ.

In the curve for vessel I. the point slightly below this value was used because the next point was clearly inaccurate. The



result so obtained was practically identical with that obtained by leaving out both the points in question. On account of the difference in the values of $\frac{L_0}{R}$ caused by omitting even single points, this value is reliable only to one figure. The results are given in the following table :—

	N_0.	$\frac{L_0}{R}$ in cm.
Vessel I.	187	·71
Vessel II.	221	·73
Vessel III.	414	·64

It has only been possible to test the values of N_0, as given by equation (4), very roughly, owing to the difficulty of finding the volume of vessel V and its connexions, with sufficient accuracy to determine the relative values of M in the three sets of experiments.

WOOD.

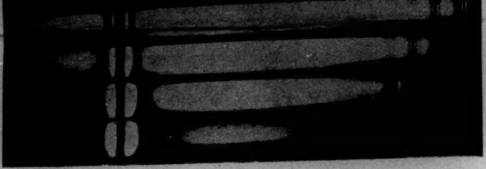


Fig. 3.

Fig. 4.

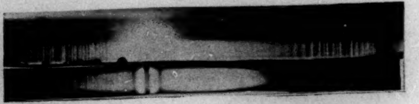

D_1D_2
Red.
Green.
Blue.

Fig. 5

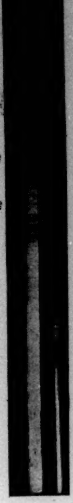

4933
4962
5001
504
5079

Fig. 6.

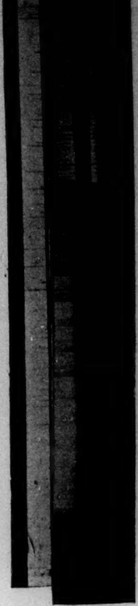


FIG. 1.

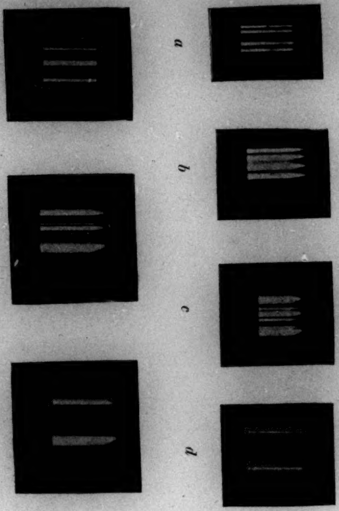


FIG. 2.

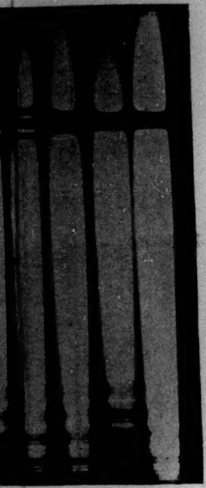


FIG. 5.

4933
4837
Absorption.
Mag. Rotation.

6.

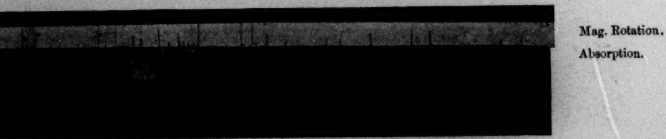

Mag. Rotation.
Absorption.

FIG. 5.

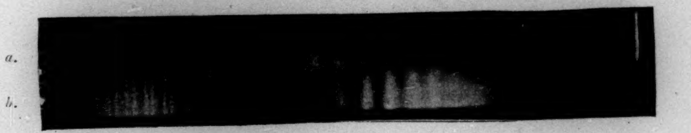


FIG. 6.

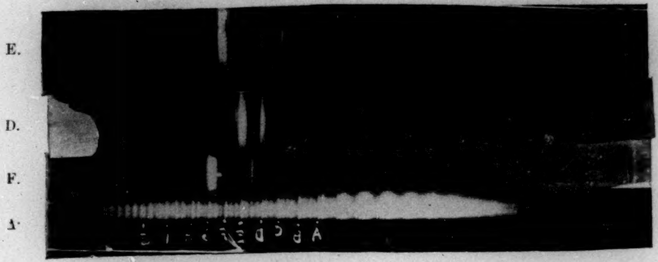
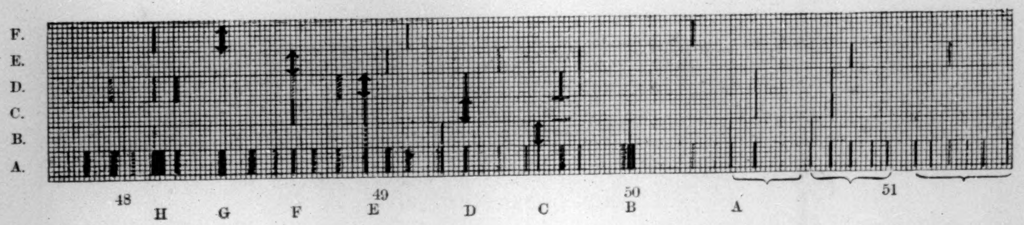


FIG. 7.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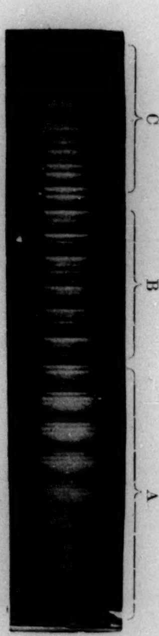

Fig. 1.

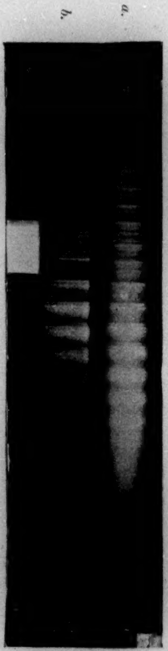

Fig. 2.

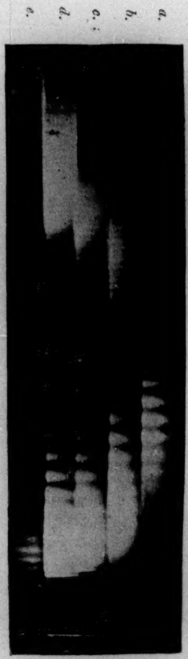
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Fig. 3.

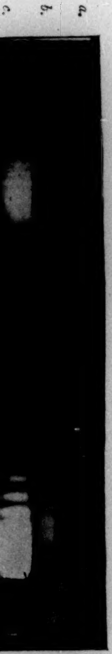

Fig. 4.

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Printed by Taylor and Francis, Red Lion Court, Fleet Street.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1903-1904.

February 27th, 1903.

Meeting held at Burlington House.

The PRESIDENT in the Chair.

The following Papers were read:—

1. The Measurement of Small Capacities and Inductances. By Dr. FLEMING and W. C. CLINTON.

2. The Thickness of the Liquid Film formed by Condensation at the Surface of a Solid. By Dr. PARKS.

March 13th, 1903.

Meeting held at Burlington House.

The PRESIDENT in the Chair.

The following Papers were read:—

1. On the Interpretation of Milne Seismograms. By Dr. FARR.

2. A Potentiometer for Thermo-couple Measurements. By Dr. LEHFELDT.

3. A Direct-reading Potentiometer for Thermoelectric Work. By Dr. Harker.

4. The Measurement of Small Resistances. By A. Campbell.

5. A Resistance Comparator. By Dr. Lehfeldt.

Mr. W. A. Price exhibited a Crompton Potentiometer.

March 27th, 1903.

Meeting held at Burlington House.

The President in the Chair.

The following were elected Fellows of the Society :—
G. H. Baillie, M. A. Codd, L. Lownds.

The following Papers were read :—

1. On the Refraction at a Cylindrical Surface. By A. Whitwell.

2. Evaluation of the Absolute Zero. By Dr. Lehfeldt.

Mr. Blakesley exhibited a form of Lens having peculiar properties.

April 24th, 1903.

Meeting held at Burlington House.

Mr. Blakesley, and subsequently Dr. Thompson, in the Chair.

The following were elected Fellows of the Society :—
L. J. M. Polley, A. F. Ravenshear, H. G. Wood.

The following Papers were read :—

1. The Dimensional Analysis of Physical Quantities and the Correlation of Units. By A. F. Ravenshear.

2. On the Dimensions of Physical Quantities. By R. J. Sowter.

Mr. Croft gave an exhibition of apparatus.

May 8th, 1903.

Meeting held at Burlington House.

The President in the Chair.

The following were elected Fellows of the Society:—

W. H. Kay, E. A. O'Keefe, O. W. Richardson.

The following Papers were read:—

1. The Mathematics of Bees' Cells. By Prof. Everett.
2. The Coloured Map Problem. By W. A. Price.
3. The Construction and Attachment of Galvanometer Mirrors. By Dr. Watson.

Mr. T. H. Blakesley gave an account of a Spectroscope of Direct Vision and Minimum Deviation.

Mr. W. A. Price exhibited some Anchor Rings, illustrating the "Coloured Map Problem."

May 22nd, 1903.

Meeting held at Burlington House.

The President in the Chair.

The following Paper was read:—

On an Instrument for Measuring the Lateral Contraction of Tie-Bars, and on the Determination of Poisson's Ratio. By J. Morrow.

Mr. J. Stöttner exhibited some Nernst Lamps.

Mr. T. H. Blakesley exhibited a diagram of Single-piece Lenses.

June 5th, 1903.

Meeting held at University College.

The President in the Chair.

The following Paper was read:—

On Radioactive Processes. By Prof. Rutherford.

June 12th, 1903.

Meeting held at Burlington House.

The **President** in the Chair.

The following was elected a Fellow of the Society:—

H. A. **Wilson**.

The following Papers were read:—

1. On a Method of Determining the Viscosity of Pitch-like Solids. By Prof. F. T. **Trouton** and Mr. E. S. **Andrews**.

2. The Positive Ionization produced by Hot Platinum in Air. By O. W. **Richardson**.

Dr. **Thompson** showed some experiments on Shadows in an Astigmatic Beam of Light.

June 26th, 1903.

Meeting at the Physiological Laboratory, London University.

The **President** in the Chair.

A Special General Meeting was held at which the following Resolution was passed:—

Resolution:—"That the Articles of Association contained in the printed document submitted to the Meeting, and, for the purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby approved: and that such Articles of Association be, and they are hereby, adopted as the Articles of Association of the Society."

An Ordinary Meeting of the Society was then held.

The following were elected Fellows of the Society:—

W. D. **Eggar**, J. S. **Townsend**.

The following Papers were read:—

1. Electrical Effects of Light upon Green Leaves. By Dr. **Waller**.

2. Blaze-Currents: (1) of a Vegetable Tissue; (2) of an Animal Tissue. By Dr. **Waller**.

3. Quantitative Estimation of Chloroform Vapour in Air by (1) Oil Absorption, (2) Densimetry. By Dr. **Waller**.

4. The Temperature of Limits of Nerve-Action in Cold-blooded and in Warm-blooded Animals. By Dr. **Waller** and Dr. **Alcock**.

July 13th, 1903.

Meeting at the Royal College of Science, South Kensington.

The PRESIDENT in the Chair.

A Special General Meeting was held at which the following Resolution was confirmed:—

Resolution:—"That the Articles of Association contained in the printed document submitted to the Meeting, and, for the purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby approved: and that such Articles of Association be, and they are hereby, adopted as the Articles of Association of the Society."

October 23rd, 1903.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Papers were read:—

1. The Bending of Magnetometer Deflection-Bars. By Dr. CHREE.

2. The Magnetism of Basalt and the Magnetic Behaviour of Basaltic Bars when heated in Air. By Dr. ALLAN.

Dr. WATSON showed some experiments on Electrical Oscillations.

November 13th, 1903.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Papers were read:—

1. Means of Electrifying the Atmosphere on a Large Scale. By Sir OLIVER LODGE.

2. An Apparatus for Driving Mercury Pumps. By Sir OLIVER LODGE and B. DAVIES.

November 27th, 1903.

Meeting held at the Royal College of Science.

The President in the Chair.

The following was elected a Fellow of the Society:—

W. Bennett.

The following Papers were read:—

1. An Electrical Thermostat. By Horace Darwin.

2. On the Occurrence of Cavitation in Lubrication. By S. Skinner.

December 11th, 1903.

Meeting held at the Royal College of Science.

Mr. J. Swinburne, Vice-President, in the Chair.

The following was elected a Fellow of the Society:—

V. Lough.

The following Papers were read:—

1. On a Method of Mechanically Reinforcing Sounds. By Rev. T. C. Porter.

2. On the Simmance-Abady "Flicker-Photometer." By F. J. Simmance and J. Abady.

Mr. Appleyard exhibited a Conductometer.

Prof. L. R. Wilberforce exhibited a model to illustrate various properties of Wave-motion.

January 22nd, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Papers were read:—

1. On Astigmatic Aberration. By W. Bennett.

2. Some New Cases of Interference and Diffraction. By Prof. R. W. Wood.

3. The Photographic Action of Radium Rays. By S. Skinner.

Mr. Price gave an exhibition of Instruments by Messrs. Crompton & Co.

Annual General Meeting.

February 12th, 1904.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Report of the Council was read by the Secretary:—

SINCE the last Annual General Meeting fourteen Science Meetings of the Society have been held. Of these, seven were held at the Rooms of the Chemical Society at Burlington House, five at the Royal College of Science, one at University College, and one, that on June 26th, was held by invitation of the Principal and Dr. A. Waller in the Physiological Laboratory of the University of London.

The number of Fellows now on the roll is 418, an increase of 11 over the corresponding number last year. Eighteen new Fellows have been elected. There have been no resignations, but the Society has to mourn the loss of six Fellows by death, namely, the Marquis of Salisbury, Sir Frederick Bramwell, Mr. E. T. Carter, Mr. W. Gleed, Mr. E. Woods, and Dr. W. Francis.

During the year changes of some importance have been made which the Council hope will further the aims for which the Society was founded and in particular tend to increase the attendance at, and interest in, the meetings. While recognizing the kindness of the Council of the Chemical Society in placing their Rooms at the disposal of this Society, of which they availed themselves for nine years, your Council has been forced to acknowledge that the convenience of the place of meeting as far as accessibility was concerned was more than counterbalanced by the absence of all equipment suitable for the conduct of physical experiments. They therefore decided to return to the original home of the Society, and meet in the Physical Laboratory of the Royal College of Science at South Kensington; and they would like to express their thanks to the Board of Education for placing the resources of that Laboratory at the disposal of the Society. At the same time they decided to make a trial of holding evening meetings. At present afternoon and evening meetings alternate, and up to now it does not appear that either one or other is markedly the more popular. The wisdom of making the change of place of meeting seems justified by the livelier interest in, and marked increase in the experimental illustrations of, the papers read.

During the year some difficulty was experienced in carrying on 'Science Abstracts,' owing to the lack of funds. The Council did not feel justified in increasing the contribution of the Society much beyond what had been given during the past few years, and the Council of the Institution of Electrical Engineers were not prepared to make up the necessary sum under the old arrangements. Finally, the Council of the Institution decided that they were willing to take over 'Science Abstracts' completely, including both financial and editorial responsibility, undertaking to supply Fellows of the Physical Society with copies of the Physical part, provided the Society make an annual contribution (with a fixed limit) towards the cost of production. It was also to be part of the scheme that your Council should be represented on the Committee of Management. As such an arrangement placed the 'Abstracts' in a strong position financially your Council decided to accept the proposal, even though it meant curtailing the control of the Society over the publication. The 'Science Abstracts' Committee has therefore been dissolved, and the management transferred to the Council of the Institution. At present the publication is in the hands of a sub-committee on which your Council are represented by Dr. Glazebrook and Mr. Elder.

The Council believe that the arrangements which have been made are sound and will work satisfactorily, and the terms of an Agreement with the Institution for three years on the lines indicated are practically settled. The Magazine is to be issued by the Institution of Electrical Engineers, being compiled and edited by the Institution in association with the Physical Society and with the co-operation of certain American Institutions.

Another change which has been inaugurated during the past year is the formation of a Students' class, which it is hoped will act as a feeder to the Society by inducing students of Physics to join the Society and attend the meetings at a time when they are not prepared to become full Fellows.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—R. T. Glazebrook, D.Sc., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; The Lord Kelvin, G.C.V.O., D.C.L., LL.D., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prin. Sir Arthur W. Rücker, M.A., D.Sc., F.R.S.; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S.; Shelford Bidwell, M.A., LL.B., F.R.S.; Prin. Sir Oliver J. Lodge, D.Sc., F.R.S.; Prof. S. P. Thompson, D.Sc., F.R.S.

Vice-Presidents.—T. H. Blakesley, M.A.; C. Chree, D.Sc., F.R.S.; Prof. J. D. Everett, D.C.L., F.R.S.; J. Swinburne.

Secretaries.—W. Watson, D.Sc., F.R.S.; W. R. Cooper, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Prof. H. L. Callendar, M.A., F.R.S.

Librarian.—W. Watson, D.Sc., F.R.S.

Other Members of Council.—C. V. Boys, F.R.S.; Prof. W. Cassie, M.A.; W. B. Croft, M.A.; H. M. Elder, M.A.; Prof. J. Perry, D.Sc., F.R.S.; A. W. Porter, B.Sc.; W. A. Price, M.A.; Prof. F. T. Trouton, D.Sc., F.R.S.; W. C. D. Whetham, M.A., F.R.S.; S. A. F. White, M.A.

Votes of thanks were passed to the Auditors, the Officers and Council, and to the Board of Education.

The President then delivered his Address.

Treasurer's Report.

The revenue of the Society from subscriptions and dividends has increased somewhat as compared with previous years, but the revenue from sales of publications has diminished, and the values of the investments have declined. The reduction of the amount paid towards 'Science Abstracts' (£256 instead of £284) has in some measure compensated for extra expenditure on the International Catalogue and on the printing of Abstracts of Proceedings. In spite of an unusually heavy printing bill, and allowing for outstanding liabilities, the year's accounts show a balance of about £20 on the right side. A further saving of upwards of £20 per annum has been effected by holding the meetings at the Royal College of Science. It appears probable that a contribution of upwards of £400 may be required of the Society, towards the expenses of 'Science Abstracts' during the ensuing year. But having regard to the growing strength of the Society, and the increased interest shown in the meetings, it is hoped that we may survive this exceptional tax on our resources. In the estimated assets of the Society there appears to be a large decline in the value of Publications in stock, which have been estimated at £600 in previous years. This is not due to any loss of stock, but to a new method of valuation. The yearly sales of back numbers etc. realize about £6 6*s*., equivalent to 3 °/₀ on a capital sum of £210, which may be taken to represent roughly the present value of the stock.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1903, TO DECEMBER 31ST, 1903.

Dr.	£	*s.*	*d.*	£	*s.*	*d.*
Jan. 1, 1903. Balance in the Bank				68	6	6
Entrance-Fees	21	1	0			
Subscriptions	375	18	0			
Extra Subscriptions	37	13	0			
Arrears	33	10	1			
Abstracts and Advance Proofs	8	0	0			
				476	2	1
Dividends:—						
Furness Debenture Stock	15	1	2			
Metropolitan Board of Works	6	12	6			
Lancaster Corporation Stock	11	7	0			
New South Wales	8	8	4			
Midland Railway	37	13	4			
Great Eastern Railway	18	16	8			
London, Brighton, & South Coast Railway	25	5	2			
				123	4	2
Sales of Publications				35	4	7
				£702	17	4

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
Science Abstracts				256	0	0
International Catalogue				9	15	6
Taylor and Francis, for printing:—						
Proceedings	98	7	0			
Miscellaneous	84	18	0			
Distribution	13	16	9			
Separate Copies	14	13	6			
				211	15	3
Periodicals				3	10	10
Attendance				4	2	6
Royal Asiatic Society (2 years)				4	4	0
Reporting				32	1	7
Commission on cheques, etc.					11	6
Binding				4	12	2
Petty Cash:						
Treasurer and Secretaries				19	12	1
Balance in Bank, Dec. 31, 1903				154	9	11
Cheque in Treasurer's hands				2	2	0
				£702	17	4

Audited and found correct.

HUGH L. CALLENDAR, *Honorary Treasurer.*

19*th January*, 1904.

R. MULLINEUX WALMSLEY, } *Auditors.*
WALTER BAILY,

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1903.

Assets.	£	s.	d.
Subscriptions due, estimated	20	0	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	463	0	0
£1600 Midland Railway 2½ per cent. Preference Stock	1184	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	202	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	352	0	0
£259 2*s.* 9*d.* New South Wales 3½ per cent. Inscribed Stock	246	0	0
£500 London, Brighton, and South Coast Railway Ordinary Stock	605	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	595	0	0
Publications in Stock, estimated	210	0	0
Balance in Bank	154	9	11
Cheque in Treasurer's hands	2	2	0
	£4033	11	11

Liabilities.	£	s.	d.
Chemical Society, Rent	14	3	6
Taylor and Francis, Printing	50	12	9
Balance	3968	15	8
	£4033	11	11

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

19th January, 1904.

R. MULLINEUX WALMSLEY,
WALTER BAILY, } *Auditors.*

OBITUARIES.

Sir FREDERICK BRAMWELL, Bart., D.C.L., LL.D., F.R.S.

Sir Frederick Bramwell was born in London on March 7th, 1818. He received a primary education at the Palace School at Enfield, and at the age of 16 was apprenticed to John Hague, a mechanical engineer. When out of his time he was engaged by the firm and rose to the position of chief draughtsman.

Later, he was associated with various engineering works, including one in the Isle of Dogs of which he was manager. At the age of 35 he commenced business on his own account, and, although for the first few years progress was slow, yet his sterling merits and unflagging energy made ultimate success certain. From the age of 40 he progressed rapidly both in his private practice and in the estimation of the engineering world.

His connection with scientific and engineering societies was wide and distinguished. In 1874 and 1875 he filled the office of President of the Institution of Mechanical Engineers. In 1884 and 1885 he was President of the Institution of Civil Engineers. He became a member of the British Association in 1865, and acted as President in 1872, 1884, and 1888. The Society of Arts received a large share of his time and business aptitude. He was member of the Council in 1875, Chairman in 1881 and 1882, and Treasurer in 1890 and 1895. The Royal Society elected him a Fellow in 1873, and he served as member of Council in 1877 and 1878. Of all scientific societies, it was perhaps to the Royal Institution that he gave his best services. He was elected a Member in 1876, and joined the Board of Managers in 1879. From 1885 to 1900 he filled the office of Honorary Secretary. His zeal and enthusiasm, coupled with exceptional business ability, and a tactful and genial manner, combined to make him one of the most useful and popular men who have ever acted in that capacity.

Sir Frederick's abilities were directed chiefly to applied science and to engineering, and he was apt to underrate his knowledge of pure science. Though never himself a schoolman and somewhat impatient at questions of merely academic interest, he nevertheless took a keen interest in educational matters. For his services in connection with the movement which led to the founding of the Technical Schools, now associated with the City Guilds, he in 1881 received the honour of knighthood; in 1889 he was created a baronet, but as he leaves no son the title expires with his death. He died on November 30, 1903.

Edward Tremlett Carter.

Edward Tremlett Carter was born in Calcutta in 1866. He came to England at an early age, and was educated privately at Bristol, afterwards at the Merchant Venturers' College in that city, and finally at the Bristol University College, where he went through the Engineering and Physics courses under Professors Hele Shaw and Silvanus P. Thompson. After holding for a short time the position of demonstrator at the Bristol University College, Mr. Carter obtained an appointment at the School of Electrical Engineering, Hanover Square, London, as assistant to the late Mr. Lant Carpenter, who was then the principal.

He was afterwards one of the lecturers at this school, where he remained until its close in 1893. During this period Mr. Carpenter was a frequent contributor to the technical press, and also carried on a small practice as a consulting engineer. In 1893 Mr. Carter joined the permanent staff of 'The Electrician,' of which Mr. A. P. Trotter was then editor, and on Mr. Trotter's retirement in 1895 he was appointed assistant-editor under Mr. W. G. Bond as editor. In 1897 Mr. Carter became editor-in-chief of 'The Electrician,' which appointment he held up to the time of his death.

Mr. Carter had never a strong constitution, and in the winter of 1899, after a severe attack of pleurisy and bronchitis, following after influenza, had to leave his work and make a two months' tour to the Mediterranean and Egypt; this set him up again temporarily, but unfortunately the improvement in his health was not permanent. In October 1902 it was found that his lungs were badly affected, and he went to a sanatorium to follow the "open-air treatment," but all efforts at cure were too late, and he succumbed to tuberculosis of the lungs on April 16th, 1903, aged 37 years; leaving a widow and three sons.

Mr. Carter was a Member of the Institution of Electrical Engineers, a Member of the Société des Ingénieurs Civils de France, and a Fellow of the Royal Astronomical Society.

Dr. William Francis.

Dr. William Francis was born in London on the 16th of February, 1817. He was educated at University College School and St. Omer.

He left St. Omer in 1834 and proceeded to Crefeldt, but in the autumn of the same year went to Gera, where he remained for about two years. In 1836 he returned to England and spent a year at University College, London, afterwards devoting some time to learing the printing business under Mr. Richard Taylor, to whom he had been apprenticed some time previously. He then went to Berlin, and thence to Giessen, where he studied under Liebig, and did much original work, chiefly on the salts of molybdenum. He took his degree of Doctor of Philosophy at Giessen in 1842.

He early developed a taste for Natural History, and in 1837 suggested to Mr. Richard Taylor the founding of the 'Annals of Natural History,' with which subsequently the well-known 'Magazine of Natural History' of London and Charlesworth was amalgamated.

While in Berlin and Giessen, Dr. Francis, in conjunction with his friend and fellow-student Henry Croft, forwarded every month a series of reports to the 'Philosophical Magazine' on the progress of chemical science on the Continent; but the space available in that Journal being limited, they, on their return to England, started the 'Chemical Gazette' in 1842. Croft was compelled to relinquish the editorship before the fourth number appeared, being appointed Professor of Chemistry at King's College, Toronto; and the 'Gazette' was carried on by Dr. Francis alone until 1859, when the pressure of other work compelled him to relinquish the task, and the 'Gazette' was incorporated with the then newly-founded 'Chemical News.'

In addition to furnishing translations of foreign scientific papers to the 'Philosophical Magazine,' he also translated many papers for Taylor's 'Scientific Memoirs,' in the conducting of which, moreover, he had a very large share, although his name did not appear on the titlepage. He also translated Beckman's 'History of Inventions' for Bohn's Scientific Series.

In 1851 his services to the 'Philosophical Magazine' over many years, both in furnishing translations and in conducting the Journal, were acknowledged by the appearance of his name on the wrapper as co-editor, where it remained until his death.

In 1841 he was elected an Associate of the Chemical Society, becoming a Fellow in the following year. He was also a Fellow of the Linnean Society (1844), of the Royal Astronomical (1851), of the Geological (1859), and of the Physical (1876).

In 1852 he joined Mr. Richard Taylor as partner in the firm of Taylor and Francis, printers and publishers. He was one of the oldest members of the Stationers' Company, having taken the Livery in 1841.

In 1862 he married Isabella Gray, daughter of Mr. Taunton, M.R.C.S., of Hatton Garden, but became a widower in 1899. He died at Richmond on the 19th of January last.

William Gleed.

William Gleed was born in October 1851, his father, who was in the Customs, dying a fortnight before he was born. He was educated at Hurstpierpoint, and, obtaining a mathematical scholarship at Peterhouse in 1870, went up to Cambridge in the October of that year. In 1874 he took his degree as 30th wrangler. After acting for a short time as mathematical master at his old school, he was appointed to one of the Instructorships in the Royal Naval College, Greenwich, where his chief duty was to assist in the instruction of Lieutenants R.N. in mathematics and mathematical physics. This office he held until his death.

At school, and also at Cambridge, Gleed was distinguished not only by his devotion to mathematical study but by his success in all games; he was captain of his College boat and also played in the College eleven. At Greenwich he was recognized as an efficient and sympathetic teacher. In December 1878 Gleed was elected a Fellow of the Physical Society, and at one time was a fairly regular attendant at our meetings. His health broke down in 1901, and in the summer of 1902 he underwent a serious operation, for the removal of an intestinal ulcer. The operation was entirely successful, and it was thought the evil was removed; but in the autumn of last year it was recognized that he was suffering from cancer, and that no further operation could save him. He died December 18th.

The Marquis of Salisbury, K.G., G.C.V.O., F.R.S.

Lord Salisbury always showed a great interest in natural science, and occupied much of his spare time in scientific experiments which he carried out as an amusement and relaxation. His attention was first drawn to chemistry, but afterwards he devoted himself to physics. He published an article on Photography in the Quarterly Review for Oct. 1864, and only one other paper, a short note in the Philosophical Magazine, xlv. pp. 241–245 (1873), on the spectrum of the light seen in the vacuous portion of a thermometer when the bulb is in contact with one pole of an induction-coil, the other pole being insulated. In a dark room a brush discharge was seen from the closed end of the thermometer. At first he attributed the light to a low-temperature discharge in the thermometer, but afterwards found that although there was very little rise of the thermometer column yet the gas in the space above was really at a high temperature. While performing these experiments he noticed that a thermometer lying near the induction-coil became luminous, probably one of the first observations of the effect of electromagnetic waves. He made some elaborate experiments on the action of magnets at a distance, and on the variation of the strength of an induced current by altering the distance between two flat coils with and without cores of soft iron.

Another series of experiments was made to endeavour to obtain two pieces of the same metal inactive when they were connected with a galvanometer and simultaneously dipped in an electrolyte.

Hatfield House was one of the first large houses in this country that was lighted by electricity, the power being obtained from a water-mill on the river Lea which runs through the park, and at an early period motors were used for various operations on the farm.

The general interest which Lord Salisbury took in science is well shown by his presidential address at the British Association at Oxford in 1894. Although the address professedly deals with a survey of our ignorance and not of our science, it exhibits a considerable acquaintance with much of the work of scientific investigators.

In recent years the affairs of the State interfered with his experimental work, but his interest in science was maintained, as shown by his sympathy with the aims of the National Physical Laboratory. After Sir Douglas Galton's address on the subject to the British Association in 1895, a committee was formed and a petition presented to Lord Salisbury. This resulted in the appointment of a Treasury Committee to consider the subject, and finally in the Government giving support to the scheme.

Lord Salisbury died in August 1903, at the age of 73.

THEORIES OF RESOLVING POWER IN A MICROSCOPE.

Presidential Address to the PHYSICAL SOCIETY, *February* 12, 1904.

By R. T. GLAZEBROOK, F.R.S.

DURING the past Session our attention has been drawn by more than one paper to the study of Geometrical Optics. At our last Meeting the importance of that study was emphasized by one who is well qualified to express an opinion; and I propose today to call your attention to one or two matters, not by any means novel, which seem to offer a fruitful field for further study and research. I wish to touch on one or two points connected with the theory of the microscope.

Geometrical Optics in its relation to instruments has been studied to greater advantage abroad; we in England have, I fear, of recent years somewhat neglected the subject, with the result that only a small share in the recent advance in lens construction has been ours.

It was in 1878, in his report on the London International Exhibition of Scientific Apparatus, that Prof. Abbe first called attention to the fact that the further perfection of the microscope as an optical instrument depended on the advance of the art of glass-making.

With the glasses then at their disposal, it was impossible for opticians to get rid of the secondary spectrum of their object-glasses: while a glass could be made achromatic for two wave-lengths, the differences in the relative dispersion of the two ends of the spectrum were such that there was an outstanding amount of colour which prevented the attainment of the highest perfection of the image.

It was to this fact that the establishment of the now celebrated firm of Schott & Company was due; and the results of Abbe's own work on microscope-lenses are summed up in the first volume of his 'Collected Papers' which has recently appeared.

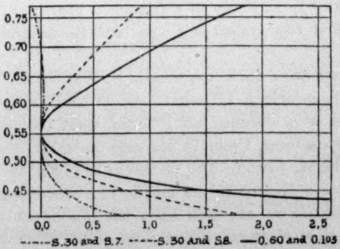
The well-known paper, "Contributions to the Theory of the Microscope and of Microscopic Perception," which forms the basis of his work, is there reprinted, and it will be interesting to consider some of the points it raises.

But first let us contrast what is now possible so far as achromatic correction is concerned with what was possible, say, twenty years ago. Table I. gives the optical constants of various kinds of glass. Various pairs of these glasses might be used; and in fig. 1 we have three curves showing the relation between focal length and wave-length of a lens having a focal length of 1 metre for the D line.

TABLE I., *giving Optical Properties of various Jena Glasses.*

No.	Description.	μ.	ν.	Δ.	α.	β.	γ.
O. 60	Silicate Crown	1·5179	60·2	00860	·643	·703	·566
O. 43	Flint	1·6202	36·2	01709	·605	·714	·609
S. 30	Barium Phosphate	1·5760	65·2	00884	·644	·703	·565
S. 8	Borate Flint	1·5736	50·8	01129	·645	·704	·571
	Fluorspar	1·4338	97·4	00455	·639	·707	·563
O. 225	Light Phosphate Crown	1·5159	70·0	00737	·658	·698	·552
O. 252	Borate Flint	1·5521	53·8	01026	·650	·703	·567
O. 527	Barium Flint	1·5718	50·4	01133	·623	·709	·582
O. 211	Heavy Baryta Crown	1·5726	57·5	00995	·633	·706	·571
O. 1209	Heaviest Baryta Crown	1·6112	57·2	01068	·636	·705	·571
O. 608	Crown of Highest Dispersion ...	1·5149	54·6	00943	·631	·706	·576
O. 381	Crown of High Dispersion	1·5262	51·3	01026	·629	·709	·582
S. 7	Borate Flint	1·6086	44·3	01375	·628	·708	·583
O. 2122	Heavy Baryta Crown	1·5899	60·8	00970	·640	·704	·563
O. 2071	Heavy Baryta Crown	1·6698	58·8	01037	·641	·704	·569
O. 2388	Telescope Crown	1·5254	61·7	00852	·641	·707	·568
O. 2001	Telescope Flint	1·5211	51·8	01007	·635	·705	·573

Fig. 1.



The curves are for the pairs

O. 60 and O. 103, a glass somewhat like O. 43,
S. 30 and S. 8,
S. 30 and S. 7,

and it is clear how much has been gained.

Attention should also be drawn to the constants of fluorspar and to the possibility of combining it with a glass like the heavy baryta crown O. 1209.

The figures for the relative dispersions are:

636 639; 707 707; 571 563.

The refractive indices are very different, 1·611 and 1·434, while the values of ν are also different, viz. 57·2 and 77·4; but the relative distribution of light is practically the same for the two, hence the possibilities of achromatic combination are great.

Again, it appears from a recent paper by Mr. H. D. Taylor that while for the Lick object-glass the difference in the focal length between D and H is over 3 inches, for an object-glass of three lenses, recently designed by himself, of the same focus—60 feet—the difference would be $\frac{1}{7}$ of an inch.

These figures are enough to show how the art of the glass manufacturer has helped the optician.

Turning now, however, to the microscope, let us, taking Abbe and his pupils as our guide, analyse the action of the parts with a view of seeing, if we can, where further improvement may be possible.

The magnifying power of an ordinary lens held close to the eye is $1+d/f$, where d is the least distance of distinct vision; if, however, we suppose the eye placed at the secondary principal focus, the magnifying power is d/f.

Taking, then, this definition, let us suppose we have two lenses of focal lengths f_1, f_2, placed at a distance Δ apart; the equivalent focal length of this system is $f_1 f_2/\Delta$, and the magnifying power is $\Delta d/f_1 f_2$.

Now we see from these expressions (i) that by the use of two lenses of appreciable focal length, say 10 mm. each, we can by placing them at a distance, say 100 mm. apart, get an equivalent lens of shorter focus, 1 mm. in the case in point, and of correspondingly higher power. The advantage of this is obvious: if for no other reason because of the greater ease of working the larger lenses; while (ii) we can look upon the instrument as a simple magnifying-glass of power d/f_1, the image of which is magnified further by the factor Δ/f_2.

Again, we can consider the function of the two lenses otherwise thus.

A pencil of strongly divergent rays falls on the object-glass and is transformed into a convergent pencil. At one period of their course these rays form a parallel pencil. Now imagine the lens separated at this position into two by a plane perpendicular to the axis. The lower part of the lens transforms the divergent rays into a parallel pencil: this is the function of a simple magnifying-glass adjusted for normal vision; thus we may look upon this lower portion as such a simple magnifying-glass. The upper portion of the object-glass forms from these parallel rays a real image, as does the object-glass of a telescope, and this image is magnified as in the telescope by the eye-piece: thus the upper portion of the object-glass and the eye-lens form a telescope, and a microscope is a telescope with a short-focus lens put on in front and used to view an image in the principal focus of that lens.

Such a method of viewing its action seems to me instructive.

Again, we owe to Abbe the introduction of the term Numerical Aperture, and the realization of its importance as defining in certain circumstances the resolving power of the instrument. Let us look into this. By numerical aperture is meant the value of quantity $\mu \sin \alpha$, where μ is the refractive index of the medium in which the object is placed, α the semivertical angle of the cone subtended at the object-glass by the point in which the axis of the instrument meets the object.

Let us suppose, then, that an object is on the stage viewed by transmitted light, and to simplify matters let us suppose the source of light at some distance.

Then, according to Abbe and his followers, in considering the image formed in the focal plane of the eye-piece we are not to start from the object as a self-luminous source and consider where the image of such a source would be, if formed by the laws of Geometrical Optics; we are instead to start from the source itself, to consider its image formed in the focal plane of the object-glass, and to treat this image as the source of light in the microscope-tube from which arises the image we see.

If the object be small, the focal image will be modified by diffraction due to the object, and according to Abbe it is on the nature of the diffraction images and number of them which are formed that the definition depends.

We will return later to the question whether it is necessary thus to consider our problem. I wish at present to develope it and to consider whether it affords us a satisfactory solution of the problem of resolving power.

Suppose now the microscope has been focussed on some object on the stage and that this object has been removed; the parallel rays from the source are brought to a focus in the focal plane of the object-glass, forming there a circular patch of light; from each point of this rays diverge, and reaching the eye produce the sensation of a uniform luminous field.

Now let the field in a focal plane be limited by diaphragms pierced with a series of small apertures.

The distribution of light in the focal plane of the eye-lens will be no longer uniform, we shall see the diffraction-pattern formed there by the apertures.

If, for example, there be but one aperture, a single narrow slit, the field will still be uniform, light diverges from the slit uniformly and no structure is seen.

If we have a number of equidistant slits, the view-plane will be crossed by a series of equidistant dark and light bars. The distance between these bars and the distribution of light between them will depend on the distance between the slits of the diaphragm and the distribution of luminosity among the slits. If this be known, the distribution of light in the view-plane can be calculated. If, for example, the distance between the slits be doubled, the distance between the maxima in the view-plane will be halved; that is to say, the number of bright bars in a given interval will be doubled. The distribution in the view-plane depends on that in the focal plane, and can be calculated from it: this is quite certain.

But now, instead of producing a variable distribution in the focal plane of the object-glass by means of diaphragms, we can do it by means of the diffraction effects of small objects on the stage.

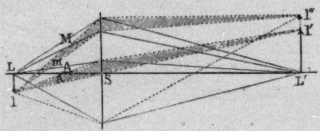
Thus, if we put on the stage a grating consisting of a series of equidistant spaces, and if ϵ be the grating distance, then, taking homogeneous light, a series of narrow bands of light, the diffraction images of the source will be produced in the focal plane with darkness between them; the central image will be on the axis, and if $\theta_1, \theta_2 \ldots .$ be the angular distances between the images and the axis, then $\sin \theta_1 = \lambda/\epsilon$, $\sin \theta_2 = 2\lambda/\epsilon$, etc.

Now a pencil starting, say, from the point where the axis of the system cuts the stage, at an angle θ_1 with the axis, is refracted at an angle θ_1' and meets the view-plane in the point in which it is cut by the axis. If the microscope is perfect, the sine law holds so that the angles θ_1, θ_1' are connected by the equation $\sin \theta_1/\sin \theta_1' = N$, where N is the linear magnification—the medium on both sides of the object-glass is supposed to be air.

Let us digress for a moment to consider this law, the proof of which, originally due to Hockin, has been discussed by many writers.

It is illustrated by fig. 2, and expresses the condition that the magnification of an object such as Ll whether formed by a central pencil or by an excentric one should be the same. If l and l' are the images of l formed by the central and an excentric zone, then for good definition l' and l'' must coincide, and if this be the case the sine condition will be satisfied.

Fig. 2.



Moreover, in a microscope θ_1' is small, so that we have

$$\theta_1' = \sin\theta_1//\mathrm{N} = \lambda/\mathrm{N}\epsilon,$$
$$\theta_2' = \sin\theta_2/\mathrm{N} = 2\lambda/\mathrm{N}\epsilon, \text{ etc.}$$

But if l be the distance between the focal plane and the view-plane and x_1, x_2 the distances from the axis of the successive diffraction images, formed in the focal plane, clearly

$$x_1 = l\theta_1' = l\lambda/\mathrm{N}\epsilon,$$
$$x_2 = l\theta_2' = 2l\lambda/\mathrm{N}\epsilon, \text{ etc.},$$

so that these diffraction images are a series of equidistant lines. Moreover, since $l = \mathrm{N}f$, where f is the focal length of the object-glass, the distance x between these lines in the focal plane is $f\lambda/\epsilon$.

We have thus recovered our series of equidistant slits, and the image in the view-plane is the result of the light from these slits; it consists of a series of uniformly spaced bright and dark bands, the distances between the maxima and the axis being

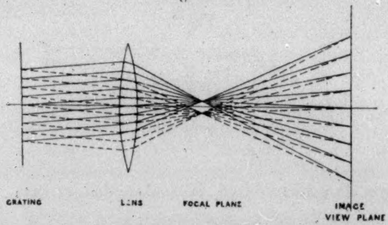
$$l\lambda/x, \quad 2l\lambda/x, \quad \text{etc.}$$

But since x is equal to $f\lambda/\epsilon$ these distances are $l\epsilon/f$, $2l\epsilon/f$, etc., and these values, since l/f is equal to N the magnifying power of the object-glass, become Nϵ, 2Nϵ, ...

Thus the image in the view-plane is the ordinary geometrical image of the grating; a result we have reached by what I venture to think is a somewhat roundabout method. The result is shown geometrically in figure 3.

It will be observed that we have not discussed the distribution of light in the interspaces between the maxima, and it is on this distribution that the question of resolving power depends. It is clear of course that if we modify the number of spectra in the focal plane we modify the image, and this is done in an ingenious way in some of the experiments arranged by Prof. Abbe's pupils to illustrate the theory.

Fig. 3.



If we cut out all but the central image the view-field is uniform, no structure is visible; if we allow the first image on either side of the central one to become effective the bands appear in the field in their proper positions, and so on. It is said to be a fundamental result of Abbe's theory that the object, the grating, can be fully resolved if one diffraction image on either side of the central one is effective.

It is clear that in this case there will be variations of intensity in the view-plane, we shall see later what these amount to.

Now the number of spectra is limited by the fact that some of the diffracted light may be so obliquely diffracted as not to enter the object-glass. If 2α be the angular aperture of the object-glass measured from the axial point of the stage, then the nth diffracted image will not appear if $\sin\theta_n$ is $> \sin\alpha$. But $\sin\theta_n = n\lambda/\epsilon$; hence for the nth image to be excluded $n\lambda/\epsilon$ must be greater than $\sin\alpha$. Now according to Abbe, for resolution the first diffracted image must appear, and hence resolution is just possible if λ/ϵ is equal to $\sin\alpha$.

It has been assumed that air is the medium on either side of the object-glass; if on the object side we have a medium of refractive index, then it is easy to show that we must replace $\sin\theta$ by $\mu\sin\theta$, and the condition of resolution is that ϵ should be equal to $\lambda/\mu\sin\theta$, or, introducing the term numerical aperture for the quantity $\mu\sin\theta$, we have the result that a grating can be resolved if the space between the lines is not less than the result found by dividing the wave-length of light by the numerical aperture.

Now while I am aware that the truth of this result can be established, the reasoning I have outlined—I trust I have not misrepresented it—does not seem to me to prove it.

In order to decide if the grating can be resolved, we must establish the law of variation of intensity in the view-plane, and then consider whether these variations are such that they can be detected by the eye. Nothing that I have seen published in Abbe's work does this; it has been done by Lord Rayleigh.

The images formed in a microscope are, like all other images, produced by interference; in considering resolving power we have to consider diffraction effects it is true, but the diffraction which concerns us mainly is that due to the aperture of the object-glass, and only indirectly that due to the object viewed; it seems to me that the size of the object on the stage only affects in a secondary manner the method in which the object is formed.

Neither is it necessary, if we know completely the distribution of the light over the stage, to go back to the source in our consideration of the problem; having given the distribution over the stage both in amplitude and in phase, we are theoretically able to determine that in the view-plane without reference to the source—difficulties of calculation may stop us it is true, but that is another matter.

Let us take again the case of a grating illuminated by plane waves, their plane being parallel to that of the grating; we have to consider the effect due to a series of equidistant lines of light: these differ, however, from a series of independent equidistant linear sources in that with the grating the phases of the various sources are the same, we have therefore to remember that interference will take place between the light from the different lines. With a series of independent lines there is no relation between the phases, we can calculate the intensity due to each source separately and superpose the whole.

Lord Rayleigh's solution of the problem which is presented when a narrow double line in a spectrum is viewed through a telescope, or when the attempt is made to resolve two close double stars, is better known than his equally valuable solution of the grating problem, and as it is simpler it will be useful to indicate it first.

The intensity in the view-plane for a single linear source, assuming for the moment that we are dealing with a telescope with a rectangular aperture, is given by a curve of the form shown in figure 4 or by the expression $\left(\frac{\sin u}{u}\right)^2$ in which $u = \frac{\pi a \zeta}{f\lambda}$, where a is the breadth of the slit, f the focal length of the lens, and ζ the distance from the central image of the point considered. If we assume a second independent source parallel to the first, we get a similar curve alongside the first. The resultant intensity is found by adding the corresponding ordinates of the two curves, and the lines will appear as double when the drop in the intensity curve shown at B in figure 5 is sufficient to be detected by the eye.

Fig. 4.

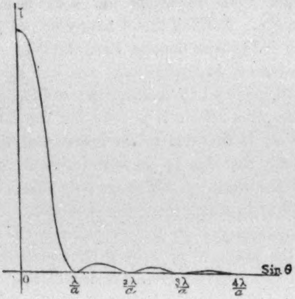
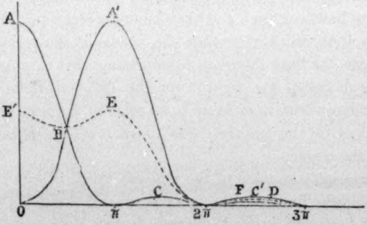


Fig. 5.

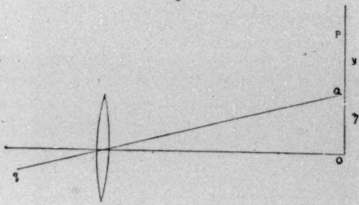


Lord Rayleigh suggested that in this case the drop would be just distinguishable when the maximum due to the second curve was superposed on the minimum due to the first, as is done in the figure, and experiment has borne this out.

In this case the two halves of the aperture send light in opposite phases to a point in the position of the first minimum, and the angular deflexion of the minimum is the angle subtended by the wave-length of light at the distance of the breadth of the aperture. Two lines which subtend a greater angle than this can be resolved.

Now similar methods have been applied by Lord Rayleigh (Phil. Mag. August 1896) to the microscope. Consider in figure 6 a luminous line q in front of the object-glass. Let Q be the geometrical image of q at a distance η from O the axial point, P a point at a distance y from O, then if the aperture be treated as a rectangle of breadth b, we have a wave of breadth b converging on to Q, and it can be shown directly that the amplitude at P is proportional to

Fig. 6.



$$\sin\frac{\pi b(y-\eta)}{\lambda l}\bigg/\frac{\pi b(y-\eta)}{\lambda l},$$

where l is the distance of O from the lens.

If we write $u=\frac{b(y-\eta)}{\lambda l}$ then the amplitude is proportional to $\sin u/u$.

The disturbance vanishes if $u=n\pi$.

The disturbance due to a source having its image at η_1 is proportional to $\sin u_1/u_1$,

$$\text{where}\quad u_1=\frac{\pi b(y-\eta_1)}{\lambda l}=u+v_1 \quad\text{if}\quad v_1=\frac{\pi b}{\lambda l}(\eta-\eta_1).$$

Thus the amplitudes from a series of sources will be proportional to

$$\frac{\sin u}{u},\quad \frac{\sin u_1}{u_1},\quad \frac{\sin u_2}{u_2},\ \text{etc.}$$

The resultant disturbance will depend on the relative phases of the sources. If they are entirely independent, like a series of self-luminous lines, the case treated above, then

$$\text{The resulting intensity} = \left(\frac{\sin u}{u}\right)^2 + \left(\frac{\sin u_1}{u_1}\right)^2 + \ldots$$

If the phases are the same throughout, the case of a grating illuminated by plane waves, then

$$\text{Intensity} = \left(\frac{\sin u}{u} + \frac{\sin u_1}{u_1} + \ldots.\right)^2.$$

If the phases are opposite, which might happen if the waves fall obliquely on the grating,

$$\text{Intensity} = \left(\frac{\sin u}{u} - \frac{\sin u_1}{u_1} + \ldots.\right)^2.$$

Now we have seen that for the case of a telescope resolution is just possible if the first maximum of the second source coincides with the first minimum of the first source. In such a case $v_1 = \pi$ and $u_1 = u + \pi$.

We have, therefore, to evaluate the intensity in such a case, *i. e.* to consider the functions :—

(1) Independent sources

$$\left\{\left(\frac{\sin u}{u}\right)^2 + \left(\frac{\sin (u+\pi)}{u+\pi}\right)^2\right\}^{\frac{1}{2}},$$

and (2) Plane waves parallel to the grating

$$\frac{\sin u}{u} + \frac{\sin (u+\pi)}{u+\pi}.$$

Lord Rayleigh, to whom the above is due, has traced these two curves in fig 7. It appears they have the form shown. In I there is a drop of 10 per cent. Resolution is possible. In II there is a gradual rise amounting in all to 27 per cent., and no trace of resolution is possible.

To find the ratio of the distance between the two images to the distance between the two sources, we have by the sine law

the Linear magnification =

$$\epsilon/(\eta - \eta_1) = \sin \beta/\sin \alpha,$$

where β is the angle the extreme refracted ray makes with the axis,

Thus $\sin \beta = b/2l,$

also $\eta - \eta_1 = \frac{\lambda l}{\pi b} v_1.$

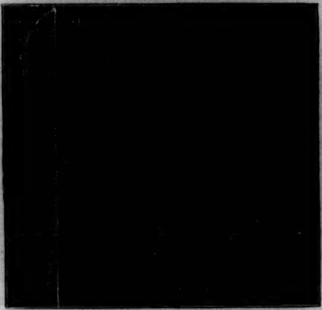
Hence

$$\epsilon \sin \alpha = (\eta - \eta_1) \sin \beta = \frac{\lambda l}{\pi b} v_1 \times \frac{b}{2l} = \frac{\lambda v_1}{2\pi},$$

and we have seen that resolution is possible if $v_1 = \pi$. Thus the limit for resolution is $\epsilon = \frac{1}{2}\lambda/\sin \alpha$.

Thus two independent double lines can be resolved if the distance between them is half that given by Abbe's theory.

Fig. 7.



Lord Rayleigh next considers the effect of a large number of equidistant sources. Let us take the image of the central source at $v=0$, and let the images of the rest be given by the following values for v, $\pm\pi$, $\pm 2\pi$, etc.

Then for independent sources he shows that

$$I = \frac{\pi}{v} + \Sigma \frac{2\pi}{v}\left(1 - \frac{\pi r}{v}\right) \cos \frac{2\pi r u}{v},$$

the series continuing so long as πr is less than v.

Thus the view-field has a periodic structure, but the amount of discontinuity gets less as v is reduced, *i. e.* as the sources get nearer together.

If v is equal to or less than π we have only the first term, and we get $I = \pi/v$.

Thus in this case, which corresponds to $\epsilon = \frac{1}{2}\lambda/\sin \alpha$, for a long series we have uniformity, while we saw above that for *two* sources there was a visible structure. If v is equal to 2π so that $\epsilon = \lambda/\sin \alpha$, we obtain a periodic structure and resolution.

Now take the grating with plane waves incident normally, then

Lord Rayleigh finds

$$A=\frac{\pi}{v}(1+2\cos\frac{2\pi u}{v}+2\cos\frac{4\pi u}{v}+\ldots\ldots)$$

the series continuing so long as $2\pi r$ is less than v.

As v diminishes the higher terms drop out. If it is equal to or less than 2π we have only the first term, the illumination is uniform, and $v=2\pi$ corresponds to Abbe's limit $\epsilon=\lambda/\sin\alpha$.

Each of the periodic terms corresponds to the effect of two of Abbe's lateral spectra: there is clearly no resolution possible unless at least the first spectrum on each side is effective. In this case, for which v is 2π,

$$A=\frac{\pi}{v}\left(1+2\cos\frac{2\pi u}{v}\right).$$

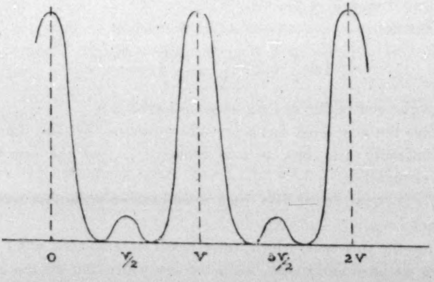
The following table gives the positions and values of the maximum amplitude in this case:

$u=0$	$v/2$	v	$3v/2$
$A=3$	-1	3	-1

with points of zero amplitude at $v/3$ . . . . . $2v/3$ . . . . .

Thus there are a series of bright bands of intensity 9 at the points corresponding to the geometrical images of the grating; midway between these are a series of less bright bands of intensity 1, while between each pair of bright bands is an absolutely dark space. The image possesses structure, but this structure is not exactly that of a grating, there is the superposed secondary series of maxima.

Fig. 8.



The intensity-curve will be as drawn, fig. 8, and in this case it is clear that if ϵ be greater than $\lambda/\sin\alpha$ there will be resolution, in accordance with Abbe's result.

Throughout the above we are dealing with a rectangular aperture. For the full discussion of the more general case reference must be made to Lord Rayleigh's paper.

In a more recent paper (Journal of the Royal Microscopical Society, June 17, 1903) Lord Rayleigh has discussed the case of a dark bar on a bright ground (*a*) when the ground is due to a plane wave advancing at right angles to the bar and parallel to the axis of the lens, (*b*) when the ground is self-luminous.

In case (*a*) he shows that the intensity will be reduced over the geometrical image of the bar by 1/8 of its maximum—and this is readily noticed even when the width of the bar is 1/32 of the minimum grating interval required for resolution—but that this reduction extends considerably outside the geometrical image, so that the apparent width is illusory.

For the second case the fall of intensity is only half what it was in the first, while the apparent breadth is again great.

Moreover, we are certain to arrive at erroneous consequences if we apply results obtained from the case of a grating of a large number of parallel slits, to a case such as that of a single small aperture through which light is coming, or a single small obstacle, obstructing the light; the diffraction pattern due to such an obstacle is entirely different to that due to a grating, and the conditions of resolution will be different also.

We may arrive at the result from considering the problem on Abbe's method in the following way. For a single slit illuminated by a plane wave the intensity in the focal plane is given by an expression of the form $\left(\frac{\sin u}{u}\right)^2$. This expression has a principal maximum for $u=0$, its other maxima are given by $u=\tan u$, whence $\sin u/u=\cos u$, so that, at the points of maximum intensity for which $u=u_1, u_2$, etc., the intensities are proportional to $\cos^2 u_1$, $\cos^2 u_2, \ldots$ But the roots of the equation $u=\tan u$ approximate more and more nearly to odd multiples of $\pi/2$ excluding the first, thus the intensities of the maxima fall off rapidly. The curves of the illumination of the source in the focal plane are different. We have no longer a series of equi-spaced sources but one of rapidly decreasing intensity.

While it is necessary to have a number of the spectra in the second case in order to reproduce our grating structure, it seems clear that in the first case the secondary spectra will have less influence on the result, the single narrow slit may still be visible although the aperture is too narrow to resolve a grating having the periodicity of the slit.

It appears, then, that while Abbe's theory of microscopic vision is undoubtedly correct in that a small object or objects on the stage will produce diffraction patterns in the focal plane of the object-glass, and the illumination in the view-plane can be inferred from these diffraction images, still this method of regarding the question is not the only possible one, and does not lead to a complete solution, while at the same time it is not necessary to go back to the original source in order to calculate the illumination.

By proceeding, however, in the way indicated by Lord Rayleigh, we can evaluate the distribution of intensity in the view-plane at any rate in certain cases, and obtain thus a numerical estimate of the resolvability.

In the above I have dealt with the grating theory as though it formed the whole of Prof. Abbe's contributions to the matter. Since the meeting my attention has been called to Dr. Czapski's remarks at a recent meeting of the Royal Microscopical Society, from which it is clear that in his lectures since 1887–88 Prof. Abbe has given a more complete theory in which he has discussed the general theory of microscopic vision. Dr. Abbe's pupils will add to the debt which is due to him if they will assist in the publication of this series of lectures.

In conclusion I have to express my great indebtedness to Mr. C. Beck and Mr. Gordon for the loan of apparatus to illustrate the grating theory of resolving power. These kindnesses enabled me to put it before the Society in a manner which would otherwise have been impossible, and for this I am most grateful.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1904-1905.

February 26th, 1904.

Meeting held at the Royal College of Science.

The President, and subsequently Mr. Blakesley, in the Chair.

The following Candidates were elected Fellows of the Society:—
H. T. Davidge, E. T. Jones, W. H. Pretty.

The following Papers were read:—

1. A Quartz-thread Vertical Force Magnetograph. By Dr. W. Watson.

2. On Stresses in a Magnetostatic Field. By Mr. G. W. Walker.

Mr. B. F. E. Keeling described a new Dilatometer, which was exhibited by Mr. B. Bonniksen.

Mr. R. J. Sowter exhibited some Simple Portable Electroscopes.

Dr. Watson gave some Hints on the Preparation of Diagrams, showing results and apparatus for their easy production.

March 11th, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Candidate was elected a Fellow of the Society:—
H. M. Hallsworth.

The following Papers were read:—

1. The Whirling and Transverse Vibrations of Shafts. By Dr. Chree.

2. Non-homocentric Pencils and the Shadows produced by them.—Part II. By Mr. W. Bennett.

Messrs. Pye & Co., of Cambridge, exhibited a number of Electrical and Mechanical Instruments.

March 25th, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Candidate was elected a Fellow of the Society:—
G. J. Burns.

The following Papers were read:—

1. Note on the Measurement of Small Inductances and Capacities, and on a Standard of Small Inductances. By Dr. J. A. Fleming.

2. A Hot-wire Ammeter for measuring very small Alternating Currents. By Dr. J. A. Fleming.

3. The Energy of Secondary Röntgen Radiation. By Mr. C. G. Barkla.

April 22nd, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Candidate was elected a Fellow of the Society:—
W. C. S. Phillips.

The following Papers were read:—

1. Calculation of Colours for Colour Sensitometers and the Illu-

mination of "Three-Colour" Photographic Transparencies by Spectrum Colours. By Sir **William Abney**.

2. Normal Piling as connected with Osborne Reynolds' Theory of the Universe. By Prof. J. D. **Everett**.

3. Note on the Diffraction Theory of the Microscope as applied to the case when the object is in motion. By Dr. R. T. **Glazebrook**.

Mr. C. E. S. **Phillips** exhibited an Automatic Gas-Pump.

Mr. **Peter Heele** gave an exhibition of Apparatus.

May 6th, 1904.

Meeting held at the Royal College of Science.

Mr. **Swinburne**, Vice-President, in the Chair.

The following Paper was read:—

Some Instruments for the Measurement of Large and Small Alternating Currents. By Mr. W. **Duddell**.

Mr. F. E. **Smith** then exhibited the following Apparatus from the National Physical Laboratory:—A Mercury-resistance standard; a 10-ohm build-up resistance-box; and an Astatic Galvanometer.

May 27th, 1904.

Meeting held at the Royal College of Science.

Mr. **Swinburne**, Vice-President, in the Chair.

The following Candidates were elected Fellows of the Society:—

E. F. **Etchells** and F. H. **Parker**.

The following Papers were read:—

1. The Law of Action between Magnets and its bearing on the Determination of the Horizontal Component of the Earth's Magnetic Field with Unifilar Magnetometers. By Dr. **Chree**.

2. The ascertained Absence of Effects of Motion through the Æther, in relation to the Constitution of Matter, and on the FitzGerald-Lorentz Hypothesis. By Prof. **Larmor**.

3. Coherence and Recoherence. By Dr. **Shaw** and Mr. C. A. B. **Garrett**.

June 10th, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Candidate was elected a Fellow of the Society:—

E. Smith.

The following Paper was read:—

A Model illustrating the Propagation of a Periodic Electric Current in a Telephone Cable, and a Simple Theory of its Operation. By Prof. J. A. Fleming.

Prof. H. L. Callendar gave a demonstration of the Projection of the Indicator Diagrams of a Petrol Motor.

Mr. M. E. J. Gheury exhibited one of Admiral Fleuriais' Gyroscopic Collimators.

June 24th, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Candidate was elected a Fellow of the Society:—

W. Liebert.

The following Papers were read:—

1. Chemical Dissociation and Electrical Conductivity. By Dr. Willows and Mr. A. E. Garrett.

2. The Magnetization of Iron in Bulk. By Dr. W. M. Thornton.

October 28th, 1904.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Papers were read:—

1. An Interference Apparatus for the Calibration of Extensometers. By Mr. John Morrow and Mr. E. L. Watkin.

2. A Sensitive Hygrometer. By Dr. W. M. Thornton.

3. Note on a Property of Lenses. By Dr. G. E. Allan.

November 11th, 1904.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Papers were read:—

1. Investigation of the Variations of Magnetic Hysteresis with Frequency. By Prof. T. R. LYLE.

2. The Determination of the Mean Spherical Candle-Power of Incandescent and Arc Lamps. By Mr. G. B. DYKE.

Mr. R. W. PAUL gave an exhibition of Electrical Instruments and other Apparatus.

November 25th, 1904.

Meeting held at Finsbury Technical College.

The PRESIDENT in the Chair.

The following Candidate was elected a Fellow of the Society:—

F. TWYMAN.

The following Paper was read:—

The Measurement of Small Differences of Phase. By Dr. W. E. SUMPNER.

Dr. C. V. DRYSDALE exhibited and described Apparatus for the Direct Determination of the Curvatures of Small Lenses.

Dr. S. P. THOMPSON gave an exhibition of Specimens of Crystals showing the phenomenon of Luminous Rings, and presented to the Society, on behalf of Dr. Hanswaldt, a series of plates, reproduced from photographs, of Interference Figures in Crystalline Plates.

December 9th, 1904.

Meeting held at the Central Technical College.

The PRESIDENT in the Chair.

The following Candidates were elected Fellows of the Society:—

B. W. CLACK, H. DAVIES, D. OWEN.

The following Papers were read:—

1. On a Rapid Method of Approximate Harmonic Analysis. By Dr. S. P. THOMPSON.

2. A High-Frequency Alternator. By Mr. W. Duddell.

Prof. Ayrton gave an exhibition of experiments showing the Retardation of the Signalling Current on a cable equivalent to 3500 miles of the Pacific Cable between Vancouver and Fanning Island.

January 27th, 1905.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Papers were read:—

1. The Action of a Magnetic Field on the Discharge through a Gas. By Dr. R. S. Willows.

2. The Action of Radium on the Electric Spark. By Dr. R. S. Willows and Mr. J. Peck.

3. The Slow Stretch in Indiarubber, Glass, and Metal Wires subjected to a Constant Pull. By Mr. P. Phillips.

4. Determination of Young's Modulus (Adiabatic) for Glass. By Mr. Chichester Bell.

5. Some Methods for Studying the Viscosity of Solids. By Dr. Boris Weinberg.

Annual General Meeting.

February 10th, 1905.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting thirteen Science Meetings of the Society have been held. Of these, eleven were held at the Royal College of Science; one, that on November 25th, was held at the Finsbury Technical College, and one, on December 9th, at the Central Technical College, South Kensington. The average attendance at the meetings has been 43, the number being somewhat larger at the evening than at the afternoon meetings. Additional interest has been given to the meetings by the increase in the number of experiments shown to illustrate the papers.

The number of Fellows now on the roll is 425, an increase of 7 over the number last year. Fourteen new Fellows have been elected. There have been four resignations, and the Society has to mourn the loss by death of one Honorary Fellow, Prof. Villari, and three Fellows, namely, Prof. J. D. Everett, W. T. Goolden, and Dr. Lawson.

The Council have co-operated with the Institution of Electrical Engineers in bringing "Science Abstracts" up to date. This has entailed a rather heavy expenditure, which, however, the Council felt was fully justified by the importance of the work. The Council are glad to be able to report that the publication has now been brought up to date, and in future the Society will be liable only for the usual annual contribution. The Agreement with the Institution referred to in the last report has not yet been executed, owing to difficulties in making an agreement exactly on the lines which were proposed. Some alterations in detail have therefore been made and a three years' agreement will shortly be executed, dating from the beginning of the present year.

During the year the President signed on behalf of the Council a petition in support of a Bill for the Metric System.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The following Candidate was elected a Fellow of the Society:—

H. H. F. HYNDMAN.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. J. H. POYNTING, Sc.D., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; The Lord KELVIN, G.C.V.O., D.C.L., LL.D., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Prof. W. E. AYRTON, F.R.S.; Prin. Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.; SHELFORD BIDWELL, M.A., LL.B., F.R.S.; Prin. Sir OLIVER J. LODGE, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.; R. T. GLAZEBROOK, D.Sc., F.R.S.

Vice-Presidents.—C. Chree, D.Sc., F.R.S.; H. M. Elder, M.A.; Prof. J. A. Fleming, M.A., D.Sc., F.R.S.; J. Swinburne.

Secretaries.—W. Watson, D.Sc., F.R.S.; W. R. Cooper, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Prof. H. L. Callendar, M.A., F.R.S.

Librarian.—W. Watson, D.Sc., F.R.S.

Other Members of Council.—T. H. Blakesley, M.A.; C. V. Boys, F.R.S.; A. Campbell, B.A.; Prof. W. Cassie, M.A.; W. B. Croft, M.A.; W. Duddell; W. A. Price, M.A.; S. Skinner, M.A.; Prof. F. T. Trouton, Sc.D., F.R.S.; Prof. S. A. F. White, M.A.

Votes of thanks were passed to the Auditors, the Officers and Council, and to the Board of Education.

The President then delivered his Address.

Treasurer's Report.

Additional expenditure was incurred last year amounting to £115 in bringing 'Science Abstracts' up to date, which has the effect of reducing the balance in the bank as compared with that at the beginning of the year. But the income from subscriptions has increased, so that the reduction of the balance is less than the extraordinary expenditure. The value of some of the securities has declined, but others have risen, so that on the whole the position of the Society remains almost unchanged.

HUGH L. CALLENDAR,
Hon. Treasurer.

The Treasurer in Account with the Physical Society, from January 1st, 1904, to December 31st, 1904.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1904. Balance in Bank				154	9	11
Entrance-Fees	13	13	0			
Subscriptions	380	2	0			
Extra Subscriptions	34	11	0			
Arrears	25	4	0			
Abstracts etc.	3	5	6			
Life Composition	31	10	0			
Students	1	11	6			
				489	17	0
Dividends:—						
Furness Debenture Stock	15	4	10			
Metropolitan Board of Works	6	13	4			
Lancaster Corporation Stock	11	8	6			
New South Wales	8	9	5			
Midland Railway	38	2	6			
Great Eastern Railway	19	1	3			
London, Brighton, & South Coast Railway	26	4	5			
				125	4	3
Sales				39	2	9
Dinner Surplus				2	7	2
				£811	1	1

Cr.	£	s.	d.	£	s.	d.
Science Abstracts				391	9	1
International Catalogue				18	5	4
Taylor and Francis:—						
Proceedings	148	15	0			
Abstracts	72	0	6			
Distribution	19	17	0			
Separate Copies	18	10	0			
				259	2	6
Periodicals				5	7	7
Attendance				6	0	0
Reporting				29	18	0
Commissions					5	7
Chemical Society (for 1903)				14	3	6
Binding				1	0	0
Petty Cash (Treasurer)				1	6	6
Balance in Bank				84	3	0
				£811	1	1

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

27*th January*, 1905.

WALTER BAILY, } *Auditors.*
ED. F. HERROUN, }

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1904.

Assets.	£	s.	d.
Subscriptions due, estimated	37	0	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	447	0	0
£1600 Midland Railway 2½ per cent. Preference Stock	1168	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	211	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	352	0	0
£254 2*s.* 9*d.* New South Wales 3½ per cent. Inscribed Stock	242	0	0
£500 London, Brighton, and South Coast Railway Ordinary Stock	680	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	595	0	0
Publications in Stock, estimated	210	0	0
Balance in Bank	84	3	0
	£4026	3	0

Liabilities.	£	s.	d.
Taylor and Francis, Printing	46	18	1
Cheques not presented	6	10	7
Balance	3972	14	4
	£4026	3	0

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

27*th January*, 1905.

WALTER BAILY,
ED. F. HERROUN, } *Auditors.*

OBITUARY.

Joseph David Everett, M.A., D.C.L., F.R.S., Vice-President of the Physical Society, was born on September 11th, 1831, at Rushmere, near Ipswich, and was the eldest son of a country gentleman belonging to an old Suffolk family of wealthy landowners. He received his early education at private schools in Ipswich, where from the first he showed an eager interest in all he undertook. He was anxious to go to Cambridge, but his father, who had a large family, and seems to have had little sympathy with his zeal for learning, was unwilling to gratify his ambition, and he therefore took a post as mathematical master in a good private school till he was sufficiently independent to make his own way. He then entered the University of Glasgow, where he graduated as M.A. in 1859 with first-class Honours in Mathematics and Natural Philosophy. He attended Lord Kelvin's class for a year only, and in after-life spoke of the immense inspiration he received in that year From Glasgow he went to Edinburgh as Secretary of the Scottish Meteorological Association and then, after a few months, was appointed Mathematical Professor at King's College, Windsor, Nova Scotia, where in addition to his class duties he made meteorological and astronomical observations. His "Reduction of the deep-sunk Thermometers" was published in the Greenwich Observations, 1860, and "Reduction of Edinburgh Underground Thermometers" in the Edin. Roy. Soc. Trans. vol. xxii. 1861. In 1864, finding the situation too remote, he resigned, and returned to Glasgow, where he assisted the Mathematical Professor, and carried on original researches for the determination of rigidities of glass, iron and copper in Lord Kelvin's laboratory; the results of which appeared in Phil. Trans. 1866–68. "Results of Observations of Atmospheric Electricity" were also printed in Phil. Trans. 1868. In 1866 he brought out a "Proportion Table," a slide-rule in parallel columns. In 1867 he was appointed Professor of Natural Philosophy at Queen's College, Belfast, a post which he held for thirty years. Among his old students who took distinguished places in after-life may be mentioned Prof. Perry, F.R.S., of the Royal College of Science, and Prof. Larmor, Sec. R.S. At Belfast, owing to the absence of a well-equipped physical laboratory, he found it impracticable to continue his experimental investigations, and henceforth turned his attention to the literary side of science. About a year after his appointment, he translated Deschanel's treatise on Physics, then just published. The translation was adopted as a standard text-book even before the last part appeared and has since passed though 16 editions, and undergone so much careful re-editing as to be practically an original work. Discussion in 'Nature' which followed its publication led him to urge the necessity for giving convenient names to some of the absolute physical units, and he induced Sections A and G of the British Association to appoint a Committee for the purpose. As Secretary of the Committee he was largely responsible for its report recommending that all kinds of physical magnitudes should be specified by means of three fundamental units, and for the launching of the C.G.S. system upon the world. The system found international adoption at the Paris Congress of Electricians, and Prof. Everett's book 'Units and Physical Constants,' containing its exposition, has been translated into most modern languages. He also did useful work in preparing the annual reports of the British Association Committee for investigating Underground Temperature, of which he was Secretary (and afterwards Chairman) from 1868 till the end of his life. At Belfast he also wrote the following works :—'Elementary Text-Book of Physics' (which reached its 11th edition in 1901), 'Vibratory Motion and Sound,' 'Outlines of Natural Philosophy,' besides many papers on various subjects, including Mirage, Brightness of Images, A new Focometer, Resultant Tones, Dynamical Illustrations of Optical Phenomena (all in Phil. Mag.), Relations between Orbits, Catenaries and Curved Rays, Linkages (B.A. Reports, 1888-89, 1894), A new Method in Kinematics (Messenger of Math. vol. xlv. 1875), The Thomson Effect ('Nature,' 1886), &c.

In 1897 he retired from the Professorship, and soon afterwards came to reside in London, where he took a keen and active interest in the proceedings of various scientific societies. The following are papers which he read at the Physical Society during the last two or three years :—"Focal Lines and Anchor-Ring Wave-Fronts," "Resolving Power of Objectives," "Comparison of Vapour Temperatures at Equal Pressures," "Bees' Cells," "Normal Piling as connected with Osborne Reynolds' Theory of the Universe." Another paper, on the Hollow Pencil given by an Annulus of an obliquely-placed Lens, was read at the Royal Society (Proc. vol. lxxi.). In 1901, his book 'Electricity' appeared, and Part III. of Deschanel, which it replaced, was withdrawn. In 1902 he published, in conjunction with Miss Everett, an English edition of